Reduced holonomy and Hodge theory

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Abstract

We develop Hodge theory for a Riemannian manifold (M, g) with a background closed 3-form, H. Precisely, we prove that if the metric connections with torsion $\pm H$ have holonomy groups $G_\pm$, then the Lie algebras $\mathfrak{g}_\pm$ of the holonomy groups act on the space of forms and the d^H-Laplacian preserves the irreducible representations of this action therefore inducing a corresponding decomposition of the d^H-cohomology.

1 Introduction

A powerful property of compact Kähler manifolds is that the decomposition of forms into $\Omega^{p,q}(M)$ determined by the complex structure induces a corresponding decomposition of the cohomology of the manifold. While this is often attributed to the identity between the Hodge Laplacian, Δ_d, and the Dolbeault Laplacians this is, in fact, not a complex geometric result, but a consequence of the reduction of the holonomy of the Levi–Civita connection, ∇, from $SO(2n)$ to $U(n)$ and similar result holds whenever the Levi–Civita connection has reduced holonomy. Indeed, the holonomy group acts on the space of k-forms, $\wedge^k T^*M$, decomposing it into irreducible representations. Even though this decomposition of forms does not come coupled with extra differential operators such as ∂ or $\overline{\partial}$, one can still use the connection to produce the *rough Laplacian*, $\nabla^*\nabla$. This Laplacian, does not agree with the Hodge Laplacian and the Weitzenböck formula expresses their difference as a tensor R defined by the Riemannian curvature:

$$\Delta_d = \nabla^*\nabla + R.$$

Using properties of the Levi–Civita connection and symmetries of the curvature tensor, one can argue that the right hand side of the equality above preserves the irreducible representations of the holonomy group on $\wedge^k T^*M$ and hence so does the Hodge Laplacian (see Theorem 3.5.3. in [4]). Therefore we get a corresponding decomposition of cohomology which, in the Kähler case, refines the decomposition of cohomology into (p, q) components.

Once one allows for metric connections with torsion, the picture becomes much less clear. In this paper we will extend the result above to connections whose torsion tensor

$$H(X, Y, Z) = g(\nabla_X Y - \nabla_Y X - [X, Y], Z)$$

is skew-symmetric and closed. Under these conditions, it is natural to consider the twisted exterior derivative $d^H = d + H\wedge$ and one has again a Weitzenböck formula relating the rough Laplacian with the d^H-Laplacian [3]

$$\Delta_{d^H} = \nabla^*\nabla + R + \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2, \tag{1.1}$$

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where R is a tensor determined by the curvature of ∇ and κ is the scalar curvature. Since connections with skew torsion do not have the same symmetries of the Levi–Civita connection, it is no longer the case that the right hand side preserves the irreducible representations of the action of the holonomy group on $\wedge^k T^*M$, in fact not even $\wedge^k T^*M$ is preserved. Indeed, despite of much progress in the theory of connections with skew symmetric torsion, in the 13 years since Bismut first proposed a Weitzenböck-type formula similar to the above in [1], this particular analogy with the torsion free case remained elusive.

In this paper we propose a new action of the Lie algebra of the holonomy group on forms. This action does not preserve the homogenous spaces $\wedge^k T^*M$, but instead mixes forms of different degrees. The irreducible representations of this action give a new decomposition of forms which is preserved by the d^H-Laplacian and hence induces a decomposition of d^H-cohomology. The proof of this fact can be achieved by the same argument outlined above using a Weitzenböck formula, and yet, it is a consequence of a simpler identity between first order operators as we show here.

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2 Basic set up

Given a manifold M^m, the generalized tangent bundle, $\mathbb{T}M = TM \oplus T^*M$, is endowed with a natural symmetric pairing:

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\eta(X)+\xi(Y)), \qquad X,Y \in T_pM\ \xi,\eta \in T_p^*M.$$

Elements of $\mathbb{T}M$ act on $\wedge^\bullet T^*M$ via

$$(X+\xi)\cdot\varphi = i_X\varphi + \xi\wedge\varphi.$$

and this extends to an action of the Clifford algebra of $\mathbb{T}M$ making $\wedge^\bullet T^*M$ a model for the space of spinors for $Spin(\mathbb{T}M)$. Given a metric g on M, we can form two subspaces of $\mathbb{T}M$, namely,

$$V_+ = \{X + g(X) : X \in TM\}, \qquad V_- = \{X - g(X) : X \in TM\}.$$

The natural pairing is positive definite on V_+, negative definite on V_- and these spaces are orthogonal to each other.

Let (M, g, H) be an oriented Riemannian manifold with closed 3-form. Since M is oriented we have the usual Hodge star operator $*$ which gives a positive definite inner product on forms:

$$(\varphi,\psi) = \int_M \varphi\wedge *\psi \qquad \text{for all } \varphi,\psi\in\Omega^\bullet(M;\mathbb{R}). \tag{2.1}$$

The natural operator to consider in the context of connections with skew torsion is $d^H = d + H\wedge$. Using integration by parts one sees that the formal adjoint of d^H according to the metric (2.1) is given by

$$d^{H*} = d^* - g^{-1}(H),$$

where $g^{-1}H$ is the trivector field obtained by the isomorphism $g : TM \longrightarrow T^*M$ defined by the metric extended to a bundle map $g : \wedge^3 TM \longrightarrow \wedge^3 T^*M$.

With d^H and d^{H*} at hand, we can form the Dirac, the signature and the Laplace operators:

$$\mathcal{D}^H_\pm = d^H \pm d^{H*}, \qquad \Delta_{d^H} = d^H d^{H*} + d^{H^*} d^H.$$

As usual, one has

$$(\mathcal{D}^H_+)^2 = -(\mathcal{D}^H_-)^2 = \Delta_{d^H}.$$

3 Reduced holonomy and Hodge theory

Given an oriented Riemannian manifold with closed 3-form, (M, g, H) there are three metric connections one can consider which are natural in terms of the available data. The first, of course, is the Levi–Civita connection, ∇. The other two are the metric connections with skew torsion $\pm H$, which we denote by $\nabla^{\pm}$.

Throughout this section we let $\{e_1, \cdots, e_m\} \in \Gamma(TM)$ be an orthonormal frame and $\{e^1, \cdots, e^m\} \in \Gamma(T^*M)$ be its dual frame. Using these, we can write explicit expressions for $\nabla^{\pm}$. Indeed, if we define $h_{ijk} = H(e_i, e_j, e_k)$, then, using Einstein summation convention and omitting the symbol for the wedge product, H is given by

$$H = \tfrac{1}{3!} h_{ijk} e^i e^j e^k.$$

and we have

$$\nabla^{\pm} = \nabla \pm \tfrac{1}{4} h_{ijk} e^i e^j e_k. \tag{3.1}$$

If the holonomy of ∇^+ is the Lie group G_+, then using the isomorphism $TM \cong V_+$, we realize its Lie algebra, $\mathfrak{g}_+$, as a sub Lie algebra of $\mathfrak{so}(V_+) = \wedge^2 V_+$. *Mutatis mutandis*, the same holds for ∇^- and we get $\mathfrak{g}_- \subset \wedge^2 V_-$. Next we notice that $\mathfrak{g}_+ \oplus \mathfrak{g}_- \subset \wedge^2 \mathbb{T}M = \mathfrak{spin}(\mathbb{T}M)$ and hence the elements of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ act on forms, thought of as spinors. Further, since V_+ is orthogonal to V_- and $\mathfrak{g}_{\pm} \subset \wedge^2 V_{\pm}$ the Lie algebra action of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ on forms commute and we get an action of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ on forms as a direct sum of the individual actions of the Lie algebras. Now we are in condition to state our main theorem.

Theorem 3.1. *Let (M, g, H) be a compact Riemannian manifold endowed with a closed 3-form. If $\nabla^{\pm}$, the metric connections with torsion $\pm H$, have holonomy in $G_{\pm}$, then the d^H-cohomology of M splits according to the decomposition of forms into irreducible representations of the action of $\mathfrak{g}_+ \oplus \mathfrak{g}_- \subset \wedge^2 V_+ \oplus \wedge^2 V_-$ on forms.*

Lemma 3.2. The connection $\nabla^{\pm}$ preserves the irreducible representations of $\mathfrak{g}_{\pm}$.

Proof. Indeed, if we denote by $\widetilde{\mathfrak{g}}_+ \subset \mathfrak{spin}(TM) \cong \wedge^2 T^*M$ the bundle of endomorphisms of TM defined by the connection ∇^+, the condition that ∇^+ has reduced holonomy implies that this bundle is preserved by parallel transport. Now $\mathfrak{g}_+$ is simply the image of $\widetilde{\mathfrak{g}_+}$ by the parallel isomorphism $\mathrm{Id} + g : T^*M \longrightarrow V_+$, hence the bundle $\mathfrak{g}_+ \subset \wedge^2 V_+$ is also parallel and its irreducible representations are preserved by the connection. □

Next we define

$$\varepsilon^i_{\pm} = e^i \pm e_i \in \Gamma(V_{\pm})$$

and

$$H_{\pm} = \tfrac{1}{3!} h_{ijk} \varepsilon^i_{\pm} \varepsilon^j_{\pm} \varepsilon^k_{\pm} \in \mathrm{Clif}^3(V_{\pm}) \subset \mathrm{Clif}^3(\mathbb{T}M).$$

To prove the theorem we need to extend to the torsion case the formulas relating the Levi–Civita connection with the exterior derivative and its adjoint (see, e.g., Chapter 7, Proposition 4.3 in [5]):

$$\begin{aligned} d &= e^i \wedge \nabla_{e_i} \\ d^* &= -\iota_{e_i} \nabla_{e_i} \end{aligned} \tag{3.2}$$

Lemma 3.3. With the definitions above, we have

$$\begin{aligned} \mathcal{D}^H_+ &= \varepsilon^i_- \nabla^+_{e_i} + H_-. \\ \mathcal{D}^H_- &= \varepsilon^i_+ \nabla^-_{e_i} + H_+. \end{aligned} \tag{3.3}$$

Proof. Since the statements are similar, we will only prove the first. The left hand side of (3.3) is

$$\begin{aligned}\mathcal{D}_+^H &= d + H + d^* + H^* \\ &= d + d^* + \tfrac{1}{3!}h_{ijk}e^ie^je^k - \tfrac{1}{3!}h_{ijk}e_ie_je_k.\end{aligned} \tag{3.4}$$

Next, we use the relation (3.1) to re-write the right hand side as

$$\varepsilon_-^i\nabla_{e_i}^+ + H_- = e^i \wedge \nabla_{e_i} - \iota_{e_i}\nabla_{e_i} + \tfrac{1}{2}h_{ijk}e^ie^je_k - \tfrac{1}{2}h_{ijk}e_ie^je_k + H_-.$$

Due to (3.2), the first two terms are $d + d^*$. Expanding H_- we get

$$\varepsilon_-^i\nabla_{e_i}^+ + H_- = d+d^*+\tfrac{1}{2}h_{ijk}e^ie^je_k-\tfrac{1}{2}h_{ijk}e_ie^je_k+\tfrac{1}{3!}h_{ijk}e^ie^je^k-\tfrac{1}{2!}h_{ijk}e^ie^je_k+\tfrac{1}{2!}h_{ijk}e^ie_je_k-\tfrac{1}{3!}h_{ijk}e_ie_je_k.$$

which equals (3.4). □

Proof of Theorem 3.1. Since ∇^+ preserves the irreducible representations of $\mathfrak{g}_+$, Lemma 3.3 implies that $\mathcal{D}_+^H$ preserves the irreducible representations of $\mathfrak{g}_+ \oplus \mathrm{Clif}(V_-)$ and hence so does the d^H-Laplacian. Similarly, $\mathcal{D}_-^H$ preserves the irreducible representations of $\mathrm{Clif}(V_+) \oplus \mathfrak{g}_-$ and hence Δ_{d^H} preserves the intersections of these representations, which are just the irreducible representations of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$. □

Using a different approach, the author established Theorem 3.1 in [2] for the cases when the holonomy of the pair (∇^+, ∇^-) is either $U(n) \times SO(2n)$ or $U(n) \times U(n)$ and showed that the decomposition of forms and cohomology for these groups is indeed finer than the usual $\mathbb{Z}_2$-grading into even and odd forms.

4 Integrability

A simple consequence of (3.2) is that if $W \subset \wedge^k T^*M$ is a representation of the holonomy group of the Levi–Civita connection, then the exterior derivative restricted to sections of W can only land in representations present in $T^*M \wedge W$. This is the Riemannian version of the claim that in a complex manifold $d : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$. Lemma 3.3 has similar implications.

Proposition 4.1. *Let (M, g, H) be a compact oriented Riemannian manifold with a closed 3-form and let $\mathfrak{g}_\pm \subset \wedge^2 V_\pm$ be the Lie algebras of the holonomy groups of the connections $\nabla^\pm$. Let W be a representation of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$, then d^H sends sections of W into sections of representations that appear in* $(\mathrm{Clif}^3(V_+) \oplus \mathrm{Clif}^3(V_-)) \cdot W$.

Proof. It follows from Lemmata 3.2 and 3.3 that

$$\mathcal{D}_\pm^H : \Gamma(W) \longrightarrow \Gamma(\mathrm{Clif}^3(V_\mp) \cdot W),$$

hence, $d^H = \frac{1}{2}(\mathcal{D}_+^H + \mathcal{D}_-^H)$ has the stated property. □

It follows from [2] that the difference between an SKT or generalized Kähler structure and parallel (almost) Hermitian or (almost) bi-Hermitian structure, respectively, is that in the former two cases if W is a representation of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ then $d^H : \Gamma(W) \longrightarrow \Gamma(\mathbb{T}_\mathbb{C}M \cdot W)$, while in the latter two cases Proposition 4.1 is the best one can say. This suggests that in general there is a subclass of the space of manifolds (M, g, H) with reduced holonomy which may be of further interest:

Definition 4.2. We say that (∇^+, ∇^-) induces an *integrable* $G_+ \times G_-$ structure if the holonomy of $\nabla^\pm$ is $G_\pm$ and

$$d^H : \Gamma(W) \longrightarrow \Gamma(\mathbb{T}_\mathbb{C}M \cdot W) \tag{4.1}$$

for every representation $W \subset \wedge^\bullet T_\mathbb{C}^*M$ of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$.

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