Group theory – Sheet 8

The exercises from the book are 15.2, 15.5, 15.6, 15.7, 15.8, 15.11, 15.12, 15.14, 15.15, 15.16.

15.1)*Let $H, K < G$. Show that the set*

$$HK = \{h \cdot k | h \in H \text{ and } k \in K\}$$

is a subgroup if and only if $HK = KH$.

Conclude that if K is normal, and $H < G$ then HK is a subgroup of G.

2) (**Upper central series**) *Given a group G, let $Z_0 = \{e\}$ and define inductively*

$$Z_i = \{g \in G : ghg^{-1}h^{-1} \in Z_{i-1}, \quad \text{for all} \quad h \in G\}.$$

1. *Show that Z_1 is the center of G, that $Z_i \subset Z_{i+1}$ and that Z_i is a normal subgroup of G for every i. Finally, Show that Z_{i+1}/Z_i is the center of G/Z_i.*

   Remark: The series

   $$\{e\} \subset Z_1 \lhd Z_2 \lhd \cdots \lhd Z_i \lhd Z_{i+1} \cdots$$

   is called the upper central series*.*

   A group G is called **nilpotent** *if there is an $n \in \mathbb{N}$ for which $Z_n = G$. The first n for which this happens is called the* nilpotency class *of G.*

2. *Compute the upper central series for G, the group of real upper triangular 3 by 3 matrices whose entries along the diagonal are 1, i.e., the elements in G look like*

   $$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

3. *Can you guess what the upper central series is for be the group of real upper triangular n by n matrices whose entries along the diagonal are 1?*

4. *Show that if the center of G is trivial, then the upper central series is given by $Z_i = \{e\}$. Compute the upper central series for D_7, D_{28} and D_8.*

3) (**Lower central series**)*Given a group G, let $G_0 = G$ and define inductively*

$$G_i = \langle ghg^{-1}h^{-1} : g \in G_{i-1}, h \in G\rangle,$$

where $\langle\cdot\rangle$ denotes "the group generated by". So, for example, G_1 is the commutator subgroup of G.

1. (1 pt) *Show that* $G_{i+1} < G_i$. *Further, show that* G_{i+1} *is a normal subgroup of* G_i *and that the quotient* G_i/G_{i+1} *is Abelian.*

   Remark: The series

$$G = G_0 \rhd G_1 \cdots \rhd G_i \rhd G_{i+1} \rhd \cdots$$

   is called the **lower central series**.

2. *Compute the lower central series for* G*, the group of real upper triangular* 3 *by* 3 *matrices whose entries along the diagonal are 1.*

3. Compute the lower central series of D_7, D_{28} and D_8.

4. *Show that if* G *is nilpotent with nilpotency class* n*, then* $G_n = \{e\}$*. Further, if there is an* n *for which* $G_n = \{e\}$*, then* G *is nilpotent.*