# Mock exam 1 – Group theory

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) For each list of groups a) and b) below, decide which of the groups within each list are isomorphic, if any:

a) $\mathbb{Z}_{20}$, $\mathbb{Z}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$.

b) $\mathbb{Z}_2 \times D_7$, $\mathbb{Z}_2 \times \mathbb{Z}_{14}$, $D_{14}$.

2) Let $G$ be the set of sequences of integers endowed with the following product operation $+ : G \times G \longrightarrow G$

$$(a_1, a_2, \cdots, a_n, \cdots) + (b_1, b_2, \cdots, b_n, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots).$$

Show that this operation makes $G$ into a group. Show that $\mathbb{Z} \times G \cong G$ and hence conclude that, for groups, it may be the case that $A \times C \cong B \times C$ even though $A \ncong B$.[1]

3) Let $n > m$ be natural numbers, $n > 4$, let $X$ be a set with $m$ elements. Show that the orbits of any action of $S_n$ on $X$ have size 1 or 2.

4) Let $G$ be a group, $S_G$ be group of bijections from $G$ into itself and $\mathrm{Aut}(G) \subset S_G$ be the group of automorphisms of $G$. Consider the map $\mathrm{Ad} : G \longrightarrow S_G$, given by

$$\mathrm{Ad}(g) : G \longrightarrow G \qquad \mathrm{Ad}(g)(x) = gxg^{-1}.$$

a) Show that $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$, i.e., for every $g \in G$, $\mathrm{Ad}(g) : G \longrightarrow G$ is an automorphism;

b) Show that $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ is a group homomorphism and that the image of Ad is a normal subgroup of $\mathrm{Aut}(G)$. The image of Ad is called the *group of inner automorphisms*.

[1] I'd never ask this in an exam, but at home you may try to prove that *for finite groups* it is true that $A \times C \cong B \times C$ implies $A \cong B$. If you just want to see a proof, take a look at Hirshon's paper On cancellation in groups.

c) Show that the kernel of $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(G)$ is the center of $G$ and conclude that the group of inner automorphisms is isomorphic to the quotient $G/Z_G$.

d) Give an example of a group which has an automorphism which is not an inner automorphism.

5) Classify all groups or order $2009 = 7^2 \cdot 41$.

6) Let $G$ be a group and $n \in \mathbb{N}$

a) Let $H_i < G$ be subgroups, for $i \in \{1, \cdots, n\}$, show that

$$\cap_{i=1}^n H_i$$

is a subgroup of $G$.

b) If $G$ is finite and $p$ be a prime. Show that the intersection of all $p$-Sylows of $G$ is a normal subgroup.

7) Let $G$ be a finite group and $K, H < G$. Prove or give a counter-example to the following claims.

a) If $K \lhd H$ and $H \lhd G$ then $K \lhd G$.

b) If $K$ is the only $p$-Sylow of $G$, then $K \cap H$ is a $p$-Sylow of $H$.