Group theory – Sheet 7

The exercises from the book are 13.2, 17.4, 17.5, 17.7, 17.10, 17.11, 17.12, 17.13, 17.14.

1) Let G be a group of order $2p$, where p is a prime number. Show that G is isomorphic to either $\mathbb{Z}_{2p}$ or D_p.

2) (Variation of exercise 17.10) Let a finite group G act on a set X and let $\tilde{X}$ be one orbit of the action. Show that G also acts on $\tilde{X}$. Show that if a group $G \neq \mathbb{Z}_2$ acts nontrivially on a set with two elements then G is not simple. Conclude that if a group $G \neq \mathbb{Z}_2$ acts on a set and has an orbit of size 2 then G is not simple.

3) (See also 13.5) Show that if $\#G = 2m$, with m odd, then G is not simple. Hint: Use the exercise from the first hand in sheet to conclude that G has an element of order 2, say, g. We have seen that the left action of G on itself gives an injective homomorphism $G \longrightarrow S_{2m}$. Show that the image of g is an odd permutation and use exercise 6.6 from Amstrong to conclude that the intersection of the image of G with A_{2m} provides a normal subgroup of G.