Group theory – Sheet 6

The exercises from the book are 6.2, 6.3, 6.6, 6.7, 6.8, 6.9, 6.10, 6.12 and 8.9.

(1) Show that if $H_i < G$ are subgroups, for i in some index set $\mathcal{I}$, then $\cap_{i\in\mathcal{I}} H_i$ is also a subgroup.

(2) Show that if n_i are pairwise coprime numbers, for $1 \leq i \leq k$, then

$$\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k} \cong \mathbb{Z}_{n_1 n_2 \cdots n_k}.$$

Conclude that an Abelian group of order $p_1 p_2 \cdots p_k$, with p_i prime numbers, is isomorphic to $\mathbb{Z}_{p_1 p_2 \cdots p_n}$.

(3) Let S be a set and G a group acting on S. Let $S_0 \subset S$ be a subset of S. Consider the following subsets of G

$$H = \{g \in G : gs = s \quad \forall s \in S_0\}$$

$$K = \{g \in G : gs \in S_0 \quad \forall s \in S_0\}$$

Show that $H \subset K$ and that H and K are subgroups of G. Show that H is a normal subgroup of K.

(4) Let G be a group of order pq with p, q be primes $p > q$.
a) Show that any subgroup of G of order p is normal.
b) Show that if q does not divide $p - 1$, then G is Abelian.
c) Conclude that the only group of order 15 is $\mathbb{Z}_{15}$. Follow the steps of item b) to find all groups of order 21.

(5) Show that if $n < p$, for some prime p then any S_n action on a set with p elements has more than one orbit, i.e., there is no transitive action of S_n on a set with p elements.