Group theory – Mock Exam 2

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are **not** allowed to consult any text book, class notes, colleagues, calculators, computers etc.
5. If you are not sure about some definition of notation you encounter in the exam, please ask.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

1) *Let H and J be subgroups of a finite group G. Show that if the orders of H and J have no common factors then their intersection is the trivial group, i. e., $H \cap J = \{e\}$.*

Let K be the intersection of H and J. Then K is a subgroup of H and and subgroup of J because the intersection of subgroups is again a subgroup. By Lagrange's theorem, the order of a subgroup divides the order of the group. Therefore the order of K divides the order of H and divides the order of J. Since the order of H and the order of J have no (nontrivial) common divisors, the order of K is 1.

2) *Decide which of the following groups are isomorphic, if any:*

a) $A_5 \times \mathbb{Z}_2$, S_5, D_{60} *and* $D_{20} \times \mathbb{Z}_3$.

b) $D_{10} \times \mathbb{Z}_3$, $D_5 \times \mathbb{Z}_6$ *and* D_{30}.

a) All the groups have the same cardinality, so we move on to look at their centers.

$$Z_{A_5 \times \mathbb{Z}_2} = Z_{A_5} \times Z_{\mathbb{Z}_2} = \mathbb{Z}_2,$$

$$Z_{S_5} = \{e\}$$

$Z_{D_{60}} = \mathbb{Z}_2$ because the center of D_n is either trivial, if n is odd, or $\mathbb{Z}_2$, if n is even, and 60 is even,

$$Z_{D_{20} \times \mathbb{Z}_3} = Z_{D_{20}} \times Z_{\mathbb{Z}_3} = \mathbb{Z}_2 \times \mathbb{Z}_3.$$

Since isomorphic groups have isomorphic centers, the only two groups of the list which still have a chance of being isomorphic are $A_5 \times \mathbb{Z}_2$ and D_{60}. However, D_{60} has an element of order 60 and $A_5 \times \mathbb{Z}_2$ does not, so those groups are not isomorphic either. So we conclude that the groups of the first list are not isomorphic to each other.

b) Again all the groups have the same order so we look at their centers again using that Z_{D_n} is trivial for n odd and $\mathbb{Z}_2$ for n even.

$$Z_{D_{10}\times\mathbb{Z}_3} = Z_{D_{10}} \times Z_{\mathbb{Z}_3} = \mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6,$$

$$Z_{D_5\times\mathbb{Z}_6} = Z_{D_5} \times Z_{\mathbb{Z}_6} = \mathbb{Z}_6,$$

$$Z_{D_{30}} = \mathbb{Z}_2$$

so D_{30} is not isomorphic to the other two groups on the list. Finally we recall that for n odd, $D_{2n} = D_n\times\mathbb{Z}_2$ so

$$D_{10} \times \mathbb{Z}_3 = D_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = D_5 \times \mathbb{Z}_6,$$

therefore those two groups are isomorphic.

3) *Let G be a finite group. Show that if G has only two conjugacy classes then G is isomorphic to $\mathbb{Z}_2$.*

Let n be the order of G. The identity is always in a conjugacy class of its own. So, since G has only two conjugacy classes we conclude that $G\backslash\{e\}$ is the second conjugacy class. But each conjugacy class is an orbit of the action of G on itself by the adjoint action, so its size must divide the order of G, by the Orbit-Stabilizer theorem. So we have that $n-1$ must divide n, that is, there is k a positive integer such that

$$\frac{n}{n-1} = k$$

The only solution to this equation is $n = 2$ and $k = 2$, since for $n > 2$

$$1 < \frac{n}{n-1} = 1 + \frac{1}{n-1} < 2$$

so it can not be an integer. Therefore we must have $n = 2$ and hence $G = \mathbb{Z}_2$ and in fact Z_2 has only two conjugacy classes.

4) *Let G be a group of order $5 \cdot 11 \cdot 13$. Show that the 11 and the 13 Sylows are normal. Show that the 13-Sylow is in the center of G.*

By Sylow's theorem, we have that the number of 11, n_{11} Sylows must be

$$n_{11} = 1, 12, 23, 34, \cdots (1 \bmod 11)$$

$$n_{11} = 1, 5, 13 \text{ or } 65(\text{ it must divide } 5 \cdot 13).$$

The only common solution to these is $n_{11} = 1$, so the 11-Sylow is normal.

Similarly the number of 13-Sylows is given by

$$n_{13} = 1, 14, 27, 40, 53, \cdots,$$

$$n_{13} = 1, 5, 11, \text{ or } 55.$$

and again the only common solution is $n_{13} = 1$ which means that the 13-Sylow is also normal.

First proof that the 13-Sylow is in the center: Now, since the 13-Sylow has 13 elements it is isomorphic to $\mathbb{Z}_{13}$ and in particular it is cyclic. Let a be one of its generators. Let $b \in G$, then by Lagrange we have that $e = b^{|}G| = b^{5\cdot 11 c \dot{1}3}$.

Since the 13-Sylow is normal we have that bab^{-1} lies in the 13-Sylow, hence there is an integer k such that $bab^{-1} = a^k$. Using that $b^{5 \cdot 11 \cdot 13} = e$ we have

$$a = b^{5 \cdot 11 \cdot 13} a b^{-5 \cdot 11 \cdot 13} = a^{k^{5 \cdot 11 \cdot 13}}$$

so we get that

$$k^{5 \cdot 11 \cdot 13} = 1 \mod 13$$

So the order of k in $(\mathbb{Z}_{13} \backslash \{0\}, \cdot)$ must divide $5 \times 11 \times 13$ and must divide 12, the order of $(\mathbb{Z}_{13} \backslash \{0\}$. Since these numbers have no common factors, the order of k is 1 and hence $k = 1$, that is,

$$bab^{-1} = a \qquad \forall b \in G$$

Since a is a generator of the 13-Sylow we conclude that the 13-Sylow is in the center of G.

Second proof that the 13-Sylow is in the center: Warning this proves the same thing that the first proof did, it uses better the concepts we have seen during the course, but it may appear more complicated.

Since the 13-Sylow is normal, we can let G act on it by conjugation:

$$G \times Syl_{13} \longrightarrow Syl_{13} \qquad (g, x) \mapsto gxg^{-1} \in Syl_{13}.$$

Notice that each element of G nor only acts on Syl_{13} but actually gives rise to an automorphism of that group because the adjoint action of an element $g \in G$ is an automorphism of G. That is, this action is not only a group homomorphism from G into the permutations of a set with 13 elements, but is actually a homomorphism between G and the automorphisms of Z_{13}:

$$\phi : G \longrightarrow Aut(\mathbb{Z}_{13}).$$

Now an automorphism of Z_{13} is determined by where it maps the number 1 and 1 can be mapped into any nonzero element because all of them are generators, so the group of automorphisms of $\mathbb{Z}_{13}$ has 12 elements. By Lagrange and isomorphism theorem, the image of this map must divide the order of G $(5 \cdot 11 \cdot 13)$ and must divide the order of $Aut(\mathbb{Z}_{13})$, i.e., 12. Since these have no common factors, the image is the trivial subgroup and the action is trivial, that is

$$gxg^{-1} = x \qquad \forall x \in Syl_{13}, \forall g \in G.$$

Hence the 13-Sylow is in the center.

5) *For this exercise, let G be a group of order $2n$, where n is an odd number greater than 1. Following the steps below or otherwise, prove that G is not simple.*

a) *Consider the action of G on itself by left multiplication. This furnishes a group homomorphism*

$$\varphi : G \longrightarrow S_{2n}.$$

Show that φ is an injection;

b) *According to Cauchy's theorem there is $x \in G$ of order 2. Show that the image of x is an odd permutation, hence $\varphi(G)$ is not contained in A_{2n};*

c) *Identifying G with its image in S_{2n} show that $G \cap A_{2n}$ is a nontrivial normal subgroup of G, hence G is not simple.*

a) Let $g \in \ker(\varphi)$ Then g gives rise to the trivial permutation, that is $gx = x$ for all $x \in G$. Multiplying on the right by x^{-1} we get that $g = e$ and hence $\ker(\varphi) = \{e\}$ and φ is injective.

b) Let g be a nonindentity element in G. Then there is no $x \in G$ such that $gx = x$. Indeed, if there was such an element, by multiplying on the right by x^{-1} we would get $g = e$. So a nonidentity element does not fix any points. Now if g has order 2, then we can write its image in S_{2n} as a product of disjoint cycles and the order of $\varphi(g)$ is the least common multiple of the orders of those disjoint cycles. Since g has order 2, each cycle must have order 2. Since $\varphi(g)$ leaves no point fixed, it must permute all the $2n$ elements of G, so $\varphi(g)$ is a product of n disjoint transpositions. Since n is odd, $\varphi(g) \notin A_{2n}$.

c) According to a) we can see G as a subgroup of S_{2n} and according to b) $G \notin A_{2n}$. Now we prove that $G \cap A_{2n}$ is a subgroup of G with half of the elements of G, i.e., is a subgroup of G of index 2 and hence is a normal subgroup. $G \cap A_{2n}$ is a subgroup since it is the intersection of two subgroups. Since $G \notin A_{2n}$, there is $g \in G \backslash A_{2n}$. Now right multiplication by g is an injection of $G \cap A_{2n}$ into $G \backslash A_{2n}$ and right multiplication by g is also an injection from $G \backslash A_{2n}$ into $G \cap A_{2n}$. Indeed, if $h \in G \cap A_{2n}$ then h is even and hence $gh \in G$ is odd therefore is in $G \backslash A_{2n}$. Conversely if $h \in G \backslash A_{2n}$ then h is odd and gh is even, hence in A_{2n}. Injectivity follows from the fact that in any group, left multiplication by a fixed element is an injective map.

Now since there are injective maps from $G \backslash A_{2n}$ to $G \cap A_{2n}$ and viceversa, these two sets have the same cardinality, so $G \cap A_{2n}$ has half of the elements in G and hence is a subgroup of index 2 and therefore normal.

If $n > 1$, then this subgroup then $G \cap A_{2n}$ has cardinality greater than 1 and hence it is nor the trivial group either, so it is a nontrivial normal subgroup of G, showing that G is not simple.