Group theory – Sheet 12

The exercises from the book are 13.9, 13.10, 13.11, 13.12, 14.11, 14.12.

1) Let $\mathbb{H}$ denote $\mathbb{R}^4$ with the quaternion product:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Given $p = p_0 + p_1 i + p_2 j + p_3 k$, we let $\overline{p} = p_0 - p_1 i - p_2 j - p_3 k$ be its quaternionic conjugate.

i) Show that $\overline{pq} = \overline{q}\,\overline{p}$;

ii) Show that $p\overline{p} = p_0^2 + p_1^2 + p_2^2 + p_3^2 = \|p\|^2$. Conclude that

- if $p \neq 0$, then $p^{-1} = \frac{\overline{p}}{\|p\|^2}$;
- if $p \in S^3$, then $p^{-1} = \overline{p}$;
- if $p \in S^3$ is purelly imaginary, then $p^{-1} = -p$, hence there is an S^2 worth of elements which square to -1.
- if $a, b \in \mathbb{H}$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^4$, show that

$$\langle a, b \rangle = \frac{1}{2}(a\bar{b} + b\overline{a})$$

- let $a, b \in S^2$ be purely imaginary quaternions of norm 1 and orthogonal to each other and let $c = ab$. Show that c is a purely imaginary quaternion of length 1 orthogonal to a and b. Further the following identities hold

$$a^2 = b^2 = c^2 = abc = -1.$$

2) Define a map $Ad : \mathbb{H}\backslash\{0\} \longrightarrow O(4)$ by

$$q \mapsto Ad_q \qquad Ad_q(x) = qxq^{-1}$$

i) Show that Ad_q is indeed in $O(4)$, i.e., $\|Ad_q(x)\| = \|x\|$ and that this is a group homomorphism,

ii) Show that the map

$$det : \mathbb{H}\backslash\{0\} \longrightarrow \mathbb{R}, \qquad q \mapsto det(Ad_q)$$

is continuous and maps $\mathbb{H}\backslash\{0\}$ into $\{\pm 1\} \subset \mathbb{R}$. Conclude that $det(Ad_q) = 1$ for all $q \in \mathbb{H}\backslash\{0\}$;

iii) Show that $\mathbb{R} \cdot 1 \subset \mathbb{H}$ is fixed by Ad_q for all q hence $Ad_q : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, where $\mathbb{R}^3$ is the vector space of imaginary quaternions (the orthogonal complement of $\mathbb{R} \cdot 1$);

iv) Given $q \in \mathbb{H}$ a quaternion with nonzero imaginary part, write $q = q_R + q_I$, where q_I is a real number and q_I is purely imaginary. Let $a = \frac{q_I}{\|q_I\|}$ so that $q = q_R + \|q_I\|a = \|q\|(\cos(\theta) + a\sin(\theta)) = \|q\|e^{a\theta}$ where a is an imaginary quaternion of norm 1 and θ in an appropriate angle.

Show that $Ad_q(a) = a$ and that in the plane in $\mathbb{R}^3$ orthogonal to a Ad_q corresponds to rotation by 2θ. Conclude that $Ad : \mathbb{H} \longrightarrow SO(3)$ is surjective.

v) Compute the kernel of Ad and conclude that $\mathbb{H}\backslash\{0\}/\mathbb{R}\backslash\{0\} \cong SO(3)$;

vi) Show that $S^3/\mathbb{Z}_2 \cong \mathbb{H}\backslash\{0\}/\mathbb{R}\backslash\{0\}$.

3) Consider S^3 with the group structure it inherits from $\mathbb{H}$ and define a map

$$\varphi : S^3 \times S^3 \longrightarrow O(4) \qquad \varphi(p, q)(x) = pxq^{-1}.$$

i) Show that φ is indeed a map into $O(4)$. Show that φ is a group homomorphism and that its kernel is isomorphic to $\mathbb{Z}_2$;

ii) Show that

$$\det \circ \varphi : S^3 \times S^3 \longrightarrow \mathbb{R}, \qquad (p, q) \mapsto det(\varphi(p, q))$$

is continuous and takes values in $\{\pm 1\}$ hence it is constant an equal to 1. Therefore $\varphi : S^3 \times S^3 \longrightarrow SO(4)$.

iii) Show that φ is surjective and conclude that $SO(4) \cong S^3 \times S^3 / \mathbb{Z}_2$. Hint: Observe that for $A \in SO(4)$, if $\tilde{p} = A(1)$, then $\tilde{p}^{-1} \circ A \in SO(4)$ and $\tilde{p}^{-1} A(1) = 1$. Then use the previous exercise.