Group theory – Mock Exam 2

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) Is $(\mathbb{R}, +)$ isomorphic to $(\mathbb{R} - \{0\}, \cdot)$?

No. $\mathbb{R} - \{0\}$ has an element of order 2, namely, -1, while $\mathbb{R}$ does not have any element of order 2. Indeed, if $x \in \mathbb{R}$ is such that $x + x = 0$ then $x = 0$.

2) Show that if a finite group G has only two conjugacy classes, then $G \cong \mathbb{Z}_2$.

Let $\#G = n$. Since G is a group, it acts on itself by conjugation and the orbits of this action are the conjugacy classes. Since $\{e\}$ is always a conjugacy class of its own and G has only two conjugacy classes, we conclude that $G - \{e\}$ is the other class, which therefore has $n - 1$ elements.

By the Orbit-Stabilizer theorem, the number of elements in a conjugacy class (orbit of the action by conjugation) must divide divide the order of the group, i.e., $n - 1$ divides n. This is only possible for $n = 2$, hence G has two elements and therefore is isomorphic to $\mathbb{Z}_2$, the only group of order two.

3) Let H be a subgroup of finite index of an infinite group G. Prove that G has a normal subgroup of finite index contained in H.

Let n be in index of H on G and let G act on the set of left H-cosets, G/H. This action corresponds to a group homomorphism $\phi : \mathcal{G} \longrightarrow S_n$. Let $\mathcal{I} < S_n$ be the image of G by this group homomorphism and $K \lhd G$ be its kernel. Then, according to the isomorphism theorem, $G/K \cong \mathcal{I} < S_n$. In particular $\#G/K$ is finite, hence K has finite index. Further, K, being the formed by the elements which act trivially, is the intersection of the stabilizers of all points in G/H. Since H is the stabilizer of $H \in G/H$, we see that $K < H$. Therefore K is a normal subgroup of finite index contained in H.

4) Given a group G, a subgroup H is a *maximal normal* subgroup if

i) H is normal and

ii) if $K < G$ is a normal subgroup and $H < K$ then $K = H$ or $K = G$, i.e., the only normal subgroup of G which contains H as a proper subgroup is G.

Show that a normal subgroup H is maximal normal subgroup if and only if G/H is a simple group.

We will start proving that if H is not maximal normal, then G/H is not simple. So, assume H is a normal subgroup which is not maximal, that is, there is $K \lhd G$ such that $H \subsetneqq K \subsetneqq G$. Let $\pi : G \longrightarrow G/H$ be the quotient map. Then, since K is a subgroup of G, $\pi(K)$ is a subgroup of G/H whose generic element is of the form kH for $k \in K$. Hence, given $gH \in G/H$ and $kH \in \pi(K)$, we have

$$gH \cdot kH \cdot g^{-1}H = gkg^{-1}H \in KH = \pi(K),$$

where we have used in the step the fact that K is normal. Therefore we conclude that $\pi(K)$ is a normal subgroup of G/H. Since $H \subsetneqq K \subsetneqq G$, this is a nontrivial subgroup of G/H, showing that G/H is not simple.

Now we prove the converse. Assume that G/H is not simple and let $\tilde{K}$ be a nontrivial normal subgroup of G/H. Now consider the set $K = \pi^{-1}(\tilde{K}) \subset G$. We claim that this set is a normal subgroup of G which contains H.

1. $H \subset K$: Indeed, since $\tilde{K}$ is a subgroup of G/H, $eH \in \tilde{K}$ and hence $H = \pi^{-1}(eH) \subset \pi^{-1}(\tilde{K}) = K$.
2. *K is a subgroup*: Indeed, from (i), we have that $e \in H \subset K$. Further, given $k_1, k_2 \in K$,$\pi(k_1 k_2^{-1}) = \pi(k_1)(\pi(k_2))^{-1} \in \tilde{K}$, since for any element $k \in K$, $\pi(k) \in \tilde{K}$ and $\tilde{K}$ is a subgroup. Therefore $k_1 k_2^{-1} \in \pi^{-1}(K)$ and K is a subgroup.
3. *K is normal*: Indeed, for $g \in G$ and $k \in K$, we have that $\pi(gkg^{-1}) = \pi(g)\pi(k)\pi(g)^{-1} \in \tilde{K}$, since $\pi(k) \in \tilde{K}$ and $\tilde{K}$ is normal. Therefore $gkg^{-1} \in \pi^{-1}(\tilde{K}) = K$ for every $k \in K$ and every $g \in G$, showing that K is normal.
4. Since $\{eH\} \subsetneqq \tilde{K} \subsetneqq G/H$, we also get $H \subsetneqq K \subsetneqq G$.

Therefore we have seen that if G/H is not simple, then H is not a maximal normal subgroup.

5) Let G be a finite group and let p be the smallest prime which divides the order of G. Show that is $H < G$ is a subgroup of index p then H is normal.

Consider G/H, the set of left H-cosets, so that $G/H = p$. Then G acts on G/H by left multiplication and this action corresponds to a group homomorphism $\varphi : G \longrightarrow S_p$.

By Lagrange's theorem and the first isomorphism theorem, we have

$$\#G = \#\ker(\varphi)\#\mathrm{Im}(\varphi),$$

so $\#\mathrm{Im}(\varphi)$ must divide $\#G$. But also, $\mathrm{Im}(\varphi) < S_p$, so $\#\mathrm{Im}(\varphi)$ must divide $p!$. Since p is the smallest prime dividing $\#G$, we see that the only common factor between $\#G$ and $p!$ is p, so $\#\mathrm{Im}(\varphi)$ can only be either 1 or p. Since the group action is nontrivial, $\#\mathrm{Im}(\varphi) = p$ and hence $\#\ker(\varphi) = \#G/p$.

Now observe that H is the stabilizer of the point $eH \in G/H$, and the kernel of φ made by the group elements which fix all points in G/H, so $\ker(\varphi) < H$. But by the argument above H and $\ker(\varphi)$ have the same number of elements, so they must coincide, i.e., $H = \ker(\varphi)$ and since H is the kernel of a group homomorphism, it is normal.

6) Show that a group or order $2 \cdot 3 \cdot 5 \cdot 29^2$ is not simple.

Let n_{29} be the number of 29-Sylows. By Sylow's theorem, n_{29} must divide $2 \cdot 3 \cdot 5$ and must be equal to 1 mod 29. There are only two numbers which satisfy these conditions, namely 1 and 30. If $n_{29} = 1$, then the 29-Sylow is a normal subgroup and hence G is not simple.

Now consider the case $n_{29} = 30$. We let G act on the set of 29-Sylows by conjugation. By Sylow's theorem, this is a nontrivial action, hence it corresponds to a nontrivial group homomorphism

$$\varphi : G \longrightarrow S_{30}.$$

We prove now that this homomorphism has a nontrivial kernel and hence G is not simple. Since the action is nontrivial, $\ker(\varphi) \neq G$. If $\ker(\varphi) = \{e\}$ then by the isomorphism theorem, G would be isomorphic to its image, a subgroup of S_{30}. By Langrange, the order of G would divide the order of S_{30}. But 30! is not divisible by 29^2, while the order of G is. Therefore it can not be that $\ker(\varphi) = \{e\}$ and hence $\ker(\varphi)$ is a nontrivial normal subgroup of G.

7) Show that in a group of order $5 \cdot 7 \cdot 13$ the 7-Sylow and the 13-Sylow are normal. Show that such group has nontrivial center.

The number of 7-Sylows, n_7 must divide $5 \cdot 13$ and be equal to 1 modulo 7. The only possibility is $n_7 = 1$. Similarly, n_{13} must divide $5 \cdot 7$ and be 1 mod 13. The only possibility again is $n_{13} = 1$ So both the 7 and the 13 Sylows are normal.

Now let H be the 13-Sylow. Since H has prime order, H is cyclic. Let h be a generator for H. Since H is normal, for any $g \in G$ we have $ghg^{-1} \in H$. Since h is a generator for H, there is n such that $ghg^{-1} = h^n$. Let m be the order of g. Since the order of any element divides the order of the group, we have that m can only be one of the following numbers:

$$\text{possible values of } m : 1, 5, 7, 13, 5 \cdot 7, 5 \cdot 13, 7 \cdot 13, 5 \cdot 7 \cdot 13. \tag{1}$$

Further, we have

$$h = g^m h g^{-1} = h^{n^m}.$$

Hence $n^m = 1$ mod 13. And, thinking of n as element of the group $\mathbb{Z}_{13} - \{0\}$, we conclude that the order of n divides m. Since the order of n must divide 12, the order of $\mathbb{Z}_{13} - \{0\}$, the order of n can be one of the following numbers: 1, 2, 3, 4, 6 or 12. Since, except for 1, the numbers in this list do not divide any of the numbers in the list (1), we see that n must have order 1 and hence $n = 1$ and $ghg^{-1} = h$ for all $g \in G$, showing that $h \in Z_G$ and therefore $H \in Z_G$.

8) Show that every element in $SO(3)$ corresponds to rotation around an axis in $\mathbb{R}^3$.

Let A be a matrix in $SO(3)$. Then A the characteristic polynomial of A is a cubic and therefore has at least one real root, i.e., A has at least one real eigenvalue, λ. Since A is orthogonal, it preserves lengths, hence $\lambda = \pm 1$. Further either A has other two real eigenvalues or complex conjugate eigenvalues: $e^{i\theta}$ and $e^{-i\theta}$.

In the first case all real eigenvalues must be ± 1 and their product must be 1, the determinant of A. This means that at least one of them must be 1 and the other two may be either both 1 or both -1. If they are both 1, A is the identity matrix which is a rotation of 0 degrees around any axis. If they are both -1, then A fixes the axis generates by the 1-eigenvector and rotates its orthogonal complement by π (multiplicaiton by -1).

In the second case, the determinant of A is $1 = \lambda e^{i\theta} e^{-i\theta} = \lambda$, so the eigenvalue must be 1 and the matrix A fixes the axis generated by the $+1$-eigenvector and rotates its orthogonal complement by θ, since in the orthogonal complement of the $+1$-eigenspace A is the orthogonal matrix with eigenvalues $e^{i\theta}$ and $e^{-i\theta}$.