Group theory – Mock Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Compute the center of D_n. Analyse carefully the cases $n = 2$, n odd and n even and greater than 2.

2) Prove that if n is odd then $D_n \times \mathbb{Z}_2 = D_{2n}$. Is the same true when n is even?

3) Which of the following groups are isomorphic?

$$A_4 \times \mathbb{Z}_2, \qquad S_4, \qquad D_{12}, \qquad \mathbb{Z}_{24}, \qquad S_3 \times \mathbb{Z}_4, \qquad \mathbb{Z}_3 \times \mathbb{Z}_8.$$

4) Show that if D_n acts on a set with p elements and $p > n$ is a prime number then the action has more than one orbit.

5) Let $\mathcal{S} \subset S_5$ be the set of 5-cycles, sitting inside the group of permutations of 5 elements. Then S_5 acts on $\mathcal{S}$ by conjugation:

$$\sigma \cdot \tau := \sigma\tau\sigma^{-1}, \qquad \sigma \in S_5 \quad \tau \in \mathcal{S}.$$

Compute the orbit and the stabilizer of the 5-cycle (1 2 3 4 5).

6) Let G be a finite group and $H < G$. Let $n = \#G/H$ be the index of H. Show that $g^{n!} \in H$ for all $g \in G$.

7) Let G be an Abelian group. Show that if the order of G is $p_1 p_2 \cdots p_n$, where $p_1 < p_2 < \cdots < p_n$ are prime numbers, then G is isomorphic to $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$.