Group theory – Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) a) Let α, β be elements of the symmetric group S_n. Show that if α and β commute and $i \in \{1, 2, \cdots, n\}$ is fixed by α, i.e., $\alpha(i) = i$, then $\beta(i)$ is also fixed by α. (0.5 pt)

b) Show that, for $n > 2$, $Z_{S_n} = \{e\}$. (0.5 pt)

c) Show that, for $n > 3$, $Z_{A_n} = \{e\}$. (0.5 pt)

d) What is the center of A_3? (0.5 pt)

2) For each of the lists below, determine which groups are isomorphic:

a) $\mathbb{Z}_4 \times \mathbb{Z}_9$, $\mathbb{Z}_6 \times \mathbb{Z}_6$, $\mathbb{Z}_{36}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. (0.75 pt)

b) $A_5 \times \mathbb{Z}_2$, S_5, D_{30}, $D_{15} \times \mathbb{Z}_2$. (0.75 pt)

3) Let G be the group generated by

$$G = \langle a, b | a^n = b^m = e; bab^{-1} = a^l \rangle$$

Show that if $l^m \neq 1 \mod n$ then the order of a is less than n. (1 pt)

4) Given a group G, a subgroup $H < G$ is called *proper* if H is neither $\{e\}$ nor G. Find a group which is isomorphic to one of its proper subgroups. (Hint: this is only possible for infinite groups). (1 pt)

5) Let G be a group. Then the *conjugacy class* of an element $x \in G$ is the set

$$\mathcal{C}_x = \{gxg^{-1} : g \in G\}$$

and the centralizer of x, denoted by $C(x)$ is the set of all elements in G which commute with x, i.e.,

$$C(x) = \{g \in G : gxg^{-1} = x\}$$

a) Show that the centralizer of x is a subgroup of G. (0.75 pt)
b) Show that, if G is finite, then index of $C(x)$ in G, i.e., the number of elements in $G/C(x)$, is the number of elements in $\mathcal{C}_x$, the conjugacy class of x. (0.75 pt).

6 a) Show that if S_n acts on a set with p elements and $p > n$ is a prime number then the action has more than one orbit (0.75 pt).
b) Let p be a prime. Show that the only action of $\mathbb{Z}_p$ on a set with $n < p$ elements is the trivial one (0.75 pt).

7) Let G be a group, S a set and $\varphi : G \times S \longrightarrow S$ be an action. Let H be the stabilizer of a point $s \in S$. Show that the stabilizer of $g \cdot s$ is gHg^{-1}. Conclude that H is a normal subgroup of G if and only if it is the stabilizer of all the points in the orbit of s. (1.5 pt)