Complex geometry

Holomorphic vector bundles, elliptic operators and Hodge theory

Gil R. Cavalcanti[1]

June 25, 2022

[1]Utrecht University; g.r.cavalcanti@uu.nl.

Contents

Chapter 1

Complex vector spaces

1.1 Different ways to describe a complex vector space

For any field, k, a k-vector space is an Abelian group $(V, +)$ endowed with a map (called *scalar multiplication*) compatible with the group operation in the sense that for $u, v \in V$ and $a, b \in k$,

$$\begin{aligned} a(bv) &= (ab)v \\ a(u+v) &= au + av \\ (a+b)u &= au + bu \\ 1u &= u. \end{aligned}$$

Since $\mathbb{C}$ is a field the definition above leads naturally to the definition of a complex vector space

Definition 1.1. A a vector space over $\mathbb{C}$ is a *complex vector space.*

What makes $\mathbb{C}$ interesting, from the linear algebraic point of view, is that it is an extension of $\mathbb{R}$ by adding a square root of -1. Hence we can try and relate the familiar notion of real vector space to the notion of complex vector space.

The easiest relation is that since $\mathbb{R} \subset \mathbb{C}$ is a subfield, every complex vector space is also a real vector space by restriction of the scalars.

Exercise 1.1. Show that if $\{v_1, \ldots, v_k\}$ is a linearly independent set/basis for a complex vector space V then $\{v_1, iv_1, \ldots, v_k, iv_k\}$ is a linearly independent set/basis for the underlying real vector space, $V_{\mathbb{R}}$. In particular

$$\dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \dim_{\mathbb{C}} V.$$

Conversely, given a real vector space V, we can try to extend scalar multiplication by real numbers to scalar multiplication by complex numbers. There are two ways to achieve this.

The first is just to declare that now we can multiply vectors by complex numbers and if, for example, $\{v_1, \cdots, v_n\}$ was a base for V over $\mathbb{R}$ then it is also a base for this 'complexified' vector space made out of V. This is extension of the scalars is what is used in a first basic course in linear algebra to diagonalise a real matrix with complex eigenvalues. The formalism used to make this construction precise is that of tensor products:

Definition 1.2. The *complexification* of a real vector space V is the space $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ with scalar multiplication given by

$$z(w \otimes v) = zw \otimes v.$$

Notice that because of the tensor product over the reals, for $a \in \mathbb{R}$, $a \otimes v = 1 \otimes av$. When dealing with $V_{\mathbb{C}}$ we typically do not write the $\otimes$ operation and do not distinguish $1 \otimes v$ from v.

Exercise 1.2. Show that if V has dimension n as a real vector space, then $V_{\mathbb{C}}$ has dimension n as a complex vector space.

Notice that if V is a complex vector space, we can restrict the scalars so it becomes a real vector space and then we can complexify this space to obtain a new complex vector space:

$$V \to V_{\mathbb{R}} \to (V_{\mathbb{R}})_{\mathbb{C}}.$$

This process doubles the (complex) dimension of the space.

The second way to make a real vector space into a complex one requires a little more data. Since the difference between $\mathbb{R}$ and $\mathbb{C}$ is the addition of a square root of -1, what we need is a linear operator $I : V \to V$ which doubles as scalar multiplication by i, that is, $I^2 = -\mathrm{Id}$. If we have one such operator, we can extend scalar multiplication from $\mathbb{R}$ to $\mathbb{C}$ by declaring

$$(a + ib)v = av + bIv,$$

and V with this operation becomes a complex vector space.

Definition 1.3. A *(linear) complex structure* on a real vector space V is a linear automorphism $I\colon V \to V$ such that $I^2 = -\mathrm{Id}$.

Exercise 1.3. Given a complex vector space V, let $V_{\mathbb{R}}$ be the underlying real vector space and let $I\colon V_{\mathbb{R}} \to V_R$ be the linear map $Iv = iv$ defined using the original complex multiplication of V. Show that V is naturally isomorphic as a complex vector space to the complex vector space obtained from the real vector space with complex structure $(V_{\mathbb{R}}, I)$.

Because of this we often do not distinguish between the complex vector space, V, and the real vector space with complex structure, $(V_{\mathbb{R}}, I)$.

There are, however, different ways to describe a complex structure, I, on a real vector space, V. To start with, we can pick a basis of $V_{\mathbb{C}}$ that renders I in its Jordan form. The condition $I^2 = -\mathrm{Id}$ not only forces all the eigenvalues of I to be $\pm i$ but also forces its Jordan form to be diagonalizable, that is, we can split $V_{\mathbb{C}}$ into the $\pm i$-eigenspaces of I:

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, \qquad Iv = iv, \quad Iw = -iw, \qquad \forall v \in V^{1,0}, \forall w \in V^{0,1}.$$

Concretely, we have

$$V^{1,0} = \{v - iIv\colon v \in V\}, \qquad V^{0,1} = \{v + iIv\colon v \in V\}.$$

Notice further that complex conjugation on $\mathbb{C}$ gives rise to complex conjugation on $V_{\mathbb{C}}$ and since I is a real operator $\overline{V^{1,0}} = V^{0,1}$.

Since knowing the $+i$-eigenspace determines the $-i$-eigenspace of I (by complex conjugation) and these together determine I (as a diagonalizable linear map is determined by its eigenspaces and corresponding eigenvalues) we have an equivalent definition of complex structure:

Lemma 1.4. *A complex structure on a real vector space V determines and is uniquely determined by a subspace $V^{1,0} \subset V_{\mathbb{C}}$ such that*

$$V^{1,0} + \overline{V^{1,0}} = V_{\mathbb{C}},$$
$$V^{1,0} \cap \overline{V^{1,0}} = \{0\}.$$

1.2 Linear maps

Linear maps between an m-dimensional complex vector space and an n-dimensional one can be identified with $n \times m$-matrices with complex entries. Endomorphisms are therefore identified with $n \times n$ matrices and automorphisms can still be characterised as those with nonzero (complex) determinant. For $\mathbb{C}^n$, these form the complex general linear group, $\mathrm{GL}\,(n;\mathbb{C})$.

From the point of view of the underlying real vector spaces with complex structure, (V, I) and (W, J) a real linear map $A\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ is induced by a complex linear map if and only if

$$A(Iv) = JAv.$$

In particular, endomorphisms of (V, I) correspond to real linear maps A for which $[A, I] = 0$ and $\mathrm{GL}\,(n;\mathbb{C}) \subset \mathrm{GL}\,(2n;\mathbb{R})$.

Exercise 1.4. Show that if V has complex dimension 1, then complex linear maps correspond to multiplication by a complex number. That is, if $A\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ satisfies $[A, I] = 0$ then A is multiplication by a complex number.

Exercise 1.5. Show that if $A\colon V \to W$ is complex linear then the induced map $A\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ satisfies

$$A\colon V^{1,0} \subset V_{\mathbb{C}} \to W^{1,0} \subset W_{\mathbb{C}}. \tag{1.2.1}$$

Conversely, if a real linear map $A\colon V_{\mathbb{R}} \to W_{\mathbb{R}}$ satisfies (1.2.1), then it is induced by a complex linear map.

1.3 Metrics and symplectic structures

It is often easier to get a handle on a complex structure if we have further structures compatible with it at our disposal. The first readily available structure is that of a Hermitian metric.

Definition 1.5. A *Hermitian metric* on a complex vector space V is as bilinear form $h\colon V \times V \to \mathbb{C}$ such that

$$h(v, v) \in \mathbb{R}^*_+, \quad \text{for } v \neq 0,$$
$$h(zv, w) = zh(v, w),$$
$$h(v, w) = \overline{h(w, v)}.$$

When considering the underlying real vector space, it is convenient to split a Hermitian metric h into its real and imaginary parts: $h = g + i\omega$.

Lemma 1.6. *If $h = g + i\omega$ is a Hermitiam metric on V, then*

- *g is a real metric on $V_{\mathbb{R}}$ for which I is an orthogonal map,*
- *ω is a nondegenerate skew-symmetric 2-form on $V_{\mathbb{R}}$ such that $\omega(Iv, Iw) = \omega(v, w)$ for all $v, w \in V_{\mathbb{R}}$,*
- *g and ω are related by $g(v, w) = \omega(Iv, w)$.*

Proof. We start with the third point:

$$g(Iv, w) + i\omega(Iv, w) = h(Iv, w) = ih(v, w) = ig(v, w) - \omega(v, w).$$

Equating the imaginary parts of both sides we have $\omega(Iv, w) = g(v, w)$.

Regarding the first claim, since to go from V to $V_{\mathbb{R}}$ all we do is restrict the scalars we see directly that the conditions imposed on a Hermitian metric h imply that its real part is a real metric:

$$g(v, v) = \mathrm{Re}(h(v, v)) \in \mathbb{R}^*_+, \qquad \text{for } v \neq 0,$$
$$g(av, w) = \mathrm{Re}(h(av, w)) = a\mathrm{Re}(h(v, w)) = ag(v, w), \qquad \text{for } a \in \mathbb{R},$$
$$g(v, w) = \mathrm{Re}(h(v, w)) = \mathrm{Re}(\overline{h(w, v)}) = \mathrm{Re}(h(w, v)) = g(w, v).$$

Further

$$g(Iv, Iw) + i\omega(Iv, Iw) = h(Iv, Iw)) = -i^2 h(v, w)) = g(v, w) + i\omega(v, w),$$

showing that I is orthogonal with respect to g and $\omega(Iv, Iw) = \omega(v, w)$.

Finally the second point is also a consequence of the previous two we proved. Firstly, since g is nondegenerate and I is an automorphism, $\omega(\bullet, \bullet) = -g(I\bullet, \bullet)$. is nondegenerate and further

$$\omega(v, w) = -g(Iv, w) = -g(I^2 v, Iw) = g(v, Iw) = g(Iw, v) = -\omega(w, v),$$

showing that ω is skew-symmetric □

The previous lemma introduced a few key concepts that we spell out now.

Definition 1.7.

- A *symplectic form* on a real vector space, V, is a nondegenerate 2-form $\omega \in \wedge^2 V^*$.
- A metric g and a complex structure I on a real vector space are *compatible* if I is an orthogonal transformation with respect to g.
- A symplectic structure ω and a a complex structure I on a real vector space are *compatible* if I is an orthogonal transformation of with respect to ω, that is $\omega(Iv, Iw) = \omega(v, w)$, and $g(\bullet, \bullet) = \omega(I\bullet, \bullet)$ is positive definite.

Lemma 1.8. *Given a vector space with complex structure, (V, I), the maps*

$$\{h\colon h \text{ is a Hermitian metric on } V\} \to \{g\colon g \text{ is a metric on } V \text{ compatible with } I\}$$
$$h \mapsto \mathrm{Re}(h)$$

$$\{h\colon h \text{ is a Hermitian metric on } V\} \to \{\omega\colon \omega \text{ is a symplectic structure on } V \text{ compatible with } I\}$$
$$h \mapsto \mathrm{Im}(h)$$

are bijections.

Proof. Indeed, say, we have for example a metric g compatible with I, then let $\omega(\bullet, \bullet) = -g(I\bullet, \bullet)$. Since $I^2 = -\mathrm{Id}$ and I is orthogonal for g, we have that ω ids skew symmetric and $g \mapsto g + i\omega$ is a left and right inverse to the first map. $\square$

Exercise 1.6. A *complex subspace* of a complex space (V, I) is a subspace $W \subset V$ preserved by I, that is, $IW = W$.

A *symplectic subspace* of a symplectic space (V, ω) is a subspace $W \subset V$ such that the pull-back of ω to W is a symplectic form on W.

Given a pair of compatible complex and symplectic structures (I, ω) in a vector space V, show that if $W \subset V$ is a complex subspace then it is also a symplectic subspace. Is the converse true?

There is a final compatibility that one can consider:

Definition 1.9. A metric g and a symplectic form ω are *compatible* if $I = \omega^{-1} \circ g$ is a complex structure where here we regard both g and ω as linear maps from V to V^* which are invertible by nondegeneracy.

Lemma 1.10. *Given a complex vector space (V, I), there is a metric compatible with the complex structure and the space of metrics compatible with the complex structure is contractible.*

Proof. We can define a retraction from the space of all metrics to the space of metrics compatible with the complex structure:

$$\tilde{g} \mapsto g, \qquad g(v, w) = \frac{1}{2}(\tilde{g}(v, w) + \tilde{g}(Iv, Iw)).$$

It is immediate to check that g is a metric compatible with I and that $\tilde{g} = g$ if and only if $\tilde{g}$ is compatible with I. Therefore this map is a retract. Since the space of all metrics is contactible (it is a convex subset of the vector space of symmetric 2-forms), so is the space of metrics compatible with the complex structure. $\square$

Corollary 1.11. *Given a complex vector space (V, I), there is a symplectic structure compatible with the complex structure and the space of symplectic structures compatible with the complex structure is contractible.*

Follows from the bijections in Lemma 1.8.

We quote here for completeness the following result on compatibility of symplectic structures and complex structures that is covered in the course Symplectic Geometry.

Lemma 1.12. *Given a symplectic vector space (V, ω), there is a complex structure compatible with ω and the space of complex structures compatible with ω is contractible.*

1.4 Exterior algebra

Given a complex vector space, V, using duals and tensor product over $\mathbb{C}$ we can create a number of associated complex vector spaces (the whole tensor algebra of V). Yet, since V can also be regarded as a real vector space, we can also consider its real tensor algebra. The decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

gives rise to a corresponding decomposition of forms:

$$\wedge^k V_{\mathbb{C}} = \oplus_{p+q=k} \wedge^p V^{1,0} \otimes \wedge^q V^{0,1} =: \oplus_{p+q=k} \wedge^{p,q} V.$$

Since as complex vector spaces $V \cong V^{1,0}$ we see that the exterior algebra of V lies as a subspace of the exterior algebra of $V_{\mathbb{C}}$ and that the latter is much bigger.

Maybe it is good to illustrate this concretely. Say we have a real basis for V of the form

$$\{e_1, f_1, \dots, e_n, f_n\} \qquad \text{with } Ie_i = f_i$$

Then

$$\{e_1 - iIf_1, \dots, e_n - iIf_n\}$$

is a basis for $V^{1,0}$ and its conjugate a basis for $V^{0,1}$. If we denote by $\partial_{z_i} = e_i - iIf_i$, then $\wedge^{p,q}V$ is generated by elements of the form $\partial_{z_J} \otimes \overline{\partial}_{z_K}$, where J is a multi-index of length p and K a multi-index of length q.

Wedge product is compatible with the bigrading:

$$\wedge\colon\ \wedge^{p,q} V \times \wedge^{p',q'} V \to \wedge^{p+p',q+q'} V.$$

If V has complex dimension n, then $\wedge^{n,0}V \subset \wedge^{\bullet}V$ is a complex line (because the top power of any vector space is a line) generated by $\partial_{z_1} \wedge \cdots \wedge \partial_{z_n}$. This line is important enough to deserve a name:

Definition 1.13. The *anti-canonical line* of a vector space with complex structure (V, I) is the subspace $\wedge^{n,0}V \subset \wedge^n V_{\mathbb{C}}$. The *canonical line* is the subspace $\wedge^{n,0}V^* \subset \wedge^n V_{\mathbb{C}}^*$.

The canonical line is generated by a decomposable element[1] $\rho \in \wedge^n V_{\mathbb{C}}$, where $n = \dim_{\mathbb{C}} V$, with the property that $\rho \wedge \overline{\rho} \in \wedge^{2n} V_{\mathbb{C}}$ is a volume form.

Exercise 1.7. Let V be a real vector space of dimension $2n$ and let $\rho \in \wedge^n V_{\mathbb{C}}$ be a decomposable n-form such that $\rho \wedge \overline{\rho} \in \wedge^{2n} V_{\mathbb{C}}$ is a volume form. Define $V^{1,0} = \{v \in V_{\mathbb{C}} \colon v \wedge \rho = 0\}$. Show that $V^{1,0}$ determines a complex structure whose canonical bundle is generated by ρ.

Exercise 1.8. Let V be a complex vector space and $A\colon V \to V$ be a complex linear map. By restricting the scalars, A is also a map of the underlying real vector space: $A_{\mathbb{R}}\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$. Show that $\det_{\mathbb{R}} A_{\mathbb{R}} = |\det_{\mathbb{C}} A|^2$.

1.5 Relation to conformal geometry

Given a metric g on a real vector space, we can define the angle between two vectors v_1 and v_2 as the angle $\theta \in [0, \pi]$ for which the following holds:

$$\cos\theta = \frac{g(v,w)}{\sqrt{g(v,v)g(w,w)}}. \tag{1.5.1}$$

If we denote the metric by $\langle \bullet, \bullet \rangle$ instead of g, and by $\| \bullet \|$ the induced norm, this becomes the familiar equality

$$\langle v, w \rangle = \|v\| \|w\| \cos\theta,$$

which is true because we define θ this way.

[1] a decomposable element is one that can be written as a product of degree 1 elements

Definition 1.14. A *conformal transformation* on a vector space with metric is a transformation that preserves angles between vectors.

It is clear that orthogonal transformations preserve angles, as they preserve the metric. Rescalings also preserve angles because the numerator and denominator in (1.5.1) scale by the same factor. In fact these are all conformal transformations.

Exercise 1.9. Show that if $A\colon V \to V$ is conformal, then A is the composition of an orthogonal transformation and multiplication by a nonzero scalar.

In real dimension 2, we can refine the notion of angle to include a sign as long as V is oriented. Indeed, for $v, w \in V$ we let $\theta \in (-\pi, \pi]$ be the angle for which

$$\langle v, w\rangle = \|v\|\|w\| \cos\theta,$$

and $\operatorname{sign}(\theta) v \wedge w$ is a non-negative volume element.

Also, if we denote by I the linear transformation that corresponds to rotation counterclockwise by $\pi/2$, then $I^2 = -\mathrm{Id}$, which makes V into a complex vector. With these two ingredients at hand we have the following low-dimensional coincidence:

Lemma 1.15. *Conformal transformations which preserve orientation of a 2-dimensional real vector space are the same as the invertible complex linear transformations.*

Proof. It follows from Exercise 1.9 that the conformal maps of V which also preserve orientation are compositions of scalings and elements of $SO(2)$, that is, rotations. It is immedaite to check that the composition of rotation by an angle θ and rescaling by the number r corresponds to complex multiplication by $re^{i\theta}$ with $r \neq 0$. □

Writing a complex number as $a + ib$ instead of $re^{i\theta}$, we see that orientation preserving conformal transformations of $\mathbb{R}^2$ are of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Chapter 2

Holomorphic functions

With the basic linear algebra out of the way, we can now focus on the notions of differentiation and integration.

2.1 Functions of one variable

Recall from the course analysis in one complex variable that a complex function $f\colon \mathbb{C} \to \mathbb{C}$ is *holomorphic at a point* $z_0 \in \mathbb{C}$ (i.e., has a complex derivative at z_0) if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Further f *is holomorphic* if it is holomorphic for all points in its domain. For those used to real analysis, we can regard f as a function from $\mathbb{R}^2$ to itself, $f(x, y) = u(x, y) + iv(x, y)$, and compute its derivative as a real function:

$$df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Then f is holomophic at a point z_0 if at that point the *Cauchy-Riemann* relations hold:

$$u_x = v_y \quad \text{ and } u_y = -v_x$$

This can be phrased equivalently by saying that

Lemma 2.1. *f is holomorphic at z_0 if and only if $df|_{z_0}$ exists and corresponds to multiplication by a complex number, namely, $u_x + iv_x$.*

Using Exercise 1.4 we can phrase the same concept in a slightly different way.

Lemma 2.2. *f is holomorphic at z_0 if and only if $df\colon T_{z_0}\mathbb{R}^2 \to T_{f(z_0)}\mathbb{R}^2$ is complex linear, that is, $I \circ df = df \circ I$.*

Where above we are using the induced complex structure on $\mathbb{R}^2$: if $\{\partial_x, \partial_y\}$ is the standard basis for $\mathbb{R}^2$, then $I\partial_x = \partial_y$ is the complex structure induced by the identitfication $\mathbb{R}^2 \cong \mathbb{C}$ (and at this stage it should hopefully be clear that we are using the linear structure of these spaces to identify them with their own tangent space at z_0).

There is one last way to describe holomorphicity, this time in terms of differential forms. But this requires a little explanation. Firstly, the identification $\mathbb{C} = \mathbb{R}^2$ gives a complex structure to $\mathbb{R}^2$ as described above: $I\partial_x = \partial_y$. Therefore we also get a dual complex structure, I^*, on $(\mathbb{R}^2)^*$, namely, if $\xi \in (\mathbb{R}^2)^*$ and $X \in \mathbb{R}^2$, then

$$(I^*\xi)(X) = \xi(IX).$$

Spelling this out in terms of the basis $\{dx, dy\}$ for $(\mathbb{R}^2)^*$ we have $Idx = -dy$ and we can form $dz = dx + idy \in (\mathbb{R}^2)^{1,0*}$ and $d\bar{z} = dx - idy \in (\mathbb{R}^2)^{0,1*}$, as outlined in Section 1.4.

Indeed, since f takes values on $\mathbb{C}$ we can think of df as a complex valued 1-form:

$$df = \partial_x f dx + \partial_y f dy = \partial_x u dx + \partial_y u dy + i(\partial_x v dx + \partial_y v dy)$$

And we can split it into its $(1,0)$ and $(0,1)$ components:

$$df = \frac{1}{2}(\partial_x - i\partial_y)(u + iv)dz + \frac{1}{2}(\partial_x + i\partial_y)(u + iv)d\bar{z}.$$

The Cauchy Riemann relations are equivalent to the vanishing of (0,1) component of df (the second summand above).

Summarising, if we denote by ∂ the composition of d with projection onto the (1,0)-forms and by $\overline{\partial}$ the composition of d with projection onto the (0,1)-forms we have

$$d = \partial + \overline{\partial}. \tag{2.1.1}$$

Lemma 2.3. *f is holomorphic at z_0 if and only if $\overline{\partial} f|_{z_0} = 0$.*

2.2 Functions of several variables and the $\overline{\partial}$-operator

The notion of derivative of a function has a few different generalisations. One of them is as the derivative of a function of several variables. Another is as the exterior derivative. We will deal with these in turn.

Definition 2.4. A continuous map $f\colon \mathbb{C}^m \to \mathbb{C}^n$ is *holomorphic at p* if its first derivative at p exists as a real map and $df|_p$ is complex linear. The map f is *holomorphic* if it is holomorphic at all points in its domain.

This can be rephrased by saying that df_p exists and each component of f satisfies the Cauchy–Riemann relations for all the variables independently, but this coordinate based characterisation of holomorphicity is not very insightful and is rarely used.

The generalization of derivative as exterior derivative requires a bit more setup. As we mentioned before given a complex function $f\colon \mathbb{C} \to \mathbb{C}$ we could think of f as a 0-form with complex coefficients and df as a one-form with complex coefficients. In $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we can consider k-forms with complex coefficients, $\Omega^k(\mathbb{C}^n;\mathbb{C})$ and the exterior derivative. Writing

$$\partial_{x_i} = \partial_{z_i} + \partial_{\bar{z}_i}, \quad \partial_{y_i} = i(\partial_{z_i} - \partial_{\bar{z}_i}), \quad dx_i = \frac{1}{2}(dz_i + d\bar{z}_i), \quad dy_i = \frac{1}{2i}(dz_i - d\bar{z}_i),$$

we can rewrite the exterior derivative in terms of the complex variables:

$$\begin{aligned} d &= \sum \partial_{x_i} dx_i + \partial_{y_i} dy_i \\ &= \tfrac{1}{2} \sum (\partial_{z_i} + \partial_{\bar{z}_i})(dz_i - d\bar{z}_i) + i(\partial_{z_i} - \partial_{\bar{z}_i}) \frac{1}{i}(dz + d\bar{z}_i) \\ &= \sum \partial_{z_i} dz_i + \partial_{\bar{z}_i} d\bar{z}_i. \end{aligned}$$

This expression for d shows that it behaves will with respect to the decomposition of complex valued forms induced by the complex structure, namely, when applied to a (p,q) form $\alpha\colon \mathbb{C}^n \to \wedge^{p,q}T^*\mathbb{C}^n$, the exterior derivative either increases the 'holormorphic degree', p, or the 'anti-holomorphic degree', q, by one and we can define operators ∂ and $\overline{\partial}$ as d composed with the corresponding projections:

$$d = \partial + \overline{\partial}$$
$$\partial = \sum \partial_{z_i} dz_i, \qquad \overline{\partial} = \partial_{\bar{z}_i} d\bar{z}_i.$$

At this stage it is worth doing a couple of concrete examples.

Example 2.5. Now we compute the exterior derivative of a few very simple functions. We take $f_1(z) = z$, $f_2(z) = \bar{z}$, $f_3(z) = |z|^2$ and $f_4(z) = \mathrm{Re}(z)$.

$$df_1 = dz \Rightarrow \partial f_1 = dz, \quad \overline{\partial} f_1 = 0.$$
$$df_2 = d\bar{z} \Rightarrow \partial f_2 = 0, \quad \overline{\partial} f_2 = d\bar{z}.$$
$$df_3 = d(z\bar{z}) = \bar{z}dz + zd\bar{z} \Rightarrow \partial f_3 = \bar{z}dz, \quad \overline{\partial} f_3 = zd\bar{z}.$$
$$df_4 = d(\tfrac{1}{2}(z - \bar{z}) = \tfrac{1}{2}dz - \tfrac{1}{2}d\bar{z} \Rightarrow \partial f_4 = \tfrac{1}{2}dz, \quad \overline{\partial} f_4 = \tfrac{1}{2}d\bar{z}.$$

Where the first two were direct computations and the other two follow from those by using the Leibniz rule and linearity of d. ■

It is convenient to place the spaces of (p,q)-forms in a lattice parametrised by the integers p and q. Once we do this we can indicate the decomposition $d = \partial + \overline{\partial}$ graphically.

Lemma 2.6. *The following relations hold:*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0 \quad \text{and} \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Proof. When acting on (p,q)-forms ∂ and $\overline{\partial}$ map to different spaces, so the relation $d^2 = 0$ translates into the three relations stated in this lemma. □

Example 2.7. For $f\colon \mathbb{C} \to \mathbb{C}$ we can compute $\partial\overline{\partial} f$ abstractly using $\partial = \partial_z dz = \frac{1}{2}(\partial_x - i\partial_y)dz$ and similarly for $\overline{\partial}$:

$$\partial\overline{\partial} f = \frac{1}{2}\partial(\partial_x f + i\partial_y f)d\bar{z} = \frac{1}{4}(\partial_x^2 f + i\partial_x\partial_y f - i\partial_y\partial_x f + \partial_y^2 f)dzd\bar{z} = \tfrac{1}{4}(\triangle f)dzd\bar{z},$$

where $\triangle$ denotes the Laplacian in $\mathbb{R}^2$.

This low dimensional coincidence is at the heart of the connection between holomorphic functions and harmonic functions. It also leads people studying 1-dimensional complex objects to call functions in the the kernel of the operator $\partial\overline{\partial}$ *harmonic*.

Yet, in higher dimensions cross terms involving $\partial_{x_i}\partial_{y_j}$ appear and neither is $\partial\overline{\partial}$ related to the Laplacian operator nor are holomorphic functions related to harmonic functions in any obvious way. ■

Example 2.8. A smooth function $f\colon \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if $\overline{\partial} f = 0$. Yet once we consider general (p,q)-forms the significance of "being in the kernel of $\overline{\partial}$" is not as obvious.

For example, a generic volume form is form $\alpha = f dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z_n}$ and is of type (n,n). Being a volume, we have $d\alpha = 0$ and hence the projections of the $d\alpha$ into further subspaces also vanish, that is $\partial\alpha = \overline{\partial}\alpha = 0$. So the condition $\overline{\partial}\alpha = 0$ places no restriction on the coefficient f. ■

One situation in which being in the kernel of $\overline{\partial}$ still has holomorphic meaning is for forms of type $(p,0)$:

Lemma 2.9. *Let $\alpha\colon \mathbb{C}^n \to \wedge^{p,0}T^*\mathbb{C}^n$ be a form, say*

$$\alpha = \sum_{|J|=p} \alpha_J dz_J.$$

Then $\overline{\partial}\alpha = 0$ if and only if each coefficient α_J is a holomorphic function.

Proof. Indeed,

$$\overline{\partial}\alpha = \sum_{|J|=p} \sum_{j=1}^{n} \partial_{\bar{z}_j} \alpha_J d\bar{z}_j dz_J,$$

and each term in the sum above lies in a different summand of $\wedge^{p+1}T^*\mathbb{C}^n$, hence the whole expression vanishes if and only if each coefficient, α_J, is holomorphic with respect to all coordinates. □

Definition 2.10. A $(p,0)$-form α is *holomorphic* if $\overline{\partial}\alpha = 0$.

2.3 Cauchy integral formula & Hartog's theorem

One of the main results of the course analysis in one complex variable is Cauchy's integral formula:

Theorem 2.11. *Let $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function, let z_0 be an interior point of Ω and let $\gamma\colon [0,1] \to \Omega$ be a null-homologous loop that does not pass through z_0 and winds around z_0 once counterclockwise. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$

Cauchy's formula has several important consequences, for example given a simply connected domain with smooth boundary, Ω, and a continuous function $g\colon \partial\Omega \to \mathbb{C}$ there is a unique holomorphic function defined on Ω which is equal to g on the boundary, namely

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{g(w)}{(w - z)} dw.$$

Another important consequence of Cauchy's integral formula is that holomorphic functions are analytic, a fact that is very surprising when we first encounter it.

There is an important version of Cauchy integral formula that does not relly on f being holomorphic.

Theorem 2.12. *Let $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ be a C^1 function, let $\gamma\colon [0,1] \to \Omega$ be a simple loop that traces the boundary of a region U counterclockwise and let $z_0 \in U$. Then*

$$f(z_0) = \frac{1}{2\pi i}\left(\int_\gamma \frac{f(z)}{z - z_0} dz - \int_U \frac{\overline{\partial} f \wedge dz}{z - z_0}\right).$$

Remark. In the case when f is holomorphic, $\overline{\partial} f = 0$, and we obtain (something very close to) the statement of Cauchy's integral formula. The proof of this more general statement relies on Stoke's Theorem, which you may have guessed is more general than Cauchy's formula, but now you can actually see what the relation between the two is.

Proof. Let D_ϵ be a disc of radius ϵ centered at z_0. For ϵ small $D_\epsilon \subset U$. We apply Stokes Theorem to the form $\frac{f(z)}{z-z_0}dz$ integrated over the boundary of $U \backslash D_\epsilon$:

$$\begin{aligned} \int_\gamma \frac{f(z)}{z - z_0}dz - \int_{\partial D_\epsilon} \frac{f(z)}{z - z_0}dz &= \int_{U\backslash D_\epsilon} d\left(\frac{f}{z - z_0}\right) \wedge dz \\ &= \int_{U\backslash D_\epsilon} \frac{\overline{\partial} f}{z - z_0} \wedge dz, \end{aligned} \tag{2.3.1}$$

where in the second equality we used that $d = \partial + \overline{\partial}$, for any function h, $\partial h \wedge dz = 0$ and that since $\frac{1}{z-z_0}$ is holomorphic, $\overline{\partial}\left(\frac{f}{z-z_0}\right) = \overline{\partial}\frac{\overline{\partial} f}{z-z_0}$.

To get the desired result we will prove the following estimates:

$$\int_{D_\epsilon} \frac{\overline{\partial} f}{z - z_0} \wedge dz \overset{\epsilon\to 0}{\longrightarrow} 0 \tag{2.3.2}$$

$$\int_{\partial D_\epsilon} \frac{f(z)}{z - z_0} dz \overset{\epsilon\to 0}{\longrightarrow} 2\pi i f(z_0). \tag{2.3.3}$$

Indeed, once these are established, we only need to take the limit of (2.3.1) as ϵ goes to zero.

To prove (2.3.2), we observe that since f is C^1, there is a constant M for which $|\partial_{\bar{z}} f| \leq M$ and hence

$$\begin{aligned} \left\|\int_{D_\epsilon} \frac{\overline{\partial} f}{z - z_0} \wedge dz\right\| &\leq \int_{D_\epsilon} \left\|\frac{\partial_{\bar{z}} f}{z - z_0}\right\| 2dx \wedge dy \\ &\leq \int_{D_\epsilon} \frac{M}{|z - z_0|} 2dx \wedge dy \\ &\leq \int_0^{2\pi}\int_0^\epsilon \frac{M}{r} 2r dr \wedge d\theta \\ &= 2\pi\epsilon M \overset{\epsilon\to 0}{\longrightarrow} 0, \end{aligned}$$

where in the first inequality we used $d\bar{z} \wedge dz = 2i dx \wedge dy$

To prove (2.3.3), we let $\epsilon_2 > 0$ and pick $\epsilon > 0$ so that if $\|z - z_0\| < \epsilon$ then $\|f(z) - f(z_0)\| < \epsilon_2$. Then we compute directly

$$\begin{aligned} \left\| \int_{\partial D_\epsilon} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right\| &= \left\| \int_{\partial D_\epsilon} \frac{f(z) - f(z_0)}{z - z_0} dz \right\| \\ &\leq \int_{\partial D_\epsilon} \left\| \frac{f(z) - f(z_0)}{z - z_0} \right\| \|dz\| \\ &\leq \int_{\partial D_\epsilon} \frac{\epsilon_2}{\epsilon} \|dz\| \\ &= 2\pi\epsilon \frac{\epsilon_2}{\epsilon} = 2\pi\epsilon_2. \end{aligned}$$

Since ϵ_2 was arbitrary, we conclude that the limit exists and is $2\pi i f(z_0)$. □

By induction, one can easily extend Cauchy Integral Formula to functions of more than one variable. To formulate the, result it is convenient to introduce the notion of a polydisc.

Definition 2.13. Given $r = (r_1, \ldots, r_n)$ with $r_i > 0$, the *polydisc of radius* r in $\mathbb{C}^n$ is the set

$$D_r = \{(z_1, \ldots, z_n) \colon |z_i| < r_i \text{ for all } i\}.$$

If we allow $r_i = \infty$ for some i we call D a *polycycilinder.* If $r_i = r$ for all i we call D a *polydisc of radius* r and the subtle difference with the first definition of these words is that now r is a real number instead of a vector in $\mathbb{R}^n$.

Corollary 2.14 (multivariable Cauchy Integral Formula)**.** *Let $f \colon \Omega \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, let $D_r \subset \Omega$ be a polydisc in the interior of Ω. Then*

$$f(z_1, \ldots, z_n) = \left(\frac{1}{2\pi i}\right)^n \int_{\|w_1\| = r_1} \cdots \int_{\|w_n\| = r_n} \frac{f(w_1, \ldots, w_n)}{(w_1 - z_1) \ldots (w_n - z_n)} dw_n \wedge \cdots \wedge dw_1.$$

The proof is done by induction using the single variable Cauchy Integral Formula for one variable at a time.

In principle one can also find a similar formula for C^1 functions just as in Theorem 2.12, but inductive step each integral would split into two and the formulas become unyielding quickly.

The fact that holomorphic functions of many variables are analytic follows from the multivariable Cauchy Integral Formula 2.14 following the same line or argument used for the single variable case.

Exercise 2.1. Every holomorphic function $f \colon \Omega \subset \mathbb{C}^n \to \mathbb{C}$ is analytic. *Hint*: Read up the proof for a single variable and adapt the estimates to the present case.

Similarly, the maximum modulus principle also holds, which can be proved either by induction using the single variable maximum modulus principle or using the multivariable Cauchy Integral Formula:

Exercise 2.2. Let $f \colon \Omega \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. If $\|f\|$ achieves its maximum in the interior of Ω, then f is constant.

While the techniques used in the previous results still belong to the realm of one complex variable, the next one is markedly a multivariable result and in fact shows that there is a stark contrast between singularities in one and in many variables. Denoting by D_r the disc of radius r in $\mathbb{C}$, in one complex variable, a holomorphic function defined on an annulus $D_2 \backslash D_1$ can have arbitrary singularities if one tries to extend it holomorphically to the smaller disc, D_1. In higher dimensions this is not the case.

Theorem 2.15 (Hartog)**.** *Let $R > r$ and let $f\colon D_R \backslash D_r \to \mathbb{C}$ be a holomorphic function defined on the complement of the polydisc D_r inside D_R. If $n > 1$, then f can be holomorphically extended to the whole polydisc D_R.*

Proof. We will only do the case $n = 2$ as the general case is completely analogous. We define two functions

$$F_1\colon D_R \to \mathbb{C}, \qquad F_1(z_1, z_2) = \frac{1}{2\pi i}\int_{|z_1|=R} \frac{f(w, z_2)}{w - z_1} dw.$$

$$F_2\colon D_R \to \mathbb{C}, \qquad F_2(z_1, z_2) = \frac{1}{2\pi i}\int_{|z_2|=R} \frac{f(z_1, w)}{w - z_2} dw.$$

By differentiating under the integral we see that F_1 and F_2 are holomorphic. Further, for each fixed z_2 with $|z_2| > r$, by Cauchy integral formula the functions $f(\cdot, z_2)$ and $F_1(\cdot, z_2)$ agree. Similarly $f(z_1, \cdot)$ and $F_2(z_1, \cdot)$ agree as long as $|z_1| > r$. In particular if $|z_1|$, $|z_2| > r$, then $F_1(z_1, z_2) = f(z_1, z_2) = F_2(z_1, z_2)$. Since F_1 and F_2 agree on the open set $U = \{(z_1, z_2)\colon |z_1| > r, |z_2| > r\}$, they are both defined on D_R and they are both holomorphic, they agree in D_R. $\square$

One way to interpret Hartog's theorem is by saying that holomorphic singularities happen in (complex) codimension one subsets.

One final important result regarding holomorphic objects in $\mathbb{C}^n$ is the $\overline{\partial}$-Poincaré lemma. Recall that in $\mathbb{C}^n$ the space of forms obtains a double grading: $\Omega^\bullet(\mathbb{C}^n) = \oplus_{p,q}\Omega^{p,q}(\mathbb{C}^n)$ and the exterior derivative also decomposes according to this double grading $d = \partial + \overline{\partial}$. Among other things, we have $\overline{\partial}^2 = 0$ (Lemma 2.6). At this fresh discovery, the more enthusiastic among us would immediately jump to the definition

Definition 2.16. The *Dolbeault cohomology* of $\mathbb{C}^n$ is

$$H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = \frac{\ker(\overline{\partial}\colon \Omega^{p,q}(\mathbb{C}^n) \to \Omega^{p,q+1}(\mathbb{C}^n))}{\mathrm{Im}(\overline{\partial}\colon \Omega^{p,q-1}(\mathbb{C}^n) \to \Omega^{p,q}(\mathbb{C}^n))}.$$

For $q = 0$, the denominator is the trivial vector space and the Dolbeault cohomology is just the kernel of the $\overline{\partial}$ operator, which, according to 2.10, is given by the holomorphic p-forms:

$$H^{p,0}(\mathbb{C}^n) = \{\alpha \in \Omega^{p,0}\colon \alpha \text{ is holomorphic}\}.$$

The natural question is: what is the Dolbeault cohomology for $q > 0$? We deal with that next.

Lemma 2.17 (Poincaré $\overline{\partial}$-Lemma in one variable)**.** *Let D be a disc and let $\alpha \in \Omega^{0,1}(D)$. Then*

$$f(z) = -\frac{1}{2\pi i}\int_D \frac{\alpha \wedge dw}{w - z}.$$

satisfies $\overline{\partial} f = \alpha$.

Proof. The form α is given by $\alpha = g\, d\bar{z}$ for some smooth function g and we need to prove that $\partial_{\bar{z}} f = g$.

To prove this we fix a reference point z_0 and use partitions of unit to write $g = g_1 + g_2$ with g_1 supported in disc of radius 2ϵ arounf z_0 and g_2 vanishing in a disc of radius ϵ around z_0. The splitting $g = g_1 + g_2$ gives a corresponding splitting $f = f_1 + f_2$. For z in the disc of radius ϵ around z_0, the integral

$$f_2(z) = \frac{1}{2\pi i}\int_D \frac{g_2(w)\, dw \wedge d\bar{w}}{w - z}.$$

is well defined and the integrand is smooth, hence we can compute the derivative of f_2 by taking the derivative under the integral:

$$\frac{\partial f_2}{\partial \bar{z}} = \frac{1}{2\pi i}\int_D \frac{\partial}{\partial \bar{z}} \frac{g_2(w)\, dw \wedge d\bar{w}}{w - z} = 0,$$

since the integrand is holomorphic in z.

Now we turn to f_1. The estimate (2.3.2) used in the proof of Cauchy's integral formula shows that f_1 is well defined and since g_1 has compact support, we have

$$\begin{aligned} f_1(z) &= \frac{1}{2\pi i}\int_D \frac{g_1(w)\, dw \wedge d\bar{w}}{w - z} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(w)\, dw \wedge d\bar{w}}{w - z} \\ &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(w+z)\, dw \wedge d\bar{w}}{w}. \end{aligned}$$

This last expression expresses f_1 is an integral that depends smoothly on the parameter z, so we can differentiate under the integral to obtain

$$\begin{aligned} \frac{\partial f_1}{\partial \bar{z}} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} \frac{g_1(w+z)\, dw \wedge d\bar{w}}{w} \\ &= g_1(z) - \frac{1}{2\pi i}\int_{\partial \tilde{D}} \frac{\frac{\partial g_1(w+z)}{\partial \bar{z}}\, dw \wedge d\bar{w}}{w} \\ &= g_1(z), \end{aligned}$$

where in second equality we used that g_1 has compact support, picked a disc $\tilde{D}$ large enough so that g_1 vanishes at the boundary of $\tilde{D}$ and applied Cauchy's integral formula.

With these two together we have

$$\overline{\partial} f(z) = \overline{\partial} f_1(z) + \overline{\partial} f_2(z) = g_1 d\bar{z} + 0 = g_1(z) d\bar{z} + g_2(z) d\bar{z} = g(z) d\bar{z} = \alpha.$$

Since z_0 was arbitrary, the result follows. □

Theorem 2.18. *The Dolbeault cohomology of any polydisc vanishes for $q > 0$.*

Proof. Consider the polydisc D_r with $r = (r_1, \ldots, r_n)$. Let $\alpha \in \Omega^{p,q}$ be a $\overline{\partial}$-closed form and expand it in multiindex notation:

$$\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J.$$

Then $\overline{\partial}\alpha = 0$ produces a number of equations but does not change the dz_I factor of each summand of α. Hence $\overline{\partial}\alpha = 0$ is equivalent to

$$\overline{\partial}(\sum_J \alpha_{IJ} d\bar{z}_J) = 0, \qquad \text{for all } I.$$

So we can restrict our attention to $(0, q)$-forms.

Say $\alpha = \sum_J \alpha_J d\bar{z}_J$ is $\overline{\partial}$-closed, we will show it is $\overline{\partial}$-exact (in D_r) by inductively subtracting exact forms from α which successively remove all terms containing $d\bar{z}_1$, $d\bar{z}_2$, etc.

Let's say that $d\bar{z}_1, \ldots, d\bar{z}_{i-1}$ do not appear in α, that is the multi-indices J appearing in the sum start at i, so we can write

$$\alpha = \sum_{J>i} \alpha_{i,J} dz_i \wedge d\bar{z}_J + \sum_{J>i} \alpha_J d\bar{z}_J,$$

where $J > i$ denotes the multi-indices whose lowest index is greater than i.

Since $\overline{\partial}\alpha = 0$, it follows that $\frac{\partial \alpha_{i,J}}{\partial \bar{z}_j} = 0$ for $j < i$. Then consider

$$\beta_{i,J}(z) = \int_{|w|<r_i} \alpha_{i,J}(z_1, \ldots, w, \ldots, z_n) \frac{dw \wedge d\bar{w}}{w - z_i}$$

By differentiating under the integral we have

$$\frac{\partial \beta_{i,J}}{\partial \bar{z}_j} = 0 \qquad \text{for all } j < i.$$

Also, by the $\overline{\partial}$-Poincaré Lemma in one variable,

$$\frac{\partial \beta_{i,J}}{\partial \bar{z}_i} = \alpha_{i,J}(z_1, \ldots, z_i, \ldots, z_n).$$

So if we form

$$\tilde{\alpha} = \alpha - \overline{\partial}\left(\sum_{J>i} \beta_{i,J} d\bar{z}_J\right),$$

we see that $\tilde{\alpha}$ is also $\overline{\partial}$-closed and $d\bar{z}_1, \ldots, d\bar{z}_i$ do not appear in $\tilde{\alpha}$.

Repeating this n times, we conclude that α is $\overline{\partial}$-exact. □

Remark. The $\overline{\partial}$-Poincaré Lemma also holds on polycylinders [1, page 25]

Chapter 3

Complex manifolds

3.1 Definition and examples

With the basics of the theory of holomorphic functions behind us, we can now transport that to the world of manifolds. The first definition of complex manifold is one that we can naturally guess: simply replace the requirement that the transition functions of an atlas are smooth by the requirement that they are holomorphic:

Definition 3.1. A *complex manifold* is a topological manifold (second countable, Hausdorff, locally Euclidean topological space), M, endowed with an atlas $\{(U_\alpha, \phi_\alpha)\colon \alpha \in A\}$ for which each chart takes values in $\mathbb{C}^n$, that is

$$\phi_\alpha\colon U_\alpha \subset M \to V_\alpha \subset \mathbb{C}^n$$

and all change of coordinates, $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic in their domain.

Each coordinate chart in a such an atlas is called a *holomorphic (coordinate) chart*.

Just with the definition at hand we can already produce several examples of complex manifolds and it is good to go through some of them so over time we create a mental library of examples with which we can test hypothesis we may eventually raise.

Example 3.2. Euclidean space $\mathbb{C}^n$ is a complex manifold covered by a single chart. In fact open subset of $\mathbb{C}^n$ inherits automatically the structure of a complex manifold. ■

Example 3.3. The space of complex linear $n \times m$-matrices is diffeomorphic to $\mathbb{C}^{nm}$ and as such it is a complex manifold. Also the space of invertible $n \times n$-matrices, $\mathrm{GL}\,(n, \mathbb{C})$ is a complex manifold. Indeed, the complex determinant

$$\det\colon M_{n\times n}(\mathbb{C}) \to \mathbb{C},$$

is a continuous function (it is a polynomial) and hence $\mathrm{GL}\,(n, \mathbb{C}) = \det^{-1}(\mathbb{C}^*)$ is open in the space of all $n \times n$-matrices. ■

Example 3.4 (Non example)**.** With the previous example at hand, we may start to get ideas... we may start to think that classic matrix groups based on matrices with complex coefficients are complex manifolds. We do not have to look far to dispel this idea. Indeed, $\mathrm{U}(1)$, the group of unitary transformations of $\mathbb{C}$, is just the complex numbers of length 1, that is the circle $S^1 \subset \mathbb{C}$, which has odd real dimension and therefore is not a complex manifold. Similarly, SU(2) is diffeomorphic to the 3-dimensional sphere and hence is also not a complex manifold. In fact

$$\dim_{\mathbb{R}} \mathrm{U}(n) = n^2, \qquad \dim_{\mathbb{R}} \mathrm{SU}(n) = n^2 - 1,$$

so, by purely dimensional reasons, half of these groups can not be complex manifolds. Interestingly, the other half are complex manifolds, but not in the obvious way that $\mathrm{GL}(n;\mathbb{C})$ is. ■

Example 3.5 (The (Riemann) sphere)**.** We can parametrize the sphere using stereographic projections. From this point of view, one chart covers the whole sphere minus the north pole, $N = (0,0,1)$, and the other chart covers the whole sphere minus the south pole, $S = (0,0,-1)$:

$$\phi_N \colon S^2 \backslash N \to \mathbb{R}^2, \qquad \phi_N(x,y,z) = \frac{(x,y)}{1-z},$$

$$\phi_S \colon S^2 \backslash S \to \mathbb{R}^2, \qquad \phi_S(x,y,z) = \frac{(x,-y)}{1+z}.$$

Notice that in the charts above we "cheated" a little and for Φ_S we composed the stereographic projection with a reflection. This choice is made so both charts induce the same orientation on the sphere.

Both maps above are surjective onto $\mathbb{R}^2$ and miss only one point on the sphere, so it is natural to write $S^2 = \mathbb{R}^2 \cup \{\infty\}$ to indicate that the sphere is the one-point compactification of $\mathbb{R}^2$ which can be concretely visualised as, say, adding the north pole to the codomain of ϕ_N.

Since $\mathbb{R}^2 = \mathbb{C}$, we can write both Φ_N and Φ_S as complex valued maps and compute the corresponding change of coordinates:

$$\Phi_N^{-1}(w) = \frac{1}{1+\|w\|^2}(2w, \|w\|^2 - 1), \qquad \phi_S(x,y,z) = \frac{(x,-y)}{1+z}$$

$$\Phi_S \circ \Phi_N^{-1}(w) = \frac{1}{w}.$$

This shows that the two charts obtained from stereographic projection from the north and south pole give rise to holomorphic change of coordinates and therefore make the sphere a complex manifold.

Independently of the computation above, an idea you may have encountered in a course of analysis in one complex variable is that one can form the one-point compactification of $\mathbb{C}$: $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In that context, to 'see' what is happening at infinity, one has to 'invert' the complex coordinate and consider $w = 1/z$[1]. Intuitively we understand that $\overline{\mathbb{C}}$ should be a sphere and the previous computation with stereographic projections makes that intuitive picture precise. ■

Example 3.6 (Projective space)**.** The complex projective space, $\mathbb{C}P^n$ is the set of lines through the origin in $\mathbb{C}^{n+1}$, or equivalently, the quotient of $\mathbb{C}^{n+1} \backslash \{0\}$ by the action of $\mathbb{C}^*$ acting by scalar multiplication, because two nonzero points are in the same line if and only if they are nonzero multiples of each other.

The action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \backslash \{0\}$ has two very desirable properties:

[1] It is very unfortunate that the same letter, z, is used for the third coordinate in $\mathbb{R}^3$ and for complex numbers. Fortunately this appearances of z in both roles in short succession is not something that will come up very often

- it is free (of fixed points), that is, $\lambda \cdot v = v$ if and only if $\lambda = 1$ and
- it is proper, that is, the map

$$\mathbb{C}^* \times \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}^{n+1}\backslash\{0\} \times \mathbb{C}^{n+1}\backslash\{0\} \qquad (\lambda, v) \to (v, \lambda v)$$

  is proper (pre-image of compact is compact).

Whenever one has a free and proper action of a Lie group on a manifold, the quotient is a manifold.

In the present case, we can produce coordinates for $\mathbb{C}P^n$ which express it as a complex manifold. From now on we denote the line in $\mathbb{C}P^n$ that goes through the point $(z_1, \dots, z_{n+1})$ by $[z_1, \dots, z_{n+1}]$. If we let $U_i \subset \mathbb{C}P^n$ be the set of lines for which the i^{th} coordinate is nonzero, we can define maps

$$\Phi_i \colon U_i \to \mathbb{C}^n, \qquad \Phi_i[z_1, \dots, z_{n+1}] = \left(\tfrac{z_1}{z_i}, \dots, \hat{z_i}, \dots, \tfrac{z_{n+1}}{z_i}\right),$$

where the hat indicates that the i^{th} term is missing in the list. Since scalling all coordinates by the same constant does not change the value Φ_i takes on the line $[z_1, \dots, z_{n+1}]$ we see that Φ_i is well defined (and continuous and a homomorphism from its domain onto $\mathbb{C}^n$).

The change of coordinates is given by

$$\Phi_j \circ \Phi_i^{-1}(z_1, \dots, z_n) = \Phi_j([z_1, \dots, z_j, \dots, \underset{i^{th}\text{ pos}}{1}, \dots, z_n]) = \left(\tfrac{z_1}{z_j}, \dots, \underset{i^{th}\text{ pos}}{\tfrac{1}{z_j}}, \dots, \tfrac{z_n}{z_j}\right), \tag{3.1.1}$$

which is a holomorphic map, making $\mathbb{C}P^n$ a complex manifold.

This is the same computation done in an introductory "differentiable manifolds" course to produce coordinates for the real projective space, $\mathbb{R}P^n$. It is an interesting fact that while we are all aware of the existence of $\mathbb{R}P^n$, we seem to on purpose avoid it, often by starting our framework with the condition that *M is an orientable manifold*. By contrast, $\mathbb{C}P^n$ is central in much of the theory of complex manifolds as we will see over and over in this course. ■

Exercise 3.1. (Random exercise) Is $\mathbb{R}P^n$ non orientable for all n?

This example of $\mathbb{C}P^n$ is so important and will reoccur often enough that we need to introduce some language to refer back to it. The description of points in $\mathbb{C}P^n$ using the square bracket notation, $[z_1, \dots, z_{n+1}]$ to describe the line through $(z_1, \dots, z_{n+1})$, is known as *homogeneous coordinates*, while describing points in the set U_i by identifying them with points $\mathbb{C}^n$ using the coordinate chart Φ_i is known as *affine coordinates*.

Example 3.7 ($\mathbb{C}P^1$)**.** In the previous example, it is worth looking carefully at the manifold $\mathbb{C}P^1$. Following the argument above, $\mathbb{C}P^1$ is parametrised by two charts, Φ_1 and Φ_2 and according to (3.1.1) the change of coordinates is given by

$$\Phi_1 \circ \Phi_2^{-1}(z) = \frac{1}{z}.$$

That is, just like S^2, $\mathbb{C}P^2$ is covered by two charts whose codomains are $\mathbb{C}$ and the transition functions agree. That is $\mathbb{C}P^1$ is diffeomorphic to S^2. In particular, $\mathbb{C}P^1$ is orientable... I know, shocker. ■

Exercise 3.2. Show that every complex manifold is orientable.

Exercise 3.3. Show that if two manifolds M_1, M_2 can be covered by open sets $M_i = \cup_\alpha U_i^\alpha$ and admit charts $\Phi_i^\alpha \colon U_i^\alpha \to V^\alpha$ such that the transition functions agree, that is

$$\Phi_1^\alpha \circ (\Phi_1^\beta)^{-1} = \Phi_2^\alpha \circ (\Phi_2^\beta)^{-1}, \qquad \text{for all } \alpha, \beta,$$

then M_1 and M_2 are diffeomorphic.

Example 3.8 (The torus)**.** If we let $\{v_1, \ldots, v_{2n}\}$ be a real basis for $\mathbb{R}^{2n}$, we define a $\mathbb{Z}^{2n}$ action on $\mathbb{R}^{2n}$ by translations:

$$(m_1, \ldots, m_{2n}) \cdot (x_1, \ldots, x_{2n}) \mapsto (x_1 + m_1, \ldots, x_{2n} + m_{2n}).$$

Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and using that "adding a constant" is a holomorphic operation, we see that this action preserves to complex structure on $\mathbb{C}^n$. This action of $\mathbb{Z}^{2n}$ is also free and proper and the quotient is the $2n$-dimensional torus, T^{2n}. Further, the quotient map $\pi \colon \mathbb{C}^n \to T^{2n}$ is a covering map, that is every point on the quotient has a neighbourhood, U with the property that $\pi^{-1}(U)$ is a disjoint union of open sets V_i with $\imath \in \mathbb{Z}^{2n}$ such that $\pi \colon V_i \to U$ is a diffeomorphism for all i. In particular, since $V_i \subset \mathbb{C}^n$ is a complex manifold, we can use π to transport that complex structure to U (because $\pi \colon V_i \to U$ is a diffeomorphism. Since the identification between V_i and V_j is by a holomorphic transformation, we see that the complex structure on U obtained this way does not depend on i and since U is arbitrary, this produces a collection of complex coordinates in the quotient whose transition functions are holomorphic (compositions of π, π^{-1} and translations). Hence the torus is a complex manifold.

One interesting feature of this example that will become apparent later is that while all real tori are diffeomorphic to $S^1 \times \cdots \times S^1$, the complex manifold obtained above does depend on the particular basis $\{v_1, \ldots, v_{2n}\}$ with which we started and there are many non-equivalent complex tori. ∎

The key fact we used in the example above was that the action of $\mathbb{Z}^{2n}$ on $\mathbb{C}^n$ is holomorphic and properly discontinuous (that is, free, proper and the quotient map is a covering space). Any time these conditions are met the quotient inherits a complex structure.

Example 3.9 (Primary Hopf manifolds)**.** Let $\alpha \in \mathbb{C}$ be a complex number of length bigger than 1 and let $\mathbb{Z}$ act on $\mathbb{C}^n \backslash \{0\}$ by

$$m \cdot z = \alpha^m z.$$

Just as in the previous example this action is holomorphic and properly discontinuous hence the quotient is a complex manifold.

Topologically $\mathbb{C}^n \backslash \{0\} = S^{2n-1} \times \mathbb{R}_+$ and the action of $\mathbb{Z}$ is by dilations on $\mathbb{R}_+$ and rotations on the sphere. Since the action of $\mathbb{R}_+$ is properly discontinuous, the argument of α just provides a way to identify the spheres over, say 1 and α and we see that the quotient $\mathbb{C}^n \backslash \{0\}/\mathbb{Z}$ is a mapping torus. The rotations are smoothly isotopic to the identity this mapping torus is trivial, that is, $\mathbb{C}^n \backslash \{0\}/\mathbb{Z}$ is diffeomorphic to $S^{2n-1} \times S^1$.

Just as with the previous example, different values of the parameter α induce different complex structures on $S^{2n-1} \times S^1$. ∎

3.2 Almost complex structures and integrability

Given a complex manifold M, we can define multiplication by i on its tangent bundle as follows: for any given point $p \in M$, let $\phi\colon U \to \mathbb{C}^n$ be a holomorphic coordinate chart in a neighbourhood p. Then define $I\colon T_pM \to T_pM$ by $I = \phi_*^{-1} \circ I_0 \circ \phi_*$, where I_0 is the standard complex structure on the vector space $\mathbb{C}^n$. That is, if $\{\partial_{x_i}, \partial_{y_i} \colon i = 1, \dots, n\}$ are the coordinate vector fields associated to ϕ, then

$$I\partial_{x_i} = \partial_{y_i} \qquad \text{and} \qquad I\partial_{y_i} = -\partial_{x_i}.$$

If $\tilde{\phi}$ is another holomorphic coordinate chart, we can use it to define a corresponding complex structure $\tilde{I} = \tilde{\phi}_*^{-1} \circ I_0 \circ \tilde{\phi}_*$ on T_pM. Since the change of coordinates is holomorphic we obtain

$$I \circ \tilde{I}^{-1} = \phi_*^{-1} \circ I_0 \circ \phi_* \circ \tilde{\phi}_*^{-1} \circ I_0{}^{-1} \circ \tilde{\phi}_* = \phi_*^{-1} \circ \phi_* \circ \tilde{\phi}_*^{-1} \circ I_0 \circ I_0{}^{-1} \circ \tilde{\phi}_* = \phi_*^{-1} \circ \phi_* \circ \tilde{\phi}_*^{-1} \circ \tilde{\phi}_* = \mathrm{Id},$$

therefore the linear complex structure $I\colon T_pM \to T_pM$ defined above does not depend on the chart used. Also in any holomoprhic coordinate chart I is given by the constant matrix, so $I\colon TM \to TM$ is a bundle automorphism such that $I^2 = -\mathrm{Id}$. Said another way, I is a linear complex structure on TM. At this stage it is not clear if the holomorphic coordinates give us anything more than a linear complex structure on TM, but as we will see soon there is a marked difference between these two concepts and it is worth giving a name to the concept we just encoutered:

Definition 3.10.

- An *almost complex structure* on a manifold M is a bundle automorphism $I\colon TM \to TM$ such that $I^2 = -\mathrm{Id}$.
- An *almost complex manifold* is a manifold with an almost complex structure.
- An almost complex structure is *integrable* if it is induced by a complex structure.

Exercise 3.4. Show that if M has two atlases that make it into a complex manifold, $\mathcal{A}_1$ and $\mathcal{A}_2$, and both atlases induce the same almost complex structure, then the identity map $\mathrm{Id}\colon (M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$ is a holomorphic map with holomorphic inverse. That is, two complex structures agree if and only if they induce the same almost complex structure.

This means that an integrable almost complex structure carries the same data as a complex structure and hence we will not make a distinction between these two concepts.

To grasp the difference between an almost complex manifold and a complex manifold, we observe that for the latter, in holomorphic coordinates, we can easily write down a local frame for the $+i$-eigenspace of I:

$$T^{1,0}M = \langle \tfrac{1}{2}(\partial_{x_i} - i\partial_{y_i}) \colon i = 1, \dots, n \rangle$$

As in Chapter 1, we denote $\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$ and by extending the Lie bracket of vector fields to vector fields with complex coefficients by requiring that it is complex linear, we have that $[\partial_{z_i}, \partial_{z_j}] = 0$ for all i, j. Therefore, if we take combinations of these vector fields with smooth coefficients we obtain:

Lemma 3.11. *Let M be a complex manifold, then $T^{1,0}M$ is an involutive subbundle, that is,*

$$[X,Y] \in \Gamma(T^{1,0}M) \qquad \text{for all } X,Y \in \Gamma(T^{1,0}M).$$

Involutivity of $T^{1,0}M$ is an infinitesimal condition (only uses first derivatives of vector fields). It is not obvious whether there is more data that is needed to decide if an almost complex structure is integrable or not. The answer is that this is all and this is the result of a deep theorem:

Theorem 3.12 (Newlander–Nirenberg [5])**.** *An almost complex structure is integrable if and only if its $+i$-eigenbundle is involutive.*

Therefore, to measure the difference between complex and almost complex manifolds one needs to quantify how much $T^{1,0}M$ is/fails to be involutive. Using Lie bracket and projections from $T_{\mathbb{C}}M$ to $T^{0,1}M$ we can produce an object that measures precisely that.

Definition 3.13. The *Nijenhuis tensor* of an almost complex structure is the operator

$$N \colon \Gamma(T^{1,0}M) \times \Gamma(T^{1,0}M) \to \Gamma(T^{0,1}M), \qquad N(X,Y) = \pi_{T^{0,1}M}[X,Y]$$

Lemma 3.14. *The Nijenhuis tensor is skew-smmetric and C^∞-linear, that is, it is indeed a tensor.*

Proof. Since the Lie bracket is skew symmetric, it is clear that N is also skew-symmetric. Further, for any smooth function f we have

$$N(X,fY) = \pi_{T^{0,1}M}[X,fY] = \pi_{T^{0,1}M} f[X,Y] + \pi_{T^{0,1}M}(\mathcal{L}_X f)Y = f\pi_{T^{0,1}M}[X,Y] = fN(X,Y),$$

where we used that $\Pi_{T^{0,1}M}$, being a bundle map, is C^∞-linear and that since $Y \in \Gamma(T^{1,0}M)$, $\Pi_{T^{0,1}M}Y = 0$. Since N is C^∞-linear, it is a tensor. □

It is clear from the definition that the $+i$-eigenbundle of an almost complex structure is involutive if and only if the Nijenhuis tensor vanishes. Since this is an immediate quantity that measures how far an almost complex structure is from being integrable, it is worth spending some time rephrasing this concept.

The first way one can repackage the Nijenhuis tensor is by observing that we have isomorphisms $TM \to T^{1,0}M$ and $TM \to T^{0,1}M$:

$$i^{1,0} \colon TM \to T^{1,0}M, \quad i^{1,0}X = X - iIX,$$
$$i^{0,1} \colon TM \to T^{1,0}M, \quad i^{0,1}X = X + iIX,$$

which have inverses given by taking the real part of a vector. So we can compose the Nijenhuis tensor introduced above with these isomorphisms to obtain a completely real operator which measures the lack of involutivity of $T^{1,0}M$:

Exercise 3.5. Let $\tilde{N} \colon TM \times TM \to TM$ be given by

$$\tilde{N}(X,Y) = \mathrm{Re}(N(i^{1,0}X, i^{1,0}Y)).$$

Show that $T^{1,0}M$ is involutive if and only if $\tilde{N}$ vanishes and that

$$\tilde{N}(X,Y) = [X,Y] + I([IX,Y] + [X,IY]) - [IX,IY]. \tag{3.2.1}$$

In most textbooks expression (3.2.1) is called the Nijenhuis tensor. We won't make much of a distinction between the two since they carry the same information.

Theorem 3.15 (Orientable surfaces)**.** *Every orientable real surface admits a complex structure*

Proof. First we observe that any almost complex structure on a real surface is integrable. Indeed, given an arbitrary almost complex structure I, if $X \in \Gamma(T^{1,0}M)$ is a nonvanishing local vector field, then by virtue of M having complex dimension one, every other local section of $T^{1,0}M$ is of the form fX for some function f. We can compute the Nijenhuis tensor of two such generic sections:

$$N(fX, gX) = fgN(X, X) = 0,$$

where in the first equality we used that N is a tensor and in the second that it is skew-symmetric. By the Newlander–Nirenberg theorem we conclude that I is integrable.

So the quest now is only to produce an almost complex structure on a surface. The idea here is that an orientation gives the notion of "counterclockwise" rotation and the metric allow us to rotate by $\pi/2$ (counterclockwise) without stretching. The operation "rotation by $\pi/2$ without stretching" squares to "rotation by π without stretching", that is, $-\mathrm{Id}$ and hence defines an almost complex structure. The formal argument making this precise is done using the Hodge star operator, so lets have a little break in our proof to introduce that.

Definition 3.16 (Hodge star operator v1)**.** Let M^n be an oriented Riemannian manifold with Riemannian metric g. Define the *Hodge star operator* on $\wedge^\bullet TM$, as follows. Fix a positive orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and define

$$\star\colon \wedge^k TM \to \wedge^{n-k}TM, \qquad \star e_I = \mathrm{sign}(I)e_{I^c},$$

where I is a multiindex of length k, I^c is the complementary multindex of I, that is, $I \cap I^c = \emptyset$ and $I \cup I^c = \{1, \ldots, n\}$ and $\mathrm{sign}(I)$ is the sign for which

$$e_I \wedge \star e_I = e_1 \wedge \cdots \wedge e_n.$$

This defines $\star$ in basis of $\wedge^\bullet TM$ and we extend it to $\wedge^\bullet TM$ by linearity. This operator does not depend on the particular choice of basis, but only on the orientation and metric g.

Back to our proof, if $\dim M = n$, a direct computation shows that when acting on $\wedge^k TM$ we have $\star^2 = (-1)^k(n-k)$. In the case when M is 2-dimensional, if we consider the action of $\star$ on TM we obtain $\star^2 = -\mathrm{Id}$. That is, for oriented real surfaces, the choice of a metric induces an almost complex structure, which completes the proof. □

A real surface with a complex structure also goes by the names *Riemann surface* or *complex curve.*

Example 3.17 (Non example)**.** The 6-sphere admits an almost complex structure. To introduce it, we recall that there are four real division algebras:

- the reals, $\mathbb{R}$ (ordered, commutative, associative, division algebra),
- the complex numbers, $\mathbb{C}$ (commutative, associative, division algebra),
- the quaternions, $\mathbb{H}$ (associative, division algebra),

- the octonions, $\mathbb{O}$ (division algebra).

The relevant one for this example is the latter. The octonions, $\mathbb{O}$, are isomorphic to $\mathbb{R}^8$ as a real vector space and can also be described as $\mathbb{H} \oplus \mathbb{H}$. The reals sit inside the octonions via the inclusion in the first factor and form the center of the octonions. Multiplication is given by

$$(p_1, q_1) \cdot (p_2, q_2) = (p_1 p_2 - \bar{q}_2 q_1, q_2 p_1 + q_1 \bar{p}_2),$$

where $\bar{\bullet}$ denotes quaternionic conjugation.

Just as with complex numbers and quaternions, we can conjugate octonions by flipping the sign of the imaginary part, $\overline{(p,q)} = (\bar{p}, -q)$, and using conjugation we can relate octonionic multiplication with the Euclidean norm of elements: for $v, w \in \mathbb{O}$, $v\bar{v} = \|v\|^2$. and $\|vw\| = \|v\|\|w\|$.

The quaterionic condition $ij = k$ has a corresponding octonionic formulation: any two orthogonal unitary imaginary octonions, v and w, generate a copy of the quaterions inside $\mathbb{O}$, namely the vector space generated by $\{1, v, w, vw\}$. In particular, the product vw is a unitary imaginary octonion orthogonal to both v and w.

Now, if we consider S^6 as the unit sphere inside the space of imaginary quaternions, then we can define an automorphism of the tangent space by

$$I \colon TS^6 \to TS^6 \qquad w \in T_v S^6 \stackrel{I}{\mapsto} vw$$

Since vw is imaginary and orthogonal to v, it is tangent to S^6 at v. Further, since v is imaginary of length 1,

$$I^2 w = v(vw) = (vv)w = -w$$

where we used that the space generated algebraically by v and w is isomorphic to the quaternions and quaternionic multiplication is associative.

This describes an almost complex structure on S^6. ■

Exercise 3.6. Show that the complex structure defined above on S^6 is not integrable.

Theorem 3.18 (Compact Lie groups)**.** *Every even dimensional compact Lie group can be made into a complex manifold.*

Proof. For this we will need some more advanced knowledge of Lie theory. This result will not re-occur in any important passage later in these notes and can be safely skipped.

Let G be an even dimensional compact Lie group. We will use Lemma 1.4 to endow G with an almost complex structure which we will prove to be integrable. Recall that the Lie algebra, $\mathfrak{g}$, of G can be indentified with the left invariant vector fields on G and the Lie bracket on $\mathfrak{g}$ agrees with the Lie bracket of vector fields on G. So to produce a complex structure on G we will produce a sub-Lie algebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes \mathbb{C}$ such that

$$\mathfrak{g}^{1,0} + \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes \mathbb{C} \quad \text{and} \quad \mathfrak{g}^{1,0} \cap \overline{\mathfrak{g}^{1,0}} = \{0\}.$$

Since G is compact it admits a bi-invariant metric (invariant under left and right multiplication). At the Lie algebra, this translates to

$$\langle [u,v], w \rangle + \langle v, [u,w] \rangle = 0.$$

Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let T act on G by conjugation. This induces an infinitesimal action of $\mathfrak{t}$ on $\mathfrak{g}$ by the Lie bracket. Since the metric is bi-invariant, $\mathfrak{t}$ acts by infinitesimal isometries. Since $\mathfrak{t}$ is invariant under its own action (because it is Abelian), so is the orthogonal complement $\mathfrak{h} = \mathfrak{t}^{\perp} \subset \mathfrak{g}$.

Since $\mathfrak{t}$ is Abelian, the representation of $\mathfrak{t}$ on $\mathfrak{h}$ decomposses $\mathfrak{h}$ into one-dimensional irreducible components: $\mathfrak{h} = \oplus_j \mathrm{spam}\{e_j\}$. For each e_j and $v \in \mathfrak{t}$ we have that that $[v, e_j] = i\alpha_j(v)e_j$ for some constant α_j which depends linearly on v. That is, the α_j is in fact a map an element of $\mathfrak{t}^*$ and we can use the α_j themselves to index the decomposition of $\mathfrak{h}$. An $\alpha \in \mathfrak{t}^*$ that appears in this decomposition is called a *root* of $\mathfrak{g}$ and we write

$$\mathfrak{h} = \bigoplus_{\alpha \text{ root of } \mathfrak{g}} \mathrm{spam}\{e_\alpha\}.$$

There are a couple of important properties of the set of all roots:

- The roots are real. Indeed, since action preserves the metric we have
$$0 = \langle [v, e_\alpha], e_\alpha\rangle + \langle e_\alpha, [v, e_\alpha]\rangle = (i\alpha(v) - i\alpha(v))\|e_\alpha\|^2.$$
- The roots come in pairs with opposite signs. Indeed, since the underlying action is real we have that, if e_α is an element of $\mathfrak{h}$ associated to the root α and $v \in \mathfrak{t}$, then
$$[v, \overline{e_\alpha}] = \overline{[v, e_\alpha]} = -i\alpha(v)\overline{e_\alpha},$$
showing that $-\alpha$ is also a root.
- If α and β are roots with associated generators e_α and e_β, then $\alpha + \beta$ is a root with associated generator $[e_\alpha, e_\beta]$. Indeed, using the Jacobi identity, for $v \in \mathfrak{t}$, we have
$$[v, [e_\alpha, e_\beta]] = [[v, e_\alpha], e_\beta] + [e_\alpha, [v, e_\beta]] = i(\alpha(v) + \beta(v))[e_\alpha, e_\beta].$$

If we pick an element $w \in \mathfrak{h}$ which is not in the kernel of any e_α, we can use w to divide the set of roots into positive and negative, depending on the sign of $\alpha(w)$ and we get

$$\mathfrak{h} = \bigoplus_{\alpha \text{ +ve root of } \mathfrak{g}} \mathrm{spam}\{e_\alpha \oplus e_{-\alpha}\} = \bigoplus_{\alpha \text{ +ve root of } \mathfrak{g}} \mathrm{spam}\{e_\alpha \oplus \overline{e_\alpha}\}\},$$

that is,

$$\mathfrak{h}^{1,0} = \bigoplus_{\alpha \text{ +ve root of } \mathfrak{g}} \mathrm{spam}\{e_\alpha\}$$

satisfies the conditions of Lemma 1.4 and therefore determines a complex structure on $\mathfrak{h}$.

Finally, if we pick an arbitrary complex structure $\mathfrak{t}^{1,0} \subset \mathfrak{t} \otimes \mathbb{C}$ we obtain an almost complex on G by declaring $\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \mathfrak{h}^{1,0}$. We claim this complex structure is integrable. To prove that we need to check that $\mathfrak{g}^{1,0}$ is involutive, which we can do just on generators. For $v_1, v_2 \in \mathfrak{t}^{1,0}$ and α and β positive roots we have

$$\begin{aligned} [v_1, v_2] &= 0 && \text{because } \mathfrak{t} \text{ is Abelian,} \\ [v_1, e_\alpha] &= i\alpha(v)e_\alpha && \text{by the definition of root,} \\ [e_\alpha, e_\beta] &= \lambda e_{\alpha+\beta} && \text{by the the properties of roots,} \end{aligned}$$

since α and β are positive roots, so is $\alpha+\beta$ and we conclude that $\mathfrak{g}^{1,0}$ is involutive, showing that the complex structure is integrable. $\square$

Returning to Example 3.4, we may have the warm feeling that now the question of for which values of n are $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ complex manifolds is settled:

	$\mathrm{U}(n)$	$\mathrm{SU}(n)$
n odd	✗	✓
n even	✓	✗

Figure 3.1: Table showing for which values of n $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ admit complex structures.

But it is important to notice that in the construction of the complex structures on these spaces we made no reference to the complex structure on $\mathrm{GL}(n;\mathbb{C})$ and as we will see next session the natural embedding of these groups in $\mathrm{GL}(n;\mathbb{C})$ is not a holomorphic map.

3.3 Results from smooth manifolds

This section we take a quick look at how several results from smooth functions translate to holomorphic functions. We start with some of the most fundamental ones.

Theorem 3.19 (Inverse Function Theorem)**.** *Let $f\colon \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic. If df is invertible at a point x, then x has a neighbourhood U such that $f\colon U \to f(U)$ is a bijection with holomorphic inverse.*

Proof. Since df is invertible, it follows that x has a neighbourhood U such that $f\colon U \to f(U)$ is a bijection with smooth inverse. Since f is holomorphic, df is complex linear and therefore $(df)^{-1}$ is also complex linear showing that f^{-1} is holomorphic. $\square$

Theorem 3.20 (Implicit Function Theorem)**.** *Let $f\colon \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ be a holomorphic function and let $x \in \mathbb{C}^n \times \mathbb{C}^m$ be such that $d_2f|_x\colon \mathbb{C}^m \to \mathbb{C}^m$ is invertible. Let $y = f(x)$. Then there a neighbourhood $U \times V$ of x and a holomorphic function $h\colon \mathbb{C}^n \supset\!\to \mathbb{C}^n \times \mathbb{C}^m$ such that*

$$f^{-1}(y) \cap U \times V = \{a, h(a)\colon a \in \mathbb{C}^n\}.$$

Proof. Consider the function $F\colon \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n \times \mathbb{C}^m$ defined by

$$F(a,b) = (a, f(a,b)),$$

and let $F(x) = (z,y)$. Then dF is invertible at x and therefore F^{-1} exists and is holomorphic in a neighbourhood of (z,y). Since the inclusion $\iota_1\colon \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^m$, $\iota(z) = (z,y)$ is holomorphic, the composition $h = F^{-1} \circ \iota_1\colon \mathbb{C}^n \supset\!\to \mathbb{C}^n \times \mathbb{C}^m$ is also holomorphic and we have $f(x') = y$ if and only if $F(x') = (z,y)$ for some z, which happens if and only if $F^{-1}(z,y) = x'$ for some z, which happens if and only if $x' \in \mathrm{Im}(h)$. $\square$

With these theorems at hand, a number of their consequences follow along the same lines as in the smooth case. We mention a few of them:

Exercise 3.7 (Immersions)**.** Let M^m and N^n be complex manifolds and let $f\colon M \to N$ be a holomorphic immersion (immersion means that $df\colon T_pM \to T_{f(p)}N$ is an injection for all $p \in M$). Then given $p \in M$, there exist neighbourhoods U of p and V of $f(p)$ and holomorphic coordinate charts $\Phi\colon U \to \mathbb{C}^m$ and $\Psi\colon V \to \mathbb{C}^m \times \mathbb{C}^{n-m}$ centred at p and $f(p)$ such that the expression for f in these charts is

$$\Psi \circ f \circ \Phi^{-1}(z) = (z, 0).$$

Exercise 3.8 (Submersions)**.** Let M^m and N^n be complex manifolds and let $f\colon M \to N$ be a holomorphic submersion (submersion means that $df\colon T_pM \to T_{f(p)}N$ is a surjection for all $p \in M$). Then given $p \in M$, there exist neighbourhoods U of p and V of $f(p)$ and holomorphic coordinate charts $\Phi\colon U \to \mathbb{C}^n \times \mathbb{C}^{m-n}$ and $\Psi\colon V \to \mathbb{C}^m \times \mathbb{C}^n$ centred at p and $f(p)$ such that the expression for f in these charts is

$$\Psi \circ f \circ \Phi^{-1}(x, y) = x.$$

A variation of the exercise above is the regular value theorem

Exercise 3.9 (Regular value theorem)**.** Let M^m and N^n be complex manifolds, let $f\colon M \to N$ be a holomorphic map and let $q \in N$ be a regular value of f, that is, for all $p \in f^{-1}(q)$, $df\colon T_pM \to T_{f(p)}N$ is a surjection. Then $f^{-1}(q)$ is am embedded complex submanifold of M whose tangent space at p is the kernel of $df\colon T_pM \to T_qN$.

Example 3.21 (Zeros of polynomials)**.** Let $p\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree d. Assume that except for 0, p has no further singular point in $p^{-1}(0)$, that is, on $\mathbb{C}^{n+1}\backslash\{0\}$ there is no solution to

$$p = 0, \quad \frac{\partial p}{\partial z_i} = 0, i = 1, \dots n.$$

Then by the regular value theorem $p^{-1}(0) \subset \mathbb{C}^{n+1}\backslash\{0\}$ is an embedded complex submanifold.

Since p is homogeneous, $z \in p^{-1}(0)$ if and only if $\lambda z \in p^{-1}(0)$ for all $\lambda \neq 0$. That is $p^{-1}(0)$ is preserved by the free and proper action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ and as such $p^{-1}(0)/\mathbb{C}^* \subset \mathbb{C}P^n$ is also a smooth embedded submanifold. ■

We finish this section with a result from smooth manoifolds that does not go through to the holomorphic setting. Whitney's Embedding Theorem states that any compact smooth manifold M^n, can be embedded in $\mathbb{R}^{2n}$. One way to prove this involves choosing partitions of unit which allow to embed M in an $\mathbb{R}^N$ for large N and then use generic projections into subspaces to cut the dimension of the codomain down to $2n$. In the holomorphic world, partitions of unity are a big no-no, but the fact that the technique used fails does not mean the result does not hold and one could still ask the question of whether it is possible to embed complex manifolds in $\mathbb{C}^n$. Unfortunately the answer is no.

Theorem 3.22. *Let M be a compact connected complex manifold and let $f\colon M \to \mathbb{C}^n$ be a holomorphic map. Then f is constant.*

Proof. Let $\Pi_i\colon \mathbb{C}^n \to \mathbb{C}$ be the projection onto the i^{th} coordinate. Then $\Pi_i \circ f\colon M \to \mathbb{C}$ is a continuous function in a compact set, hence it attains a maximum. Since both f and Π_i are holomorphic, so is their composition, which, by the Maximum Principle 2.2, must be constant. Since i is arbitrary, f has constant coordinates, hence is the constant function. □

As an aside, we mention that there is an alternative proof of this fact for immersions that relies on the symplectic structure of $\mathbb{C}^n$ and therefore has an extension to symplectic immersions instead of holomorphic maps.

Theorem 3.23. *Let M be a compact connected complex manifold and let $f\colon M \to \mathbb{C}^n$ be a holomorphic immersion. Then M is a point.*

Proof. Consider the standard symplectic form in $\mathbb{C}^n$:

$$\omega = \frac{1}{2i}\sum dz_i \wedge d\bar{z}_i = \sum dx_i \wedge dy_i = d\left(\sum x_i\, dy_i\right) =: d\alpha$$

Let $f\colon M \to \mathbb{C}$ be a holomorphic immersion and assume by contradiction that $\dim M = 2m > 0$, then, by Exercise 1.6, $df(T_pM)$ is a symplectic vector subspace of $T_{f(p)}\mathbb{C}^n$ for every $p \in M$, which is equivalent to saying that $(f^*\omega)_p$ is compatible with the linear complex structure on T_pM. Then $(f^*\omega)^m$ is a volume form on M, therefore we have

$$0 \neq \int_M (f^*\omega)^m = \int_M (f^*d\alpha)^m = \int_M d(f^*(\alpha \wedge (d\alpha)^{m-1})) = \int_{\delta M} f^*(\alpha \wedge (d\alpha)^{m-1}) = 0.$$

Where above we used that $m > 0$ to write $\omega^m = d\alpha \wedge (d\alpha)^{m-1}$ and Stokes Theorem to obtain a contradiction. □

3.4 Decomposition of forms, $\overline{\partial}$ and further integrability

As we have seen, an almost complex structure, I, on a manifold, M, gives rise to a decomposition of the tangent space into $\pm i$-eigenbundles of I:

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

which in turn decomposes the exterior algebra of TM (the multivector fields) and leads to define $\mathcal{X}^{p,q}(M) := \Gamma(\wedge^p T^{1,0}M \otimes \wedge^q T^{0,1}M)$.

It also induces a dual complex structure I^* on T^*M defined by

$$I^*\alpha(\bullet) = \alpha(I\bullet).$$

And we have a corresponding decomposition

$$T^*_{\mathbb{C}}M = T^{*1,0}M \oplus T^{*0,1}M.$$

where $T^{*1,0}M$ can be described equivalently as either as the $+i$-eigenbundle of I^* or the annihilator of $T^{0,1}M$.

This decomposition of $T^*_{\mathbb{C}}M$ gives rise to a decomposition of the exterior algebra of T^*M, into subbundles, as we saw in Section 1.4:

$$\wedge^k T^*_{\mathbb{C}}M = \oplus_{p+q=k} \wedge^{p,q} T^*M,$$

and we denote the sections of these subbundles by $\Omega^{p,q}(M) = \Gamma(\wedge^{p,q}T^*M)$.

The exterior derivative behaves well with respect to this decomposition.

$$\begin{tikzcd}
\Omega^{0,2} & & \\
 & \Omega^{1,1} & \\
 & \Omega^{1,0} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] \arrow[uul, "\overline{\mathcal{N}}"] & \Omega^{2,0}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^{0,2} & & \\
\Omega^{0,1} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] \arrow[drr, "\mathcal{N}"] & \Omega^{1,1} & \\
 & & \Omega^{2,0}
\end{tikzcd}$$

Figure 3.2: Since $\Omega^2(M;\mathbb{C}) = \Omega^{2,0}(M) \oplus \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$, the stated decomposition of d is automatic for 1-forms.

Lemma 3.24. *Given an almost complex manifold (M, I) the exterior derivative satisfies*

$$d\colon \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p-1,q+2}(M).$$

Proof. We start with 1-forms. Given $\alpha \in \Omega^{1,0}(M)$, the lemma does not say much about $d\alpha$: it is a 2-form and hence has potentially three components all accounted for in the statement. The same applies to $(0,1)$-forms

To prove the statement of the lemma it is enough to consider decomposable forms, since any (p,q)-form can be written as sum of decomposable ones. Given a decomposable (p,q)-form, say $\phi = \alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q$, with $\alpha_i \in \Omega^{1,0}(M)$ and $\beta_i \in \Omega^{0,1}(M)$, then

$$d\phi = \sum_i (-1)^i \alpha_1 \wedge \cdots \wedge d\alpha_i \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q + \sum_i (-1)^{p+i} \alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge d\beta_i \wedge \cdots \wedge \beta_q.$$

The first summand lies in $\Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M) \oplus \Omega^{p-1,q+2}(M)$ while the second summand lies in $\Omega^{p+2,q-1}(M) \oplus \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$, therefore proving the statement. □

Recall that a degree k linear operator in a graded vector space $V^\bullet$ is a linear map $A\colon V \to$ such that $A\colon V^i \to V^{i+k}$. Given an operator A of degree k and an operator B of degree l on $V^\bullet$ the *graded commutator* of these operators is

$$\{A, B\} = AB - (-1)^{kl} BA.$$

The space of forms on a manifold is a graded vector space (graded by degree) and the operators $d, \partial, \overline{\partial}, \mathcal{N}$ and $\overline{\mathcal{N}}$ all have degree 1. The condition $d^2 = 0$ can be rewritten as $\{d, d\} = 0$ and by decomposing d in components we obtain

Corollary 3.25. *The following relations (and their complex conjugate) hold in an almost complex manifold*

$$\{\mathcal{N}, \mathcal{N}\} = 0, \quad \{\mathcal{N}, \partial\} = 0, \quad \partial^2 + \{\overline{\partial}, \mathcal{N}\} = 0, \quad \{\overline{\partial}, \partial\} + \{\overline{\mathcal{N}}, \mathcal{N}\} = 0.$$

Proof. Follow the arrows. The sum of all arrows arriving at a lattice point must vanish since it is one of the components of d^2. □

This lemma allows us to decompose d into four operators

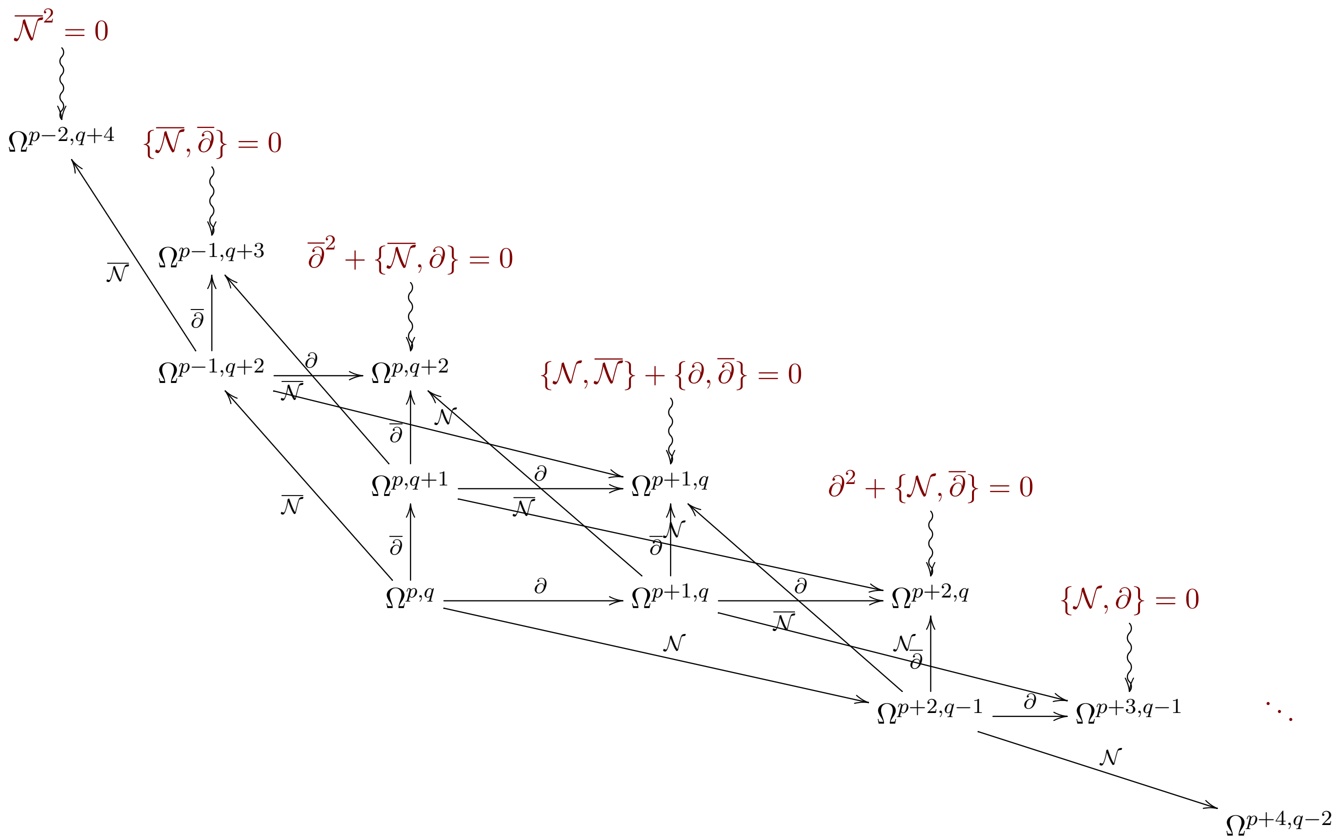

Figure 3.3: The condition $d^2 = 0$ gives rise to a web of relations between the operators $\mathcal{N}$, ∂, $\overline{\partial}$ and $\overline{\mathcal{N}}$.

Definition 3.26. We denote the different components of d acting on (p,q)-forms by $\mathcal{N}$, ∂, $\overline{\partial}$ and $\overline{\mathcal{N}}$:

$$d = \mathcal{N} + \partial + \overline{\partial} + \overline{\mathcal{N}},$$

$$\begin{aligned} \mathcal{N}\colon \Omega^{p,q}(M) \to \Omega^{p+2,q-1}(M), \quad \overline{\mathcal{N}}\colon \Omega^{p,q}(M) \to \Omega^{p-1,q+2}(M),\\ \partial\colon \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \quad \overline{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M). \end{aligned}$$

Since Lie brackets and exterior derivative are closely related, it is no surprise that we can recover the Nijenhuis tensor from the action of d on 1-forms:

Proposition 3.27. *Let (M, I) be an almost complex manifold and $\alpha \in \Omega^{0,1}(M)$. Then*

$$\mathcal{N}(\alpha)(X, Y) = -\alpha(N(X, Y)).$$

In particular I is integrable if and only if

$$d\colon \Omega^{0,1}(M) \to \Omega^{0,2}(M) \oplus \Omega^{1,1}(M).$$

Proof. We compute the $(2,0)$-component of $d\alpha$. For that we let $X, Y \in \mathcal{X}^{1,0}(M)$:

$$(\mathcal{N}\alpha)(X, Y) = (d\alpha)(X, Y) = \mathcal{L}_X \iota_Y \alpha - \mathcal{L}_Y \iota_X \alpha - \alpha([X, Y]) = -\alpha(N(X, Y)) = 0,$$

where in the first equality used that the remaining components of $d\alpha$ vanish when applied to two $(0,1)$-vectors, in the second equality we used the expression for computing the exterior derivative in terms of the Lie bracket of vector fields, in the third equality we used that α is of type $(0,1)$ and both X and Y are of type $(1,0)$, hence the first two summands vanish and the third only depends on the projection of $[X, Y]$ onto $T^{0,1}M$, which is the definition of the Nijenhuis tensor of I. □

Proposition 3.28. *Let (M, I) be an almost complex manifold. Then I is integrable if and only if $\overline{\partial}^2 = 0$.*

Proof. If $\mathcal{N} = 0$, then $\overline{\mathcal{N}} = 0$ and it follows from Corolary 3.25 that

$$2\overline{\partial}^2 = \{\overline{\partial}, \overline{\partial}\} = \{\overline{\partial}, \overline{\partial}\} + \{\partial, \overline{\mathcal{N}}\} = 0.$$

Conversely, assume that $\overline{\partial}^2 = 0$, let $X, Y \in \mathcal{X}^{0,1}(M)$ and $f\colon M \to \mathbb{C}$ be smooth, then

$$\begin{aligned} 0 &= (\overline{\partial}^2 f)(X, Y) = (d\overline{\partial} f)(X, Y)\\ &= \mathcal{L}_X(\overline{\partial} f(Y)) - \mathcal{L}_Y(\overline{\partial} f(X)) - \overline{\partial} f([X, Y])\\ &= \mathcal{L}_X(df(Y)) - \mathcal{L}_Y(df(X)) - df(\Pi^{0,1}[X, Y])\\ &= \mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f - \mathcal{L}_{\Pi^{0,1}[X,Y]} f\\ &= \mathcal{L}_{[X,Y]} f - \mathcal{L}_{\Pi^{0,1}[X,Y]} f\\ &= \mathcal{L}_{\Pi^{1,0}[X,Y]} f = \mathcal{L}_{\overline{N}(X,Y)} f \end{aligned}$$

Since $X, Y \in \mathcal{X}^{0,1}(M)$ are arbitrary we conclude that $\overline{N} = 0$ and hence the Nijenhuis tensor vanishes. □

Exercise 3.10. Let (M^{2n}, I) be an almost complex manifold. Show that for $\alpha \in \Omega^{p,q}(M)$ and $f \in C^\infty(M;\mathbb{C})$, we have

$$\mathcal{N}(f\alpha) = f\mathcal{N}\alpha,$$

that is, $\mathcal{N}$ is tensorial.

Exercise 3.11. Let (M^{2n}, I) be an almost complex manifold. Show that I is integrable if and only if $d\colon \Omega^{n,0}(M) \to \Omega^{n,1}(M)$.

As we just saw, in a complex manifold we have $\overline{\partial}^2 = 0$. This allows us to define a corresponding coholomology:

Definition 3.29. The *Dolbeault cohomology* of a complex manifold, M, are spaces

$$H^{p,q}(M) = \frac{\mathrm{Ker}(\overline{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M))}{\mathrm{Im}(\overline{\partial}\colon \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))},$$

and the *Hodge numbers* of M are their dimensions:

$$h^{p,q}(M) = \dim(H^{p,q}(M)).$$

Definition 3.30. A form $\rho \in \Omega^{p,0}(M)$ is *holomorphic* if its local representative is holomorphic in all (holomorphic) charts of M.

Exercise 3.12. Show that $\overline{\partial}$ satisfies the Leibniz rule:

$$\overline{\partial}(\alpha \wedge \beta) = (\overline{\partial}\alpha) \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \overline{\partial}\beta.$$

Conclude that wedge product of forms induces a product in Dolbeault cohomology

$$H^{p,q}(M) \times H^{p',q'}(M) \to H^{p+p',q+q'}(M).$$

Exercise 3.13 (Integration by parts)**.** Let M^{2n} be a compact almost complex manifold, let $\alpha \in \Omega^k(M;\mathbb{C})$, $\beta \in \Omega^{2n-k-1}(M;\mathbb{C})$ and let $\mathcal{D}$ denote any of the operators ∂, $\overline{\partial}$, $\mathcal{N}$ or $\overline{\mathcal{N}}$. Show that

$$\int_M (\mathcal{D}\alpha) \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge \mathcal{D}\beta.$$

Exercise 3.14. Show that $H^{p,0}(M)$ corresponds to the vector space of (globally defined) holomorphic $(p,0)$-form on M. Conclude that if M is compact and connected $H^{0,0}(M) = \mathbb{C}$.

Exercise 3.15. Show that if M is a compact Riemann surface, $\alpha \in \Omega^{1,0}(M)$ is not identically zero and satisfies $\overline{\partial}\alpha = 0$, then α is d-closed and represents a nontrivial de Rham cohomology class.

Exercise 3.16. Extend the previous result to higher dimensions: Show that if $\dim_{\mathbb{C}} M = n$, $\alpha \in \Omega^{n,0}(M)$ is not idetically zero and satisfies $\overline{\partial}\alpha = 0$, then α is d-closed and represents a nontrivial de Rham cohomology class. Conclude that for all $n \in \mathbb{N}$, $H^{2n+1,0}(\mathbb{C}P^{2n+1}) = \{0\}$.

Exercise 3.17. Compute the Dolbeault cohomology of $\mathbb{C}P^1$.

Dolbeault cohomology of complex manifolds is the first simple invariant one encounters that is directly associated to the presence of complex structures. At this moment we only have a definition which does not lend itself to immediate computations. One of the main objectives of these notes is to develop tools to compute this cohomology in concrete cases. The close relation between $\overline{\partial}$ and the exterior derivative might suggest that there is at least a slight relation between Dolbeault and de Rham cohomology. We will spend some time exploring and exploiting this relation.

A this point it is also good to give a clear warning. Our experience from de Rham cohomology is that to define it one needs a smooth structure on the manifold, but eventually one proves that it is isomorphic to $\mathbb{R}$-valued singular cohomology of the underlying topological space and hence independent of the smooth structure used to define it. For general complex manifolds, Dolbeault cohomology will not have this behaviour and in fact there are examples of different complex structures on a fixed differential manifold for which the Dolbeault cohomologies do not agree.

3.4.1 The d^c-operator.

Finally we mention that there is another differential operator associated to a complex manifold, namely one obtained by twisting d by the complex structure. To give a precise account we observe that there are two related actions of a complex structure on forms. The first is the 'Lie algebra' action:

$$I\colon \wedge^{p,q} T^M \to \wedge^{p,q} T^M, \qquad I \cdot \alpha^{p,q} = (p-q)i\alpha.$$

This can be exponentiated to obtain the corresponding 'Lie group' action:

$$e^{\frac{\pi I}{2}}\colon \wedge^{p,q} T^M \to \wedge^{p,q} T^M, \qquad e^{\frac{\pi I}{2}} \cdot \alpha^{p,q} = i^(p-q)\alpha.$$

The latter is the familiar action

$$e^{\frac{\pi I}{2}} \cdot \alpha(X_1, \dots, X_{p+q}) = \alpha(IX_1, \dots, IX_{p+q}).$$

which sends real forms to real forms.

The twisting of d by the complex structure is then given by

Definition 3.31. Given a complex manifold (M, I) we let

$$d^c = [I, d] = e^{\frac{\pi I}{2}} d e^{-\frac{\pi I}{2}} = i(\partial - \overline{\partial}).$$

Exercise 3.18. Show that $(d^c)^2 = 0$ and that the d^c-cohomology is isomorphic to the de Rham cohomology.

Exercise 3.19. Show that

$$d^c(\alpha \wedge \beta) = d^c(\alpha) \wedge \beta) + (-1)^k \alpha \wedge d^c(\beta),$$

where k is the degree of α.

3.5 Results on integrability summarised

Given an almost complex manifold (M, I), we can define its Nijenhuis tensor $N\colon \wedge^2 T^{0,1}M \to T^{1,0}M$,

$$N(X,Y) = \Pi^{1,0}[X,Y].$$

Also the exterior derivative decomposes $d = \mathcal{N} + \partial + \overline{\partial} + \overline{\mathcal{N}}$ where $\mathcal{N}$ and $\overline{\mathcal{N}}$ are tensorial components determined by the Nijenhuis tensor.

Theorem 3.32. *The following are equivalent*

1. *There are holomorphic coordinate charts for M which induce the almost complex structure I on tangent spaces.*
2. *$T^{1,0}M$ is an involutive subbundle,*
3. *$N = 0$,*
4. *$d = \partial + \overline{\partial}$,*
5. *$\overline{\partial}^2 = 0$,*
6. *$d\colon \Omega^{1,0}(M) \to \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$,*
7. *$d\colon \Omega^{n,0}(M) \to \Omega^{n,1}(M)$.*

Chapter 4

Complex manifolds and algebraic topology

Complex manifolds and holomorphic maps play nicely with a number of concepts from topology. In this short chapter we focus on three such concepts: the degree of a map, Euler characteristic of surfaces and de Rham cohomology.

4.1 Degree

On manifolds there are two flavours of de Rham cohomology: the usual one and the compact support one. While the usual de Rham cohomology is the quotient of the space of closed forms by the space of exact forms, the compact support cohomology relies on the observation that the space of forms with compact support, $\Omega_c(M)$, together with the exterior derivative form a subcomplex of the space of forms

$$\cdots \xrightarrow{d} \Omega_c^{k-1}(M) \xrightarrow{d} \Omega_c^k(M) \xrightarrow{d} \Omega_c^{k+1}(M) \xrightarrow{d} \cdots$$

and we can form the quotient

$$H_c^k(M) = \frac{\ker(\Omega_c^k(M) \xrightarrow{d} \Omega_c^{k+1}(M))}{\mathrm{Im}(\Omega_c^{k-1}(M) \xrightarrow{d} \Omega_c^k(M))}.$$

Of course, for compact manifolds, $\Omega_c^k(M) = \Omega^k(M)$ and therefore $H_c^k(M) = H^k(M)$, so this distinction is only relevant for noncompact manifolds.

In the same way that one can pull back forms via smooth maps, one can pull back a form, ρ with compact support, but for the result to have compact support we need that the inverse image of $\mathrm{supp}(\rho)$ is compact. Requiring this for all forms with compact support leads us to a special class of maps:

Definition 4.1. A continuous map between topological spaces, $f\colon M \to N$, is *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset N$.

An alternative definition for metric spaces is:

Exercise 4.1. A *divergent sequence* is one without converging subsequences. Show that a continuous map between metric spaces is proper if and only if the image of every divergent sequence is a divergent sequences.

If M is compact, every continuous map $f\colon M \to N$ is proper, while if N is compact f is proper if and only if M is compact.

A key fact from algebraic topology is that an n-dimensional, orientable connected manifold M has $H^n_c(M) \cong \mathbb{R}$ and a choice of orientation for M fixes the isomorphism

$$H^n_c(M;\mathbb{R}) \stackrel{\cong}{\to} \mathbb{R}, \qquad [\alpha] \mapsto \int_M \alpha.$$

Therefore any smooth proper map $f\colon M^n \to N^n$ induces a map $f^*\colon H^n_c(N) \to H^n_c(M)$, which, once we fix the isomorphism with $\mathbb{R}$ is just multiplication by a scalar:

Definition 4.2. Let $f\colon M^n \to N^n$ be a smooth and proper map between oriented manifolds, and let $[\rho] \in H^n_c(N)$ be a nontrivial class, then *degree of* f is

$$\deg(f) = \frac{\int_M f^*\rho}{\int_N \rho}.$$

The degree gives a measure of how many times 'M covers N'. So for example if f is not surjective, the answer is none:

Exercise 4.2. Let $f\colon M^n \to N^n$ be a smooth and proper map between oriented manifolds. Show that if f is not surjective, then $\deg(f) = 0$.

Hand-in hand with the definition of degree is the definition of local degree:

Definition 4.3. Let $f\colon M^n \to N^n$ be a smooth and proper map between oriented maniflolds and let $x \in M$. Assume that x and $y = f(x)$ have contractible neighbourhoods, U and V, respectively, such that $f\colon U \to V$ is proper. Then the *local degree of* f *at* x is

$$\deg_x(f) = \frac{\int_U f^*\rho}{\int_V \rho},$$

where $[\rho] \in H^n_c(V)$

There is a clear relation between the degree of f and the sum of its local degrees.

Proposition 4.4. *Let $f\colon M^n \to N^n$ be a smooth and proper map between oriented manifolds . Fix $y \in N$, and let $f^{-1}(y) = \{x_1, \dots, x_k\}$ be its pre-image. Assume that each x_i has a contractible neighbourhood U_i for which the restriction $f|_{U_i}\colon U_i \to V_i$ is proper for some neighbourhood V_i of y, then*

$$\deg(f) = \sum_i \deg_{x_i} f.$$

Proof. Indeed, let $V = \cap V_i$. Shrinking V further we can ensure that if $f(z) \in V$ then $z \in \cup_i U_i$. Indeed, assume by contradiction that this was not the case. Then we would be able to find a sequence $\{y_j\}_{j\in\mathbb{N}}$ converging to y with pre-images $\{z_j\}_{j\in\mathbb{N}}$ such that $z_j \notin \cup_i U_i$. There are two possibilities for the sequence $\{z_j\}_{j\in\mathbb{N}}$: it has a convergent subsequence or diverges. If it had a converging subsequence (which we still denote by $\{z_j\}_{j\in\mathbb{N}}$, say with limit z), then, $z \notin \cup_i U_i$ and by continuity of f, $f(z) = y$, which is a contradiction. Therefore $\{z_j\}_{j\in\mathbb{N}}$ is a divergent sequence, but by construction $f(z_j)$ converges to y, which is a contradiction to f being proper.

Now, let ρ be an n-form with compact support on V. Then $f^*\rho$ is a form with compact support in $\cup_i U_i$ and

$$\deg(f) = \frac{\sum_j \int_{U_j} f^*\rho}{\int_V \rho} = \sum_i \deg_{x_j} f.$$

$\square$

If we restrict ourselves to regular values of f, the degree has a geometric interpretation as the number of pre-images counted with signs.

Corollary 4.5. *Let $f\colon M^n \to N^n$ be a smooth and proper map between oriented manifolds and let $y \in N$ be a regular value of f with pre-image $f^{-1}(y) = \{x_1, \dots, x_k\}$. Define the sign of f at x_i to be $\epsilon(f; x_i) = \pm 1$ according to whether $df_{x_i}\colon T_{x_i}M \to T_yN$ is orientation preserving or reversing.*

$$\deg(f) = \sum_i \epsilon(f; x_i).$$

Proof. Indeed, due to the inverse function theorem f is a diffeomorphism between a neighbourhood of x_i and a neighbourhood of y, so we can compute its local degree and it is ± 1 depending precisely on whether f is orientation preserving or reversing in that neighbourhood, a fact that can be checked using $df|_{x_i}$. $\square$

With the basics of degree out of the way, we can make the relevant observation about holomorphic maps:

Theorem 4.6. *Let $f\colon M \to N$ be a proper holomorphic map between connected complex manifolds of the same dimension. Then for a regular value $y \in N$, $\deg(f) = \#f^{-1}(y)$. In particular, if df is an isomorphism at one point, then its degree is positive and f is surjective.*

Proof. Indeed, if y is a regular value and $x \in f^{-1}(y)$, then $df\colon T_xM \to T_yN$ is complex linear and therefore preserves orientations, so $\epsilon(f; x) = 1$. It follows from Corollary 4.5 that $\deg(f) = \#f^{-1}(y)$.

Further, if there is x for which $df|_x$ is an isomorphism, then by the inverse function theorem there is a neighbourhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. By Sard's theorem, there is a regular value $y \in f(U)$ and $f^{-1}(y)$ contains at least one point, so $\deg(f) > 0$ and f is surjective by Exercise 4.2. $\square$

Degrees and local degrees are particularly useful for maps between Riemann surfaces.

Lemma 4.7. *Let $f\colon U \subset \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic and let $p \in U$. Let k be the natural number defined by the condition*

$$f'(p) = \cdots = f^{(k-1)}(p) = 0, \qquad f^{(k)}(p) \neq 0.$$

Then there is a biholomorphism $\psi\colon U \to \mathbb{C}$ defined in a neighbourhood of p such that

$$f \circ \psi(p+z) = f(p) + z^k.$$

and in particular $\deg_p(f) = k$.

Proof. Without loss of generality we may assume that $p = f(p) = 0$. Since the first $k-1$ derivatives of f vanish, we can expand f in power series as

$$f(z) = z^k(\sum_{j\geq k} a_j z^{j-k}) = z^k h(z),$$

where h is defined by the equality above and satisfies $h(0) \neq 0$. Picking a branch of the k-th root we can find a holomorphic function $\tilde{h}$ defined in the neighbourhood of 0 such that $(\tilde{h}(z))^k = h(z)$, hence we have

$$f(z) = (z\tilde{h}(z))^k.$$

Since $\tilde{h}(0) \neq 0$, we see that the function $z \mapsto z\tilde{h}(z)$ has nonzero derivative at 0 and hence has an inverse defined in a neighbourhood of 0, $\psi\colon U \to V$, $\psi(z\tilde{h}(z)) = z$. For this map we have

$$(f \circ \psi)(\psi^{-1}(z)) = f(z) = (z\tilde{h}(z))^k = (\psi^{-1}(z))^k,$$

that is, $f \circ \psi(w) = w^k$.

Finally, the function $f(z) = z^k$ maps the open disc of radius r properly to the open disc of radius r^k and we can compute its degree by counting the number of points in the inverse image of any regular value, which is k, that is, $\deg_0 f = k$. □

Because of the local behaviour of the function $z \mapsto z^k$, the local degree of a map f between Riemann surfaces at point p is also called the *branching number of f at p*.

Corollary 4.8. *Let $f\colon \Sigma_0 \to \Sigma_1$ be a proper holomorphic map of degree one between Riemann surfaces, then f is a biholomorphism.*

Proof. Since f has degree 1 it is surjective (by Exercise 4.2). Further, if f had a singular point, say, x, by Lemma 4.7, $\deg_x f > 1$ and hence $\deg f > 1$ which is a contradiction, so f has no singular points. Finally, since the degree counts the number of pre-images of any given regular value and $\deg(f) = 1$, we conclude that f is injective. That is f is a bijection with invertible derivative, therefore a biholomorphism. □

4.2 Euler characteristic

There are several equivalent ways to define the Euler characteristic of a compact manifold. Probably the most geometric version consists of taking a triangulation of the manifold and doing the alternating sum of the number of simplices of each dimension used in the triangulation. For surfaces, 0-dimensional simplices are called vertices, 1-dimensional simplices are called edges and 2-dimensional simplices are called, well, faces and the number of each of them used in a triangulation is typically denoted by V, E and F, respectively. So the Euler characteristic of a triangulated Riemann surface, Σ, is

$$\chi_\Sigma = V - E + F.$$

This number is independent of the triangulation and is also equal to the alternating sum of the Betti numbers of Σ.

Euler characteristic, local and total degree of a map come together nicely in the Riemann–Hurwicz formula.

Theorem 4.9 (Riemann–Hurwicz). *Let $f\colon \Sigma_0 \to \Sigma_1$ be a non constant holomorphic map between compact connected Riemann surfaces. Then*

$$\chi_{\Sigma_0} = (\deg f)\chi_{\Sigma_1} - \sum_{x\colon f'(x)=0} (\deg_x f - 1).$$

The quantity

$$\sum_{x\colon f'(x)=0} (\deg_x f - 1)$$

is also referred to as the *total branch number of f.*

Sketch of the proof. To prove the theorem one shows that there is a triangulation of Σ_1 with the following properties:

- all the critical values of f are vertices of the triangulation,
- taking the inverse image of the triangulation gives a triangulation of Σ_0.

Counting faces, vertices and edges, we conclude that each edge and each face on Σ_1 gives rise to $\deg(f)$ edges and faces on Σ_0: $E(\Sigma_0) = \deg(f)E(\Sigma_1)$, $F(\Sigma_0) = \deg(f)F(\Sigma_1)$. For 'generic' vertices the same holds, but if a vertex y is a singular value, then $f^{-1}(y)$ will have $\deg(f) - \sum_{x\in f^{-1}(y)}(\deg_x f - 1)$ points because each singular point x counts only once as a vertex but has a contribution $\deg_x f$ for the computation of the degree. Hence

$$\begin{aligned}\chi_{\Sigma_0} &= \deg(f)V(\Sigma_1) - \sum_{x\colon f'(x)=0} (\deg_x f - 1) - \deg(f)F(\Sigma_1) + \deg(f)F(\Sigma_1) \\ &= (\deg f)\chi_{\Sigma_1} - \sum_{x\colon f'(x)=0} (\deg_x f - 1).\end{aligned}$$

□

This result restricts the existence of holomorphic maps between Riemann surfaces.

Exercise 4.3. Denoting by Σ_g the orientable surface of genus g (with an arbitrary complex structure), show that there is no nonconstant holomorphic map $f\colon \mathbb{C}P^1 \to \Sigma_g$, for $g > 0$.

Exercise 4.4. Denoting by Σ_g the orientable surface of genus g (with an arbitrary complex structure), show that there is no nonconstant holomorphic map $f\colon \Sigma_g \to \Sigma_h$, for $h > g > 0$.

4.3 Frölicher spectral sequence

The last topic we treat is a first attempt to relate Dolbeault cohomology to de Rham cohomology. Since at the moment we are armed only with the definition of Dolbeault cohomology, the decomposition of forms $\oplus_k \Omega^k(M;\mathbb{C}) = \oplus_{p,q}\Omega^{p,q}(M)$ and the corresponding decomposition of the exterior derivative, $d = \partial + \overline{\partial}$, we will just have to magic something out of very few ingredients.

The remark that gets us kick started is that if we are given a form $\rho \in \Omega^k(M;\mathbb{C})$ and let q_0 be the highest value of q for which its (p,q) component $\rho^{p,q}$ is nonzero, then one of the conditions that is necessary for ρ to be closed is $\overline{\partial}\rho^{k-q_0,q_0} = 0$. Further, if we change ρ by an exact form, $d\tau$, such that $q_0 - 1$ is the highest value of q for which $\tau^{q,k-q-1}$ is nonzero, then the Dolbeault cohomology class of $\rho^{q_0,k-q_0}$ does not change (see Figure 4.1). In a way, this is saying that the Dolbeault cohomology is a coarse first approximation to the de Rham cohomology.

$$\begin{tikzcd}
0 & & & \\
0 \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & 0 & & \\
0 \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \rho^{k-q_0,q_0} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & ? & \\
& \tau^{k-q_0,q_0-1} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \rho^{k-q_0+1,q_0-1} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & ? \\
& & \tau^{k-q_0+1,q_0-2} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \ddots \arrow[u, "\overline{\partial}"]
\end{tikzcd}$$

Figure 4.1: A first approximation to ρ being closed is $\overline{\partial}\rho^{k-q_0,q_0} = 0$, while a first approximation to ρ being exact if $\rho^{k-q_0,q_0} = \overline{\partial}\tau^{p-q_0,q_0-1}$.

The idea next is to improve this coarse approximation in two ways: we consider the "first two" nonvanishing components of ρ and when changing by exact elements we allow τ to have one more component (see Figure 4.4). Then we notice that the questions leading to whether ρ is closed/exact (at least if if ignore terms in $d\rho$ with $q < q_0$) do not depend on ρ itself, but on its Dolbeault cohomology class of ρ^{k-q_0,q_0} and the cohomology ∂ induces in $H_{\overline{\partial}}(M)$. Explicitly, we need that $\partial\rho^{k-q_0,q_0} = -\overline{\partial}\rho^{k-q_0+1,q_0-1}$, or , in Dolbeault cohomology, $\partial[\rho^{k-q_0,q_0}] = 0$.

As we successively increase the number of components of ρ and τ that we consider we are eventually left with the de Rham cohomology of M. An interesting aspect of this construction is that each step can be phrased in terms of data present in the previous step, therefore creating a "sequence of cohomologies" that starts with the Dolbeault cohomology and shrinks progressively until it *converges* to the de Rham cohomology after n steps, where $n = \dim_{\mathbb{C}} M$.

It turns out that the algebraic situation we just described using Dolbeault and de Rham cohomologies pops up in different contexts with some frequency, so before we delve into the precise steps outlined above to relate these two cohomologies, we present the general framework which they represent.

$$\begin{tikzcd}
0 & & & \\
0 \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & 0 & & \\
\tau^{k-q_0-1,q_0} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \rho^{k-q_0,q_0} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & 0 & \\
& \tau^{k-q_0,q_0-1} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \rho^{k-q_0+1,q_0-1} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & ? \\
& & \tau^{k-q_0+1,q_0-2} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \ddots \arrow[u, "\overline{\partial}"]
\end{tikzcd}$$

Figure 4.2: A a second approximation to ρ being closed/exact is that the Dolbeault cohomology class $[\rho^{k-q_0,q_0}]$ is closed/exact for the operator induced by ∂ in Dolbeault cohomology.

The first step is to introduce the basic spaces and differentials.

Definition 4.10.

- A *differential double complex of vector spaces* is a collection of vector spaces $\{E_0^{p,q}\}_{p,q\in\mathbb{N}}$ together with linear maps $d_0\colon E_0^{p,q} \to E_0^{p,q+1}$ and $d_1\colon E_0^{p,q} \to E_0^{p+1,q}$ such that $d_0^2 = d_0^2 = [d_0, d_1] = 0$.
- The *total grading* and the *total differential* of $(E_0^{p,q})_{p,q\in\mathbb{N}}$ are given by

$$E^k = \oplus_{p+q=k} E_0^{p,q}, \qquad d = d_0 + d_1.$$

- In this situation, $d\colon E^k \to E^{k+1}$ and $d^2 = 0$, and the *total cohomology* of $\{E_0^{p,q}\}_{p,q\in\mathbb{N}}$ is the cohomology of the complex $(E^\bullet, d)$.

Remark. Because we are following the analogy with Dolbeault cohomology, we will use vector spaces, but the whole theory works with a few changes for R-modules or even just Abelian groups with the differentials being module/group morphisms. The more algebraic structure we have at the start, the more structure we will have at the end.

It should be readily recognizable that the total grading plays the role of the natural grading of forms by degree and the (p,q)-grading plays the role of the bigrading induced by the complex structure. The objective here is to relate the d_0-cohomology to the d-cohomology. The answer will be rather elaborate and it comes in the form of a spectral sequence.

Definition 4.11. A (first quadrant) *spectral sequence of vector spaces* is a collection of complexes of vector spaces $\{(E_r^{p,q}, d_r)\}_{p,q,r\in\mathbb{N}}$,

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1},$$

such that $E_{r+1}^{p,q}$ is the cohomology of d_r in degree p,q.

For each fixed r, the complex $(E_r^{p,q}, d_r)$ is called the r^{th} page of the spectral sequence. It is useful to think of a spectral sequence as a book whose pages we turn as we increase the value of r. Figures 4.3, 4.4 and 4.5 show the 0^{th}, 1^{st} and 2^{nd} pages of a spectral sequence. The entries of the next page are the cohomology of the previous page and the new differential points one column further to the right and one row further down.

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots & E_0^{p-1,q} \arrow[u, "d_0"] & E_0^{p,q} \arrow[u, "d_0"] & \cdots \\
\cdots & E_0^{p-1,q-1} \arrow[u, "d_0"] & E_0^{p,q-1} \arrow[u, "d_0"] & \cdots \\
 & \vdots \arrow[u, "d_0"] & \vdots \arrow[u, "d_0"] &
\end{tikzcd}
$$

Figure 4.3: Page 0: The original complex with differential d_0.

$$
\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, "d_1"] & E_1^{p-1,q} \arrow[r, "d_1"] & E_1^{p,q} \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r, "d_1"] & E_1^{p-1,q-1} \arrow[r, "d_1"] & E_1^{p,q-1} \arrow[r, "d_1"] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}
$$

Figure 4.4: Page 1: The d_0-cohomolgy with differential induced by d_1.

Because we require $p, q \geq 0$ we see that for $r > p + q + 1$ the arrows arriving and leaving $E_r^{p,q}$ come from and point to trivial spaces, that is the process of taking cohomology has stopped and the space $E_r^{p,q}$ does not change anymore.

Definition 4.12. A spectral sequence $\{(E_r^{p,q}, d_r)\}_{p,q,r\in\mathbb{N}}$ *converges* to a bigraded space $\{E_\infty^{p,q}\}_{p,q\in\mathbb{N}}$, if $E_r^{p,q} \cong E_\infty^{p,q}$ for all r large enough. Alternatively, we say that $\{E_\infty^{p,q}\}_{p,q\in\mathbb{N}}$ is the *limit* of the spectral sequence.

We just argued that every spectral sequence converges to a bigraded space (and here it was important the indices p and q were bounded from below). Keeping an eye on the prize, de Rham cohomology or the total cohomology of a double complex do not inherit a double grading so they

$$
\begin{tikzcd}
 & \vdots \arrow[drr, "d_2"] & \vdots \arrow[drr, "d_2"] & \vdots & & \\
\cdots \arrow[drr, "d_2"] & E_2^{p-1,q} \arrow[drr, "d_2"] & E_2^{p,q} \arrow[drr, "d_2"] & E_2^{p+1,q} \arrow[drr, "d_2"] & \cdots & \\
\cdots & E_2^{p-1,q-1} \arrow[drr, "d_2"] & E_2^{p,q-1} \arrow[drr, "d_2"] & E_2^{p+1,q-1} \arrow[drr, "d_2"] & \cdots & \\
 & \vdots & \vdots & \vdots & &
\end{tikzcd}
$$

Figure 4.5: Page 2: The cohomology induced by d_1 in the d_0-cohomology with a new differential.

cannot be the limit of a spectral sequence as defined above. The notion that allows us to phrase in which sense the limit of a spectral sequence is related to the total cohomology of the double complex is that of a filtration.

Definition 4.13.

- A *(decreasing) filtration of a (graded) vector space*, $H^\bullet$, is a collection of subspaces $\mathcal{F}^p H^\bullet$ such that
$$\dots \mathcal{F}^{p+1} H^\bullet \subset \mathcal{F}^p H^\bullet \subset \mathcal{F}^{p-1} H^\bullet \subset \dots$$
- A filtration of a differential complex $(E^\bullet, d)$ is a filtration of $E^\bullet$ such that $d\colon \mathcal{F}^p E^\bullet \to \mathcal{F}^p E^\bullet$ for all p.
- A filtration is *bounded* if there are p_0 and P_0 such that $\mathcal{F}^{p_0} H^\bullet = \{0\}$ and $\mathcal{F}^{P_0} H^\bullet = H^\bullet$.

Example 4.14. The complex of forms on a complex manifold has a filtration, namely $\mathcal{F}^p \Omega^\bullet(M;\mathbb{C}) = \sum_{p' \geq p; q \in \mathbb{N}} \Omega^{p',q}(M;\mathbb{C})$. Similarly the de Rham cohomology inherits a filtration:

$$\mathcal{F}^p H^\bullet = \{a \in H^\bullet \colon \exists \alpha \in \mathcal{F}^p \Omega^\bullet(M;\mathbb{C}) \text{ such that } [\alpha] = a\}.$$

Similarly a double complex of vector spaces, $\{E_0^{p,q}, d_0, d_1\}_{p,q\in\mathbb{N}}$, admits a filtration $\mathcal{F}^p E_0^{\bullet,\bullet}(M;\mathbb{C}) = \sum_{p' \geq p, q \in \mathbb{N}} E^{p',q}(M;\mathbb{C})$ and so does its total cohomology

$$\mathcal{F}^p H^\bullet = \{a \in H^\bullet \colon \exists \alpha \in \mathcal{F}^p E_0^{\bullet,\bullet}(M;\mathbb{C}) \text{ such that } [\alpha] = a\}.$$

■

Definition 4.15. A spectral sequence $\{(E_r^{p,q}, d_r)\}_{p,q,r\in\mathbb{N}}$ *converges* to filtered graded space $H^\bullet$ if $\mathcal{F}^p H^k / \mathcal{F}^{p+1} H^k \cong E_\infty^{p,k-p}$ for all p and k.

Remark. Since we are dealing with vector spaces, this notion of convergence is equivalent to $H^k \cong \oplus_p E_\infty^{p,k-p}$, but for R-modules or Abelian groups this last conclusion may not hold.

Now we have introduced all the algebraic gadgetry to be able to state the main theorem of this session:

Theorem 4.16. *Let $\{E_0^{p,q}, d_0, d_1\}_{p,q\in\mathbb{N}}$ be a differential double complex. Then there is a spectral sequence $\{(E_r^{p,q}, d_r)\}_{p,q,r\in\mathbb{N}}$ such that*

- *first page is the d_0-cohomology of $\{E_0^{p,q}\}_{p,q\in\mathbb{N}}$ with differential induced by d_1 and*
- *the sequence converges to the total cohomology of $\{E_0^{p,q}, d\}_{p,q\in\mathbb{N}}$.*

Proof. We consider the filtration of $E^\bullet$ from Example 4.14:

$$\mathcal{F}^p E^k = \sum_{p'\geq p} E^{p',k-p'}.$$

So

$$\{0\} = \mathcal{F}^{k+1}E^k \subset \mathcal{F}^{k-1}E^k \subset \cdots \subset \mathcal{F}^0 E^k = E^k.$$

Further $d\colon \mathcal{F}^p E \to \mathcal{F}^p E$, that is, d preserves this filtration. Next we try to compute the cohomology of d, step by step by ignoring certain terms that appear in $d\rho$ and restricting the forms we allow as primitives using the filtration introduce above.

To be precise, for $r\in\mathbb{N}$, we define

$$\begin{aligned}
Z_r^{p,q} &= \{\rho \in \mathcal{F}^p E^{p+q}\colon d\rho \in \mathcal{F}^{p+r}E^{p+q+1}\}\\
B_r^{p,q} &= \{d\rho \in \mathcal{F}^p E^{p+q}\colon \rho \in \mathcal{F}^{p-r}E^{p+q-1}\}\\
Z_\infty^{p,q} &= \{\rho \in \mathcal{F}^p E^{p+q}\colon d\rho = 0\}\\
B_\infty^{p,q} &= \{d\rho \in \mathcal{F}^p E^{p+q}\colon \rho \in E^{p+q-1}\}
\end{aligned}$$

In words, $Z_r^{p,q}$ contains elements that are closed "up to higher terms in the filtration" (the index r determines how high) and $B_r^{p,q}$ are exact elements with primitives "not too far down the fitration" (again the index r determines how low). So, for example $B_0^{p,q} = d(\mathcal{F}^p E^{p+q-1})$ and $Z_0^{p,q} = \mathcal{F}^p E^{p+q}$.

The spaces $Z_\bullet^{\bullet,\bullet}$ and $B_\bullet^{\bullet,\bullet}$ are nested:

$$B_0^{p,q} \subset B_1^{p,q} \subset \cdots \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset \ldots Z_1^{p,q} \subset Z_0^{p,q}.$$

Further, from the definition of the filtration of cohomology, we have that

$$\mathcal{F}^p H^k = \frac{Z_\infty^{p,k-p}}{B_\infty^{p,k-p}} \quad \text{and}$$

$$\frac{\mathcal{F}^p H^k}{\mathcal{F}^{p+1}H^k} = \frac{Z_\infty^{p,k-p}}{Z_\infty^{p+1,k-p-1} + B_\infty^{p,k-p}}. \tag{4.3.1}$$

We define $E_r^{p,q}$ as

$$E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}} \tag{4.3.2}$$

and to define d_r we observe that d induces a map $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ because, from the definition of $Z_r^{p,q}$ and $B_r^{p,q}$,

$$d(Z_r^{p,q}) \subset B^{p+r,q-r+1} \subset Z^{p+r,q-r+1} \qquad \text{and}$$

$$d(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) = d(Z_{r-1}^{p+1,q-1}) + \{0\} \subset B_{r-1}^{p+r,q-r+1} \subset Z_{r-1}^{p+r+1,q-r+1} + B_{r-1}^{p+r,q-r+1}.$$

So we let d_r be the map induced by d

$$d_r \colon E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}} \to \frac{Z_r^{p+r,q-r+1}}{Z_{r-1}^{p+r,q-r+1} + B_{r-1}^{p+r,q-r+1}} = E_r^{p+r,q-r+1}$$

From (4.3.1) and the definition (4.3.2) we have

$$\frac{\mathcal{F}^p H^k}{\mathcal{F}^{p+1} H^k} = E_\infty^{p,q},$$

which means that whatever we are constructing converges to the total cohomology.

Also, since d_r is induced by d, it is clear that $d_r^2 = 0$. So to prove the theorem we need to check three facts:

1. $E_1^{p,q}$ is the d_0-cohomology in degree (p,q),
2. $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ is induced by the differential d_1 of the original double complex,
3. $E_{r+1}^{p,q}$ is the cohomology of d_r at $E_r^{p,q}$.

Fact 1. Follows directly from the definition of $E_1^{p,q}$. Indeed, letting $k = p+q$, the numerator of $E_1^{p,q}$ is

$$Z_1^{p,q} = \{\sum_{p' \geq p} \rho^{p',k-p'} : d_0\rho^{p,q} = 0\} = \ker(d_0 \colon E_0^{p,q} \to E_0^{p,q+1}) \oplus_{p'>p} E_0^{p',k-p'}$$

and the denominator is

$$Z_0^{p+1,q-1} + B_0^{p,q} = \oplus_{p'>p} E_0^{p',k-p'} + d_0(E_0^{p,q-1})\},$$

so the quotient is precisely the d_0-cohomology at level (p,q).

Fact 2. Following the recipe above, to obtain the 'spectral' d_1 we just need to apply d to a d_0-closed form and take the d_0-cohomology class of the result, which is precisely how d_1 from the double complex induces a map on the d_0-cohomology (because of the commuting relation $d_0 d_1 = d_1 d_0$).

Fact 3. Is again a direct computation, but because of all the indices, it is much less intuitive.

□

This is not a powerful theorem. More often than not, the point of view is that the d_0-cohomology is something computable and the total cohomology is something desirable. In the best situations we can compute the first or second page of the spectral sequence of a double complex and use the general shape of those pages (for example, the existence of very few nontrivial spaces), to conclude something about the final page. This is referred to as lacunary phenomena.

Exercise 4.5. Show that if $H_{d_0}^{p,q} = \{0\}$ for $p+q = 1 \mod 2$, then the d_0-cohomology is isomorphic to the total cohomology of the double complex.

Of course at the moment our main problem when it comes to relating Dolbeault to de Rham cohomology is that the first page (Dolbeault cohomology) is the unknown one. But there is a couple of pieces of information we can still get from the spectral sequence.

Proposition 4.17. *In a complex manifold M we have*

$$b^k \leq \sum_{p=0}^{k} h^{p,k-p},$$

where $b^k = \dim H^k(M)$ is the k^{th}-Betti number of M. Further, if the Dolbeault cohomology of M is finite dimensional we have

$$\chi_M = \sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q},$$

where χ_M is the Euler characteristic of M.

Proof. In the process of turning the pages in a spectral sequence, at each step we take the kernel of an operator and then divide it by the image of another operator. Neither of these operations increases dimensions, hence $\dim E_{r+1}^{p,q} \leq \dim E_r^{p,q}$. Therefore,

$$b^k = \dim H^k(M) = \sum_{p=0}^{k} \dim E_\infty^{p,k-p} \leq \sum_{p=0}^{k} \dim E_1^{p,k-p} = \sum_{p=0}^{k} h^{p,k-p}.$$

The second claim follows from the fact that if

$$0 \to V^0 \overset{d_0}{\to} V^1 \overset{d_1}{\to} \dots \overset{d_{n-1}}{\to} V^n \to \{0\}$$

is a complex of vector spaces, that is, each map d is linear and $d_i \circ d_{i-1} = 0$, then we can define the cohomology, $H^\bullet$, of this sequence. If each V^i is finite dimensional, then

$$\sum i = 0^n \dim V^i = \sum i = 0^n \dim H^i.$$

Indeed, by the rank nullity theorem we have that $\dim V_i = \dim \ker(d_i) + \dim \mathrm{Im}(d_i)$, while by definition $\dim H^i = \dim \ker(d_i) 0 \dim \mathrm{Im}(d_{i-1})$. Writing down the sums under consideration we have

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \dim H^i &= \sum_{i=0}^{n} (-1)^i \dim \ker(d_i) - (-1)^i \dim \mathrm{Im}(d_{i-1}) \\
&= \sum_{i=0}^{n} (-1)^i \dim \ker(d_i) + (-1)^{i-1} \dim \mathrm{Im}(d_{i-1}) \\
&= \sum_{i=0}^{n} (-1)^i \dim \ker(d_i) + (-1)^i \dim \mathrm{Im}(d_i) \\
&= \sum_{i=0}^{n} (-1)^i \dim V_i
\end{aligned}$$

Since at each turn of page of a spectral sequence we take the cohomology of the previous page, the alternating sum of the dimensions of the spaces remains constant and can be computed from the first page whose spaces are all finite dimensional. $\square$

Chapter 5

Vector bundles – part I

5.1 Smooth vector bundles

A special type of manifold that appear frequently in many different situations are vector bundles. We will spend some time and energy studying those and in this section we introduce them, their basic properties and constructions.

Let us start by recalling the definition of smooth vector bundle.

Definition 5.1. A *real vector bundle* of rank k over a manifold M is a manifold E together with a submersion $\pi\colon E \to M$ such that

- the level sets $E_p = \pi^{-1}(p)$ have the structure of a k-dimensional real vector space,
- for every point $p \in M$ there is a neighbourhood U of p and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ for which the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"] & U \times \mathbb{R}^k \arrow[d, "\pi_1"] \\ U \arrow[r, "\mathrm{Id}"] & U, \end{tikzcd} \tag{5.1.1}$$

  where π_1 is projection onto the first factor and

- $\Phi|_{E_q}\colon E_q \to \mathbb{R}^k \times \{q\}$ is a linear isomorphism for every $q \in U$.

The level set E_p is also called the *fiber over* p and any such map Φ as above is called a *local trivialisation*. We have a similar notion of complex vector bundle over a smooth manifold if we replace the vector space $\mathbb{R}^k$ by the vector space $\mathbb{C}^k$ and require the local trivialisation to be fiberwise complex linear. From our discussion in Chapter 1, it should be clear that a complex vector bundle of rank k is the same thing as a real vector bundle of rank $2k$, $E \to M$ with a smooth fiberwise complex structure $I\colon E_p \to E_p$, $I^2 = -\mathrm{Id}$, for all p.

From this point of view, an almost complex structure is a way to turn the tangent bundle (which is a real vector bundle) into a complex vector bundle.

Vector bundles with trivialisations can be equivalently described by their *transition functions.* Namely, if we have two trivialisations, $\Phi_i\colon \pi^{-1}U_i \to U_i \times \mathbb{C}^k$ for $i = 0, 1$, in the overlap $U_{0,1} = U_0 \cap U_1$ we can compare the two trivialisations:

$$\begin{tikzcd} U_{0,1} \times \mathbb{C}^k \arrow[d, "\pi_1"] & \pi^{-1}(U_{0,1}) \arrow[l, "\Phi_0"'] \arrow[r, "\Phi_1"] \arrow[d, "\pi"] & U_{0,1} \times \mathbb{C}^k \arrow[d, "\pi_1"] \\ U_{0,1} \arrow[r, leftrightarrow, "\mathrm{Id}"] & U_{0,1} \arrow[r, leftrightarrow, "\mathrm{Id}"] & U_{0,1}, \end{tikzcd} \tag{5.1.2}$$

Because the diagram above commutes, there is a map $g_1^0\colon U_{0,1} \to \mathrm{GL}\,(k;\mathbb{C})$ such that

$$\Phi_1 \circ \Phi_0^{-1}(p, v) = (p, g_1^0(p)v)$$

The map g_1^0 is the *transition function* from the trivialisation Φ_0 to the trivialisation Φ_1.

If we use a collection of trivialisations to identify fibers overs different open sets and eventually cycle back to the first trivialisation, as in

$$(\Phi_{\alpha_0} \circ \Phi_{\alpha_n}^{-1}) \circ (\Phi_{\alpha_n} \circ \Phi_{\alpha_{n-1}}^{-1}) \circ \cdots \circ (\Phi_{\alpha_1} \circ \Phi_{\alpha_0}^{-1}),$$

then cancelling the middle terms, $\Phi_{\alpha_i}^{-1} \circ \Phi_{\alpha_i}$, we see that we obtain the identity map. This translates to the fact that transition functions satisfy

$$g_\alpha^\alpha = \mathrm{Id}, \qquad g_{\alpha_0}^{\alpha_n} \cdot g_{\alpha_n}^{\alpha_{n-1}} \cdot \cdots \cdot g_{\alpha_1}^{\alpha_0} = \mathrm{Id}, \tag{5.1.3}$$

where the $\cdot$ indicates matrix multiplication. Notice that the second identity in (**??**) follows from the one that involves only three transition functions

$$g_\alpha^\alpha = \mathrm{Id}, \qquad g_{\alpha_0}^{\alpha_2} \cdot g_{\alpha_2}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0} = \mathrm{Id}, \tag{5.1.4}$$

for any choice of three overlapping trivialisations. Indeed, taking $\alpha_2 = \alpha_1$ in the triple overlap we obtain that

$$g_{\alpha_0}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0} = g_{\alpha_0}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0} = \mathrm{Id}$$

and, for example, for four overlapping trivialisations we can just write

$$\begin{aligned} g_{\alpha_0}^{\alpha_3} \cdot g_{\alpha_3}^{\alpha_2} \cdot g_{\alpha_2}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0} &= g_{\alpha_0}^{\alpha_3} \cdot g_{\alpha_3}^{\alpha_2} \cdot (g_{\alpha_2}^{\alpha_0} \cdot g_{\alpha_0}^{\alpha_2}) \cdot g_{\alpha_2}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0} \\ &= (g_{\alpha_0}^{\alpha_3} \cdot g_{\alpha_3}^{\alpha_2} \cdot g_{\alpha_2}^{\alpha_0}) \cdot (g_{\alpha_0}^{\alpha_2} \cdot g_{\alpha_2}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0}) \\ &= \mathrm{Id} \cdot \mathrm{Id} \end{aligned}$$

We summarise this discussion in the following lemma:

Lemma 5.2. *Given a vector bundle $E \to M$, an open cover $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ of M and a collection of trivialisations of E over each U_α, $\Phi_\alpha\colon E|_{U_\alpha} \to U\alpha \times \mathbb{C}^k$, the transition functions related to these trivialisations satisfy*

$$g_\alpha^\alpha = \mathrm{Id}, \qquad g_{\alpha_0}^{\alpha_2} \cdot g_{\alpha_2}^{\alpha_1} \cdot g_{\alpha_1}^{\alpha_0} = \mathrm{Id}, \tag{5.1.5}$$

The first condition in (5.1.5) is referred to as *skew-symmetry* and the second is the *cocycle condition.*

The converse to this result also holds.

Lemma 5.3. *Given a manifold M, an open cover $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ of M and a collection of functions defined on double overlaps*

$$\check{g} = \{g^\alpha_\beta \colon U_{\alpha\beta} \to \mathrm{GL}\,(k;\mathbb{C})\},$$

satisfying (5.1.5)*, there is a vector bundle $E \to M$ with trivialisations Φ_α over U_α for which the transition functions are the functions in the family $\check{g}$.*

Proof. Define E to be the quotient space

$$E = \dot\cup_{\alpha\in A} U_\alpha \times \mathbb{C}^k / \cong,$$

where $(x,v) \in U_\alpha \times \mathbb{C}^k$ is declared to be equivalent to $(y,w) \in U_\beta \times \mathbb{C}^k$ if $x = y$ and $w = g^\alpha_\beta(x)v$. The projections onto the first factor $U_\alpha \times \mathbb{C}^k \to U_\alpha \subset M$ patch together to give rise to a map $\pi \colon E \to M$.

The cocycle condition implies that if $(x,v),(y,w) \in U_\alpha \times \mathbb{C}^k$ are equivalent, then $x = y$ and $v = w$. That is, we have natural inclusions $\Phi_\alpha \colon U_\alpha \times \mathbb{C}^k \to E$, which not only express the level sets of $\pi \colon E \to M$ as vector spaces but also provide local trivialisations of E, making it into a vector bundle over M. Finally, by construction the transition function obtained by comparing the trivialisations Φ_α and Φ_β are precisely the ones used to define the equivalence relation, that is g^α_β. □

Using descriptions of bundles via transition functions we see that several constructions of vector spaces associated to a given one, such as taking duals, tensor products or direct sums, carry over to vector bundlesholomorphic setting.

Exercise 5.1. Show that if $E, F \to M$ are holomorphic vector bundles then E^*, $E \oplus F$, $E \otimes F$ inherit a natural structure of holomorphic vector bundle over M. Concretely, describe how the transition functions for E and F determine transition functions for the proposed bundles.

Exercise 5.2. Let $E \to M$ be a complex vector bundle over a smooth manifold M. Define $\overline{E}$ by declaring that multiplication by complex scalars gets conjugated, that is, as spaces $\overline{E} = E$ and the projection map to M for both agree, but

$$\lambda \cdot_{\overline{E}} v := \overline{\lambda} \cdot_E v,$$

where $\cdot_{\overline{E}}$ and $\cdot_E$ are the scalar multiplication operationof each of the bundles.

Show that $\overline{E}$ is isomorphic to E^* as complex vector bundles.

Continuing with the relationship between vector bundles and families of transition functions, we see that Lemmas 5.2 and 5.3 say that isomorphisms classes of vector bundles together with trivialisations over an open cover $\mathcal{U}$ are in equivalence with families of skew-symmetric transition functions satisfying the cocycle condition. There are two issues with this picture. The first is the dependance on (the existence of a trivialisation over) the cover $\mathcal{U}$. The second is that vector bundles are the object of interest, and the a classification we obtained includes as data specific trivialisations. So next we deal with these shortcomings.

Vector bundles over contractible sets are trivialisable, so as long as we work with a cover $\mathcal{U}$ of M by discs, the requirement that a bundle $E \to M$ has trivialisations over the elements of $\mathcal{U}$ is not a restriction. The second issue requires us to study what happens when we change trivialisations.

Theorem 5.4 (Classification of isomorphism classes of vector bundles)**.** *Let M be a manifold and let $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ be an open cover of M by discs. Isomorphism classes of rank k complex vector bundles over M are in one-to-one correspondence with equivalence classes of skew symmetric cocycles $\check{g} = \{g^\alpha_\beta\colon U_{\alpha\beta} \to \mathrm{GL}\,(k;\mathbb{C})\colon \alpha,\beta \in A\}$ where $\check{g}$ and $\check{g}'$ are equivalent if there is a family of functions $\check{f} = \{f^\alpha\colon U_\alpha \to \mathrm{GL}\,(k;\mathbb{C})\colon \alpha \in A\}$ such that*

$$(\check{g}')^\alpha_\beta = f^\beta \cdot \check{g}^\alpha_\beta \cdot (f^\alpha)^{-1}. \tag{5.1.6}$$

Proof. We prove only one direction. Say we are given two collections of trivialisation $\{\Phi_\alpha\colon \alpha \in A\}$ and $\{\Phi'_\alpha\colon \alpha \in A\}$ with transitions functions $\check{g}$ and $\check{g}'$ respectively. Over each U_α we can compare the trivialisation Φ_α with Φ'_α to obtain the transition function between these two:

$$\Phi'_\alpha \circ (\Phi_\alpha)^{-1}(x,v) = (x, f^\alpha(x)v).$$

With this we can relate the transition functions of the two families:

$$\begin{aligned}(x, {g'}^\alpha_\beta(x)v) &= \Phi'_\beta \circ (\Phi'_\alpha)^{-1}(x,v)\\ &= \Phi'_\beta \circ \Phi_\beta^{-1} \circ \Phi_\beta \circ \Phi_\alpha \circ (\Phi_\alpha^{-1}) \circ (\Phi'_\alpha)^{-1}(x,v)\\ &= (x, f^\beta(x)\cdot g^\alpha_\beta(x)\cdot (f^\alpha(x))^{-1}v),\end{aligned}$$

showing that $\check{g}$ and $\check{g}'$ are equivalent as in (5.1.6). □

5.2 Holomorphic vector bundles

If the base manifold is complex we can introduce the notion of holomorphic vector bundles by requiring that the maps involved are holomorphic (and the fibers are complex vector spaces).

Definition 5.5. A *holomorphic vector bundle* over a complex manifold M is a complex manifold E which is a vector bundle over M, $\pi\colon E \to M$, and for which

- the map π is holomorphic,
- for every point $p \in M$ there is a neighbourhood U and a local trivialisation $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^k$ which is a biholomorphism.

The fact that local trivialisations are biholomorphisms translates into the fact that the underlying transition functions

$$g^\alpha_\beta\colon U_{\alpha\beta} \to \mathrm{GL}(k;\mathbb{C})$$

are holomorphic, where $\mathrm{GL}\,(k;\mathbb{C})$ has the structure of complex manifold described in Example 3.3 (which makes matrix multiplication a holomorphic operation).

Using descriptions of bundles via transition functions we see that several constructions of vector bundles associated to a given one, such as taking duals, tensor products or direct sums, carry over to vector the holomorphic setting. Further, all the arguments used in the previous section to classify vector bundles in terms of a cocycle $\check{g}$ of transition functions defined on double overlaps extend to the holomorphic setting with the requirement that $\check{g}$ is made of holomorphic functions.

Exercise 5.3. If $E \to M$ is a holomorphic vector bundle over a complex manifold M, does $\overline{E}$ admit a natural holomorphic structure?

In the theory of vector bundles over real manifolds, there was no preferred way to take derivatives. To do that, one needed to introduce a connection, ∇. With a connection at hand to replace the exterior derivative, we can extend it to a map, still denoted by the same symbol,

$$\nabla\colon \Omega^k(M;E) \to \Omega^{k+1}(M;E).$$

by demanding that it satisfies the Leibniz rule. At this stage it was natural to try to define a corresponding cohomology theory. However the curvature, F_∇, got on the way: since $\nabla^2 = F_\nabla$ meant that it did not make sense to divide the kernel of ∇ by its image. The exception was the case of flat bundles, that is, pairs of bundle with connection, (E,∇) , such that $F_\nabla = 0$. These can be equivalently described by the existence of trivialisations whose transition functions are locally constant, which indicates that this type of bundle does not occur frequently.

Once we go to the holomorphic world, however, we have seen that holomorphic functions play the role of constants from the point of view of the $\overline{\partial}$ operator. In particular, for a holomorphic bundle we can define a $\overline{\partial}$ operator on the space of sections in a local holomorphic trivialisation. Given a local trivialisation $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^k$, define

$$\overline{\partial}\colon \Gamma(E) \to \Gamma(\wedge^{0,1}T^*M \otimes E), \qquad \overline{\partial}s = (\mathrm{Id} \otimes \Phi^{-1})(\overline{\partial}(\Phi \circ s)).$$

Since the transition functions are holomorphic, this does not depend on the choice of trivialisation. Further, we can use the Leibniz rule to extend this operator to

$$\overline{\partial}\colon \Gamma((\wedge^{0,q}T^*M \otimes E) \in \Gamma(\wedge^{0,q+1}T^*M \otimes E), \qquad \overline{\partial}(\alpha \otimes s) = \overline{\partial}(\alpha) \otimes s + (-1)^{\deg(\alpha)}\alpha \wedge \overline{\partial}s.$$

We denote the space $\Gamma((\wedge^{0,q}T^*M \otimes E)$ by $\Omega^q(M;E)$

Lemma 5.6. *Let $E \to M$ be a holomorphic vector bundle. Then $\overline{\partial}^2\colon \Omega^q(M;E) \to \Omega^{q+1}(M;E)$ satisfies $\overline{\partial}^2 = 0$.*

Definition 5.7. Given a holomorphic vector bundle $E \to M$, the Dolbeault cohomology of M with coefficients in E is given by

$$H^q(M;E) := \frac{\mathrm{Ker}(\overline{\partial}\colon \Omega^q(M;E) \to \Omega^{q+1}(M;E))}{\mathrm{Im}(\overline{\partial}\colon \Omega^{q-1}(M;E) \to \Omega^q(M;E))}.$$

Exercise 5.4. Show that $\Omega^q(M;\Omega^{p,0}(M)) = \Omega^{p,q}(M)$ and that the two definitions of $\overline{\partial}$ on this space agree.

Exercise 5.5. Compute $H^0(T\mathbb{C}P^1)$.

5.3 Line bundles and divisors

While isomorphism classes of general vector bundles can be described in terms of transition functions, as in Theorem 5.4, for line bundles a more geometric picture quickly emerges: a line bundle is roughly determined by the vanishing locus of a generic section. The underlying concept is that of a *divisor*.

Definition 5.8. A *divisor* on a manifold M is an ideal $\mathcal{I}_Y \subset C^\infty(M)$ which is locally generated by a single function and whose vanishing locus, Y, is nowhere dense on M.

In the life of a differential geometer the ideal defining a divisor may be defined by a real or a complex function. In these notes, we will focus on complex line bundles and hence our ideals will be defined by complex functions.

To describe how divisors arise we first recall a notion of transversality.

Definition 5.9. Let E be a smooth manifold and let $\iota_i \colon M_i \to E$, $i = 0, 1$, be two immersed submanifolds. Let $p_i \in M_i$ be such that $\iota_0(p_0) = \iota_1(p_1)$. We say that the immersions are *transverse at* p_0 *and* p_1 if $\iota_{0*}(T_{p_0}M_0) + \iota_{1*}(T_{p_1}M_1) = T_{\iota_0(p_0)}E$. The immersions ι_0 and ι_1 are *transverse* if they are transverse at all pairs p_0 and p_1 for which $\iota_0(p_0) = \iota_1(p_1)$.

Exercise 5.6. Show that if $\iota_i \colon M_i \to E$ are transverse embeddings, then $\iota_0(M_0) \cap \iota_1(M_1)$ is an embedded submanifold of codimension $\operatorname{codim}(M_0) + \operatorname{codim}(M_1)$.

Definition 5.10. Let $E \to M$ be a vector bundle over M and let $s_i \colon M \to E$, $i = 0, 1$ be two sections. We say that the sections s_0 and s_1 are *transverse* or s_0 *is transverse to* s_1 if they are transverse as immersions $s_i \colon M \to E$.

A very common condition to consider is that of a section $s \colon M \to E$ transverse to the zero section. When this happens, the zero locus of s, say Z, is an embedded submanifold of M whose codimension in M is the rank of E, by exercise 8.8. Further, there is a relation between E and the normal bundle of M

Proposition 5.11. *Let $s \colon M \to E$ be a section transverse to the zero section and let $Z \subset M$ be the zero locus of s, then*

$$d^\nu s = \pi_E \circ ds \colon TM \to E|_Z$$

has kernel TZ and induces an isomorphism between $\mathcal{N}_Z$, the normal bundle of Z, and $E|_Z$.

For p a zero of s, the composition of $ds_p \colon T_pM \to T_0E \cong T_pM \oplus E_p$ with the projection onto E_p, $d^\nu s = \pi_E \circ ds$, is known as the *vertical derivative of* s.

Proof. It follows directly from the definition of transversality that s is transverse to the zero section if and only if the vertical derivative of s is surjective at every point $p \in Z$ (since the tangent space of the zero section already takes up all the "horizontal" directions). Further, since s vanishes over Z, we have that for all $X \in TZ$, $d^\nu s(X) = 0$, so the vertical derivative descends to a surjective bundle map, which we still denote by $d^\nu s$

$$d^\nu s \colon \mathcal{N}_Z = TM/TZ \to E|_Z.$$

Since these bundles have the same rank, this map is a bundle isomorphism. □

When $E \to M$ is a line bundle, a divisor arises:

Proposition 5.12. *Let $L \to M$ be a line bundle and let $s \colon M \to L$ be a section transverse to the zero section, then*

$$\mathcal{I}_s = \{\sigma(s) \colon \sigma \in \Gamma(L^*)\}$$

Exercise 5.7. Let (Σ, I) be a Riemann surface. Let $\{p_1, \dots, p_n\}$ be a finite collection of points of Σ and let $\{m_1, \dots, m_n\}$ be a collection of integers. Consider the following cover of Σ. For each i let U_i be a disc centered at p_i, that is, U_i is the domain of a coordinate chart centered at p_i whose image is a disc. Assume that the sets U_i are chosen small enough so that $U_i \cap U_j = \emptyset$ for $i \neq j$. Let $U_0 = \Sigma \backslash \{p_1, \dots, p_n\}$, and define transition functions $g_i^0(z) = z^{m_i}$.

Show that these transition functions give rise to a holomorphic line bundle, E, over Σ which admits a meromorphic section s with zeros and poles only at the points $\{p_1, \dots, p_n\}$. Further for $m_i \geq 0$, p_i is a zero of order m_i of s and for $m_i < 0$, p_i is a pole of order $|m_i|$.

Finally argue that holomorphic sections of E correspond to meromorphic functions on Σ with poles of order at most $|m_i|$ at p_i for all i such that $m_i < 0$ and zeros of order at least $|m_i|$ at p_i for all i such that $m_i \geq 0$.

Chapter 6

Blow-ups

A once we introduce an object and obtain a few examples, it is natural to ask if there ways to modify those examples to obtain yet more examples. For smooth manifolds, a typical procedure that follows this paradigm is taking connected sums. In that case, from a sphere and a torus we can produce all orientable surfaces.

Complex manifolds are more rigid objects because of their analytic nature and a number of differentiable constructions have no analogue in the holomorphic world. Yet there is one construction that has great power and that we study in this chapter: the complex blow-up.

6.1 The tautological bundle and the blow-up of the origin

The tautological bundle over $\mathbb{C}P^n$ lies at the heart of the blow-up construction, so we spend this session studying it in some detail.

Consider the subspace $\tau_n \subset \mathbb{C}^{n+1} \times \mathbb{C}P^n$ given by the natural incidence relation: a pair consisting of a point (in $\mathbb{C}^{n+1}$) and a line (in $\mathbb{C}P^n$) are in τ_n if the point is in the line, or, in symbols,

$$\tau_n = \{(z, l) \in \mathbb{C}^{n+1} \times \mathbb{C}P^n \colon z \in l\}.$$

The subset τ_n is in fact an embedded submanifold and we can readily cover it with a few parametrisations. Namely, for $1 = 1, \dots, n+1$, we define

$$\phi_i \colon \mathbb{C}^{n+1} \to \tau_n, \quad \phi_i(\lambda, z_1, \dots, z_n) = (\lambda(z_1, \dots, \underset{i^{th}\text{ pos}}{1}, \dots, z_n), [z_1, \dots, \underset{i^{th}\text{ pos}}{1}, \dots, z_n]),$$

One can readily compute the change of parametrizations. For example, for $j < i$, we have

$$\begin{aligned}
\phi_j^{-1} \circ \phi_i(\lambda, z_1, \dots, z_n) &= \phi_j^{-1}(\lambda(z_1, \dots, z_j, \dots, \underset{i^{th}\text{ pos}}{1}, \dots, z_n), [z_1, \dots, z_j, \dots, \underset{i^{th}\text{ pos}}{1}, \dots, z_n]) \\
&= \phi_j^{-1}(\lambda z_j(\frac{z_1}{z_j}, \dots, \underset{j^{th}\text{ pos}}{1}, \dots, \underset{i^{th}\text{ pos}}{\frac{1}{z_j}}, \dots, \frac{z_n}{z_j}), [\frac{z_1}{z_j}, \dots, \underset{j^{th}\text{ pos}}{1}, \dots, \underset{i^{th}\text{ pos}}{\frac{1}{z_j}}, \dots, \frac{z_n}{z_j}]) \\
&= (\lambda z_j, \frac{z_1}{z_j}, \dots, \frac{1}{z_j}, \dots, \frac{z_n}{z_j}).
\end{aligned}$$

This shows that τ_n is a complex manifold as it has holomorphic change of coordinates.

The ambient space $\mathbb{C}^{n+1}\times\mathbb{C}P^n$ has two natural projections which we can restrict to τ_n. Let's consider them one at a time.

Projection onto the second factor and τ_n as a line bundle. The level sets of the projection $\pi_2\colon \tau_n \to \mathbb{C}P^n$ are lines. Namely, $\pi_2^{-1}(l)$ consists of all the points in l, including 0. This strongly suggests that $\tau_n \to \mathbb{C}P^n$ is a line bundle and in fact the parametrizations ϕ_i defined above provide local trivialisations (the parameter λ parametrizes the level sets in a linear way) and from the expression for the change of parametrizations we see that the transition functions for this line bundle are given by $g^i_j(z_1,\dots,z_n) = z_j$.

There is another space that is closely related to τ_n, namely,

$$H_n = \mathbb{C}P^{n+1}\backslash[1,0,\dots,0].$$

We have a natural map

$$\pi\colon H \to \mathbb{C}P^n, \quad \pi([z_0,\dots,z_{n+1}]) = [z_1,\dots,z_{n+1}]$$

.

We can cover H_n with the same coordinates we covered $\mathbb{C}P^{n+1}$ (Example 3.6) but since the line $[1,0,\dots,0]$ has been removed we can do without the chart Φ_0. Changing the name of the variable z_0 to λ, for $0<j<i$ we have, from Example 3.6, that

$$\Phi_j\circ\Phi_i^{-1}(\lambda,z_1,\dots,z_{n+1}) = \Big(\frac{\lambda}{z_j},\frac{z_1}{z_j},\dots,\underbrace{\frac{1}{z_j}}_{i^{th}\text{ pos}},\dots,\frac{z_n}{z_j}\Big),$$

which is nearly the same as the change of coordinates for τ_n, except that now λ is divided by z_j instead of multiplied.

We conclude that $H_n \to \mathbb{C}P^n$ is a line bundle, the affine coordinate charts in $\mathbb{C}P^{n+1}$ express H as a line bundle with transition functions $\frac{1}{z_j}$.

Lemma 6.1. *The line bundles τ_n and H_n are dual to each other. In particular, as real line bundles they are diffeomorphic via a fiberwise orientation reversing bundle map.*

Proof. Since the transition functions of H_n and τ_n are inverse of each other, H_n is the dual of τ_n and the choice of a Hermitian metric gives the real isomorphism between these bundles conjugating the scalar action of $\mathbb{C}$. □

Another way of rephrasing this result (with a little less information) is

Lemma 6.2. *There is an orientation preserving diffeomorphism between τ_n and $\overline{\mathbb{C}P^{n+1}\backslash[1,0,\dots,0]}$, where the bar indicates that we take the orientation opposite to the one induced by the complex structure on that space.*

Projection onto the first factor and τ_n as blow-up. Projecting τ_n onto the $\mathbb{C}^{n+1}$ factor is a more interesting map. To get a geometric idea of what is going on there we first observe that there is a copy of $\mathbb{C}P^n$ inside τ_n as the zero section of $\tau_n \to \mathbb{C}P^n$. Explicitly,

$$\iota\colon \mathbb{C}P^n \to \tau_n, \qquad \iota(l) = (0,l).$$

If we consider the complement of the zero section, $\tau_n \backslash \iota(\mathbb{C}P^n)$, we see that image of projection onto the first factor misses the origin in $\mathbb{C}^{n+1}$ and

$$\pi_1 \colon \tau_n \backslash \iota(\mathbb{C}P^n) \to \mathbb{C}^{n+1} \backslash \{0\}, \qquad \pi_1(z,l) = z \tag{6.1.1}$$

is a diffeomorphism since every nonzero point is in a unique line. This is usually phrased by saying that τ_n is 'obtained by removing the origin from $\mathbb{C}^{n+1}$ and replacing it by $\mathbb{C}P^n$.'

To get a more geometric picture of how $\mathbb{C}P^n$ sits inside τ_n from this new point of view it is useful to consider what π_1 does to lines. Say we fix a point $z \in \mathbb{C}^{n+1} \backslash \{0\}$ and consider $l = \{\lambda z \colon \lambda \in \mathbb{C}\}$ and $l^* = \{\lambda z \colon \lambda \in \mathbb{C}^*\}$. Since π_1 is a diffeomorphism outside of the zero section, the pre-image $\pi_1^{-1}(l_1^*)$ is a punctured line in τ_n. When we take the closure of this set, we obtain

$$\overline{\pi_1^{-1}(l^*)} = \overline{\{(w,l) \colon w = \lambda z, \lambda \in \mathbb{C}^*\}} = \{(w,l) \colon w = \lambda z, \lambda \in \mathbb{C}\}.$$

That is, the closure of the pre-image of l^* touches the zero section precisely at the point $l \in \mathbb{C}P^n$. This leads to a more complete description of τ_n in plain English: 'τ_n is obtained by removing the origin from $\mathbb{C}^{n+1}$ and replacing it by the set of directions one can use to approach the origin'.

Because of this description we call τ_n *the blow-up of* $\mathbb{C}^{n+1}$ *at the origin* and correspondingly $\mathbb{C}^{n+1}$ *is the blow-down of* τ_n and $\pi_1 \colon \tau_n \to \mathbb{C}^{n+1}$ is the *blow-down map.*

6.2 Blowing-up a point

Change of notation warning: in the previous session it made sense to use n as the dimension of $\mathbb{C}P^n$ and hence we called the central object of that session, the tautological bundle over $\mathbb{C}P^n$, τ_n. Yet the final outcome was a description of the blow-up of $\mathbb{C}^{n+1}$ at the origin. Now we want to describe the blow-up of a complex manifold at a point, but it is aesthetically unpleasant to talk about an $(n+1)$-dimensional manifold, so we will shift n in the previous discussion by one.

To blow-up a point p in a manifold M, what we want to do is to pick a holomorphic chart centered at p, $\psi \colon U \subset M \to \mathbb{C}^n$, then replace U by a copy of the tautological bundle:

Definition 6.3. Let M^n be a complex manifold and $p \in M$. Let $\psi \colon U \subset M \to \mathbb{C}^n$ be a chart centered at p and let $\tilde{U} = \pi_1^{-1} \circ \psi(U) \subset \tau_{n-1}$, where $\pi_1 \colon \tau_{n-1} \to \mathbb{C}^n$ is the blow-down map. The *blow-up* of M at $p \in M$ is the manifold

$$\widetilde{M} = (M \backslash \{p\}) \cup_{\psi^{-1} \circ \pi_1^{-1}} \tilde{U},$$

where $\cup_{\psi^{-1} \circ \pi_1^{-1}}$ denotes the identification of points in $U \backslash \{p\}$ with points in $\tilde{U} \backslash \mathbb{C}P^{n-1}$ via the biholomorphism

$$\psi^{-1} \circ \pi_1^{-1} \colon U \backslash \{p\} \to \tilde{U} \backslash \mathbb{C}P^{n-1}.$$

Since both $M \backslash \{p\}$ and $\tilde{U}$ are complex manifolds and the gluing map is holomorphic, $\widetilde{M}$ is also a complex manifold. Notice that in this process we replaced p with a copy of $\mathbb{C}P^{n-1} \subset \widetilde{M}$ which represents the set of directions at which one can approach p. We will need to refer to this copy of $\mathbb{C}P^{n-1}$ so often that it deserves a name:

Definition 6.4. Let $\widetilde{M}$ be the blow-up of M at p. The embedded copy of $\mathbb{C}P^{n-1} \subset \widetilde{M}$ obtained by the blow-up procedure is called the *exceptional divisor.*

Using the blow-down map $\pi_1 \colon \tau_{n-1} \to \mathbb{C}^n$, we obtain a map

$$\pi \colon \widetilde{M} \to M, \qquad \pi(z) = \begin{cases} z & \text{for } z \in M \backslash \{p\}, \\ \psi^{-1} \circ \pi_1(z) & \text{for } z \in \tilde{U}. \end{cases} \tag{6.2.1}$$

In the overlapping region where we glue the spaces $M\backslash\{p\}$ and $\tilde{U}$ together, these definitions agree and hence π is well defined. Further both local expressions for π are holomorphic, hence π is a holomorphic map between $\widetilde{M}$ and M.

Exercise 6.1. Show that π is a biholomorphism between $\widetilde{M}\backslash\mathbb{C}P^{n-1}$ and $M\backslash\{p\}$.

The map π defined above is called the *blow-down map.*

In this whole discussion one nagging point has been silently left hidden. Namely, even though we call $\widetilde{M}$ *the* blow-up of M at p, the description we used clearly depends on a choice of coordinate chart, ψ. It would be nice if the resulting complex manifold $\widetilde{M}$ and map $\pi \colon \widetilde{M} \to M$ did not depend on this choice. Of course, a different choice of holomorphic coordinates boils down to composing ψ with a local biholomophism $\alpha \colon V \subset \mathbb{C}^n \to W \subset \mathbb{C}^n$ with $\alpha(0) = 0$. The relation between such maps and blow-ups is the content of the next lemma.

Lemma 6.5. *Let $\alpha \colon V \subset \mathbb{C}^n \to W \subset \mathbb{C}^n$ be a biholomophism with $\alpha(0) = 0$, then there is a unique biholomorphism $\tilde{\alpha} \colon \widetilde{V} \subset \tau_{n-1} \to \widetilde{W} \subset \tau_{n-1}$ such that the following diagram commutes*

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi_1"] & \widetilde{W} \arrow[d, "\pi_1"] \\ V \arrow[r, "\alpha"] & W \end{tikzcd} \tag{6.2.2}$$

Proof. Since the question regards the behaviour of $\tilde{\alpha}$ in a neighbourhood of the exceptional divisor, we may just as well assume that V is a polydisc, D_r.

Let us first assume that one such lift $\tilde{\alpha}$ exists. In this case it is unique, since in the complement of the exceptional divisor, where π_1 is a diffeomorphism, it must be given by $\tilde{\alpha} = \pi_1^{-1} \circ \alpha \circ \pi_1$. Also, this expresses the lift $\tilde{\alpha}$ as a composition of holomorphic maps, therefore $\tilde{\alpha}$ is holomorphic in an open and dense set and therefore it is holomorphic (recall that a map is holomorphic if its derivative commutes with the almost complex structure, and this is a closed condition). Finally, $\tilde{\alpha}$ is a diffeomorphism. Indeed, if $V = W$ and $\alpha = \mathrm{Id}|_V$, then one can readily check that $\tilde{\alpha} = \mathrm{Id}_{\tilde{V}}$ is the unique lift described in the lemma. For the general case, we can apply the lemma for α^{-1}, which is also a diffeomorphism, to obtain a lift $\tilde{\alpha}^{-1}$ and observe that $\tilde{\alpha}^{-1} \circ \tilde{\alpha}$ is another lift for $\mathrm{Id}|_V$ and similarly $\tilde{\alpha} \circ \tilde{\alpha}^{-1}$ is a lift for $\mathrm{Id}|_W$. By uniqueness, these maps are the identity and therefore $\tilde{\alpha}$ is a diffeomorphism.

So the main difficulty is to prove that $\tilde{\alpha}$ exists. To do so we first expand α around the origin, using that $\alpha(0) = 0$:

$$\alpha(z) = d\alpha|_0(z) + h(z),$$

where $h \colon D_r \to \mathbb{C}^n$ is holomorphic, $h(0) = 0$ and $dh(0) = 0$. In particular, the function

$$(\lambda, z) \in D_1 \times D_r \mapsto \frac{h(\lambda z)}{\lambda}$$

is holomorphic and vanishes for $\lambda = 0$[1], or, written in terms of α,

$$(\lambda, z) \in D_1 \times D_r \mapsto \frac{\alpha(\lambda z)}{\lambda} - d\alpha|_0(z)$$

is holomorphic and vanishes for $\lambda = 0$.

Since α is a biholomorphism $d\alpha|_0(z) \neq 0$ for $z \neq 0$. Hence, for $z \neq 0$, the function $(\lambda, z) \mapsto \frac{\alpha(\lambda z)}{\lambda}$ maps $D_1 \times D_r \backslash \{0\}$ to $\mathbb{C}^n \backslash \{0\}$ and we can compose it with projection onto $\mathbb{C}P^{n-1}$ to obtain a holomorphic map:

$$(\lambda, z) \in D_1 \times D_r \backslash \{0\} \mapsto \left[\frac{\alpha(\lambda z)}{\lambda}\right] \in \mathbb{C}P^{n-1}.$$

To check that $\tilde{\alpha}$ exists, we write its expression in the complement of the exceptional divisor in the parametrization ϕ_i for the domain (note that we do not need to use a parametrization for the co-domain because $\tau_{n-1} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ is an embedding and hence it is enough to check that $\tilde{\alpha}$ exists as a map to the ambient space $\mathbb{C}^n \times \mathbb{C}P^{n-1}$):

$$\begin{aligned}
\pi_1^{-1} \circ \alpha \circ \pi_1 \circ \phi_i(\lambda, z_1, \dots, z_{n-1}) &= \pi_1^{-1} \circ \alpha \circ \pi_1(\lambda(z_1, \dots, 1, \dots z_{n-1}), [z_1, \dots, 1, \dots z_{n-1}]) \\
&= \pi_1^{-1} \circ \alpha(\lambda(z_1, \dots, 1, \dots z_{n-1})) \\
&= (\alpha(\lambda(z_1, \dots, 1, \dots z_{n-1})), [\alpha(\lambda(z_1, \dots, 1, \dots z_{n-1}))]), \\
&= \left(\alpha(\lambda(z_1, \dots, 1, \dots z_{n-1})), \left[\frac{\alpha(\lambda(z_1, \dots, 1, \dots z_{n-1}))}{\lambda}\right]\right),
\end{aligned}$$

and we see that both components of $\tilde{\alpha}$ extend holomorphically to $\lambda = 0$. $\square$

Exercise 6.2. In the notation of the previous lemma, describe the restriction of $\tilde{\alpha}$ to the exceptional divisor.

This lemma allows us to determine how unique the blow-up is.

Proposition 6.6. *Let M be a complex manifold $p \in M$ and ψ_1, ψ_2 be two holomorphic charts centered at p. Denote by $\widetilde{M}_i$ the blow-up of M obtained using the chart ψ_i. There is a unique biholomorphic map $\widetilde{\mathrm{Id}}\colon \widetilde{M}_1 \to \widetilde{M}_2$ for which the following diagram commutes*

$$\begin{tikzcd}
\widetilde{M}_1 \arrow[r, "\widetilde{\mathrm{Id}}"] \arrow[d, "\pi_1"] & \widetilde{M}_2 \arrow[d, "\pi_2"] \\
M \arrow[r, "\mathrm{Id}"] & M,
\end{tikzcd} \tag{6.2.3}$$

where π_i are the respective blow-down maps.

Proof. Let U be the domain of the charts ψ_1 and ψ_2 (arranged to be the same by taking an intersection) and let $V_i = \psi_i(U)$. By Lemma 6.5, the change of coordinate, $\psi_2 \circ \psi_1^{-1}\colon V_1 \to V_2$

[1]By this we mean that the singularity of $(\lambda, z) \mapsto \frac{h(z)}{\lambda}$ at 0 is removable and therefore we can extend this function to 0 by continuity to obtain a holomorphic function.

induces a corresponding map $\widetilde{\psi_2 \circ \psi_1^{-1}} \colon \widetilde{V_1} \to \widetilde{V_2}$ to the blow-up of V_1 and V_2 at the origin. We define $\widetilde{\mathrm{Id}} \colon \widetilde{M_1} \to \widetilde{M_2}$ by defining what it does in the two parts that make up $\widetilde{M_1}$:

$$\widetilde{\mathrm{Id}} \colon \widetilde{M_1} \backslash \{p\} \to \widetilde{M_2} \backslash \{p\} \qquad \widetilde{\mathrm{Id}}(z) = z,$$
$$\widetilde{\mathrm{Id}} \colon \widetilde{V_1} \to \widetilde{V_2} \qquad \widetilde{\mathrm{Id}}(z) = \widetilde{\psi_2 \circ \psi_1^{-1}}(z).$$

It remains to check that $\widetilde{\mathrm{Id}}$ is well defined, that is, in the overlapping region between M_1 and $\tilde{V_1}$ both definitions agree. But by definition a point z in $\tilde{V_1}$ is in the overlapping region if it is not in the exceptional divisor, so the fact that $\widetilde{\mathrm{Id}}$ is well defined follows by observing that, by construction, the following diagram commutes and all the maps are diffeomorphisms

$$\begin{tikzcd}
\widetilde{V_1} \backslash \mathbb{C}P^{n-1} \arrow[r, "\widetilde{\psi_2 \circ \psi_1^{-1}}"] \arrow[d, "\pi"] & \widetilde{V_2} \backslash \mathbb{C}P^{n-1} \arrow[d, "\pi"] \\
V_1 \backslash \{0\} \arrow[r, "\psi_2 \circ \psi_1^{-1}"] & V_2 \backslash \{0\} \\
U \backslash \{p\} \arrow[r, "\mathrm{Id}"] \arrow[u, "\psi_1"] & U \backslash \{p\}, \arrow[u, "\psi_2"']
\end{tikzcd}$$

□

6.3 The topology of a blow-up

With the blow-up construction at hand, we can produce many new examples of complex manifolds by blowing up any discrete collection of points in a complex manifold. This process can be iterated, and we can blow up points in the exceptional divisors introduced by previous blow-ups. The positioning of the points in most cases will influence the resulting complex manifold, that is, potentially different choices of points lead to non equivalent complex manifolds. Yet it is also desirable to take a step back and have a differential-topological understanding of what blowing up amounts to. Natural questions one can ask include: What is the cohomology of the blow-up? Can we identify the diffeomorphism type of the blow-up? Notice that since in a connected manifold thre is always an isotopy that maps a preassigned point to a preassigned image the answer to these question should be independent of the choice of point.

We will tackle these questions in this section, starting with the diffeomorphism type. To phrase the answer we need to recall the connected sum operation.

Definition 6.7. Let M_1 and M_2 be oriented connected n-dimensional smooth manifolds. The *connected sum* of M_1 and M_2 is the manifold

$$M_1 \# M_2 = M_1 \backslash \{p_1\} \cup_{\widetilde{\mathrm{inv}}} M_2 \backslash p_2,$$

where $p_i \in M_i$ and $\widetilde{\mathrm{inv}}$ is the map defined with help of orientation preserving coordinates centered at p_i by

$$\widetilde{\mathrm{inv}}(x_1, \dots, x_n) = \frac{1}{\|x\|^2}(-x_1, x_2, \dots, x_n).$$

The differential type of the connected sum does not depend on the points nor the charts chosen and the resulting manifold has a natural orientation determined by the orientations of each M_i, since the map used to identify them is orientation preserving.

If we were to use a coordinate chart that is not compatible with the orientation, say, in M_2, then we can produce the connected sum using the simple inversion on the sphere

$$\mathrm{inv}(x) = \frac{x}{\|x\|^2}.$$

Theorem 6.8. *Let $\widetilde{M}$ be the blow-up of a complex manifold M of complex dimension n at a point p. Then $\widetilde{M}$ is diffeomorphic to $M\#\overline{\mathbb{C}P^n}$.*

Proof. □

With this differential geometric description of the blow-up, determining its cohomology is an exercise with the Mayer–Vietoris sequence:

Exercise 6.3. Let M_1 and M_2 be oriented, connected n-dimensional manifolds. Show that

$$H^i(M_1\#M_2;\mathbb{Z}) = \begin{cases} H^i(M_1;\mathbb{Z}) \oplus H^i(M_2;\mathbb{Z}), & \text{if } i \neq 0, n, \\ \mathbb{Z}, & \text{if } i = 0, n. \end{cases}$$

Conclude that if $\widetilde{M}$ is the blow-up of M at a point, the

$$H^i(\widetilde{M};\mathbb{Z}) = \begin{cases} H^i(M;\mathbb{Z}) \oplus \mathbb{Z}, & \text{if } 0 < i < n \text{ and } i \text{ is even}, \\ H^i(M;\mathbb{Z}), & \text{otherwise}. \end{cases}$$

6.4 Blowing-up a submanifold

When dealing with general submanifolds, we blow-up "normal directions". The argument used is a parametrised version of the blow-up of a point and the local model we need to develop is the blow-up of

$$\mathbb{C}^{n-k} \cong \{0\} \times \mathbb{C}^{n-k} \subset \mathbb{C}^n.$$

That is, we regard $\mathbb{C}^{n-k}$ as the vanishing locus of the first k coordinates.

Similar to the case of a point, we now form the normal tautological bundle:

$$\tau \subset \mathbb{C}^k \times \mathbb{C}^{n-k} \times \mathbb{C}P^{k-1}, \qquad \tau = \{(z, w, l) \in \mathbb{C}^k \times \mathbb{C}^{n-k} \times \mathbb{C}P^{k-1} \colon z \in l\} = \tau_{k-1} \times \mathbb{C}^{n-k}.$$

Once again we have projections onto different factors, with the more relevant one being the blow-down map

$$\pi_1 \colon \tau \to \mathbb{C}^n, \qquad \pi_1(z, w, l) = (z, w).$$

And just as before π_1 is a holomorphic map which is also a diffeomorphism in the complement of the exceptional divisor:

$$\pi_1 \colon \tau \backslash (\mathbb{C}P^{k-1} \times \mathbb{C}^{n-k}) \xrightarrow{\cong} \mathbb{C}^k \backslash \{0\} \times \mathbb{C}^{n-k}.$$

This gives a local model for the blow-up of a complex codimension-k submanifold, at least in a neighbourhood of a point. For the blow-up of a point the question that arose was how unique it was and to deal with that question we needed to understand the effect of a change of coordinates. Now the same question arises but has a deeper meaning. Namely if we try to use this local model to blow up a neighbourhood of a point, then a neighbourhood of the proposed blow-up will invariably meet other similar neighbourhoods and whether or not we can systematically glue these together (and if there are choices involved) will determine if the blow up exists and is unique. The analogue of Lemma 6.5 that tells us about existence and uniqueness is

Lemma 6.9. *Let $\alpha\colon V \subset \mathbb{C}^n \to W \subset \mathbb{C}^n$ be a biholomorphism with*

$$\alpha(0,w) \in \{0\} \times \mathbb{C}^{n-k} \text{ for all } w \in \mathbb{C}^{n-k},$$

then there is a unique biholomorphism $\widetilde{\alpha}\colon \widetilde{V} \subset \tau \to \widetilde{W} \subset \tau$ such that the following diagram commutes

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi_1"] & \widetilde{W} \arrow[d, "\pi_1"] \\ V \arrow[r, "\alpha"] & W, \end{tikzcd} \tag{6.4.1}$$

where $\widetilde{V} = \pi_1^{-1}(V)$ and $\widetilde{W} = \pi_1^{-1}(W)$.

Proof. The proof is similar to that of Lemma 6.5. Since the question regards the behaviour of $\tilde{\alpha}$ in a neighbourhood of the exceptional divisor, we may just as well assume that V is a polydisc, D_r.

The same arguments used before can be repeated to show that if $\tilde{\alpha}$ exists then it is unique and is a biholomorphism. So the main difficulty is to prove that $\tilde{\alpha}$ exists. To prove that we follow the same route as before and expand α in normal directions around points of the form $(0,w)$, using that $\alpha(0,w) = (0,y)$:

$$\alpha(z,w) = d_1\alpha|_{(0,w)}(z) + h(z,w),$$

where $d_1\alpha$ denotes the derivative of α with respect to the first k coordinates, where $h\colon D_r \to \mathbb{C}^n$ is holomorphic, $h(0,w) = (0,y)$ and $d_1h(0,w) = 0$. In particular, the function

$$(\lambda, z, w) \in D_1 \times D_r \mapsto \frac{h(\lambda z, w)}{\lambda}$$

is holomorphic and vanishes for $\lambda = 0$[2], or, written in terms of α,

$$(\lambda, z, w) \in D_1 \times D_r \mapsto \frac{\alpha(\lambda z)}{\lambda} - d_1\alpha|_{(0,w)}(z)$$

is holomorphic and vanishes for $\lambda = 0$.

Since α is a biholomorphism $d_1\alpha|_{(0,w)}(z) \neq 0$ for $z \neq 0$. Hence, for $z \neq 0$, the function $(\lambda, z, w) \mapsto \frac{\alpha(\lambda z, w)}{\lambda}$ maps $D_1 \times D_r \backslash \{0\}$ to $\mathbb{C}^n \backslash \{0\}$ and we can compose it with projection onto $\mathbb{C}P^{k-1} \times \mathbb{C}^{n-k}$ to obtain a holomorphic map:

$$(\lambda, z, w) \in D_1 \times D_r \backslash \{0\} \mapsto \left[\frac{\alpha(\lambda z, w)}{\lambda}\right] \in \mathbb{C}P^{k-1} \times \mathbb{C}^{n-k}.$$

[2]By this we mean that the singularity of $(\lambda, z) \mapsto \frac{h(z)}{\lambda}$ at 0 is removable and therefore we can extend this function to 0 by continuity to obtain a holomorphic function.

As before, to check that $\tilde{\alpha}$ exists, we write its expression in the complement of the exceptional divisor in the parametrization ϕ_i for the domain:

$$\begin{aligned}\pi_1^{-1} \circ \alpha \circ \pi_1 \circ \phi_i(\lambda, z_1, \ldots, z_{k-1}, w) &= \pi_1^{-1} \circ \alpha \circ \pi_1(\lambda(z_1, \ldots, 1, \ldots z_{k-1}), w, [z_1, \ldots, 1, \ldots z_{k-1}])\\ &= \pi_1^{-1} \circ \alpha(\lambda(z_1, \ldots, 1, \ldots z_{k-1}), w)\\ &= (\alpha(\lambda(z_1, \ldots, 1, \ldots z_{k-1}), w), [\alpha(\lambda(z_1, \ldots, 1, \ldots z_{k-1}), w)]),\\ &= \left(\alpha(\lambda(z_1, \ldots, 1, \ldots z_{k-1}), w), \left[\frac{\alpha(\lambda(z_1, \ldots, 1, \ldots z_{k-1}), w)}{\lambda}\right]\right),\end{aligned}$$

and we see that both components of $\tilde{\alpha}$ extend holomorphically to $\lambda = 0$. □

With this result at hand, we can construct the blow-up of a complex manifold M along a complex codimension k, closed submanifold N. Indeed, for each point $p \in N$, we pick a holomorphic coordinate chart $\psi \colon U \to \mathbb{C}^n$ defined in a neighbourhood of p for which the points in N correspond to the vanishing locus of the first k coordinates, that is,

$$\psi(N \cap U) = \{(0, z) \in \psi(U) \subset \mathbb{C}^k \times \mathbb{C}^{n-k}\}.$$

We then cover N with a collection of such charts $\{\psi_i \colon U_i \to \mathbb{C}^n \colon i \in I\}$, let $\widetilde{V}_i \subset \tau \subset \mathbb{C}^n \times \mathbb{C}P^{k-1}$ be the parametrized blow-ups constructed above and we define the blow up of M along N to be

$$\widetilde{M} = M\backslash N \cup (\dot{\bigcup}_{i\in I} \widetilde{V}_i)/\sim$$

where a point $x \in M\backslash N$ is declared to be equivalent to a point $y \in \widetilde{V}_i$ if $x \in U_i$ and $\psi_i(x) = \pi_1(y)$ and a point $x \in V_i$ is equivalent to a point $y \in V_j$ if $\widetilde{\psi_j \circ \psi_i^{-1}}(x) = y$.

6.5 Resolving singularities

A typical use of blow-ups is to resolve singularities of potential submanifolds. It is useful to illustrate the idea in a concrete example first.

Example 6.10 (Desingularizing a node)**.** Consider the map $f \colon \mathbb{C}^2 \to \mathbb{C}$ given by $f(z, w) = zw$ and let $\Sigma_\epsilon = f^{-1}(\epsilon)$. Notice that f is holomorphic and $df = zdw + wdz$, which is nonzero as long as either z or w is not zero. That is $(0, 0)$ is the only critical point of f and 0 is the only critical value. By the regular value theorem, Σ_ϵ is a smooth embedded submanifold of $\mathbb{C}^2$ for $\epsilon \neq 0$, but Σ_0 is clearly not an embedded submanifold, as it is made up of two lines intersecting transversely at the origin. One way to desingularise Σ_0 is as follows:

1. blow-up $\mathbb{C}^2$ at the origin,
2. take the inverse image of Σ_0 minus the singular point via the blow-down map and
3. consider the closure of this pre-image in $\widetilde{\mathbb{C}^2}$.

In mathematical symbols, we consider:

$$\widetilde{\Sigma_0} = \overline{\pi^{-1}(\Sigma_0 \backslash \{(0,0)\})}.$$

The picture to have in mind is that the with the blow-up procedure, we replace the origin with the possible directions one can use to approach it. Since Σ_0 is made of two lines approaching the origin from different directions, in the blow-up these two 'branches' of Σ_0 should separate and $\widetilde{\Sigma_0}$ should be made of two lines that do not intersect and hence $\widetilde{\Sigma_0}$ should be an embedded submanifold and the blow-down map restricted to $\widetilde{\Sigma_0}$, $\pi\colon \widetilde{\Sigma_0} \to \Sigma_0$ should be a biholomorphism except for the pre-image of the singular point in Σ_0

In the picture we tried to describe above, the reason to remove $(0,0)$ from Σ before taking the pre-image is that $\pi^{-1}(0,0)$ is the whole exceptional divisor and including it would never yield a smooth manifold nor something which we would like to describe as a desingularization.

It is useful to perform the operation described above explicitly and see how the mental picture given manifests itself with equations.

First, we parametrize $\widetilde{\mathbb{C}P^2}$ and write local expressions for the blow-down map. We need two charts to parametrize $\widetilde{\mathbb{C}P^2}$, the first misses the line $[0,1]$, while the second misses the line $[1,0]$:

$$\psi_1\colon \mathbb{C}^2 \to \tau, \qquad \psi_1(u_1,v_1) = (u_1, u_1v_1, [1,v_1]),$$
$$\psi_2\colon \mathbb{C}^2 \to \tau, \qquad \psi_1(u_2,v_2) = (u_2v_2, v_2, [u_2,1]).$$

The change of parametrization is

$$(u_2,v_2) = \psi_2^{-1} \circ \psi_1(u_1,v_1) = \psi_2^{-1}(u_1, u_1v_1, [1,v_1]) = (v_1^{-1}, u_1v_1^{-1}).$$

The blow-down map in each of these parametrizations is

$$(z,w) = \pi \circ \psi_1(u_1,v_1) = (u_1, u_1v_1), \qquad (z,w) = \pi \circ \psi_2(u_2,v_2) = (u_2v_2, v_2).$$

Now we turn our attention to Σ_0. We compute $\pi^{-1}(\Sigma_0 \backslash \{(0,0)\})$ in each parametrization while keeping in mind that $\pi \circ \psi_1$ misses the vertical line and $\pi \circ \psi_2$ misses the horizontal line.

$$\begin{aligned} \psi_1^{-1} \circ \pi^{-1}(\{(z,w)\colon zw = 0, z \neq 0\}) &= \psi_1^{-1} \circ \pi^{-1}(\{(z,w)\colon w = 0, z \neq 0\}) \\ &= \{(u_1,v_1)\colon u_1v_1 = 0, u_1 \neq 0\}) \\ &= \{(u_1,v_1)\colon v_1 = 0, u_1 \neq 0\}). \end{aligned}$$

Similarly, in the second coordinate set we obtain

$$\psi_1^{-1} \circ \pi^{-1}(\{(z,w)\colon zw = 0, w \neq 0\}) = \{(u_2,v_2)\colon u_2 = 0, v_2 \neq 0\}.$$

Taking the closure we see that $\widetilde{\Sigma_0}$ is made up of one line in each coordinate chart and these lines do not overlap. ■

To continue with the theory we should define the terms we have been using.

Definition 6.11 (Embedded singular manifold)**.** A *singular complex submanifold*, P, of a complex manifold M is a subset $P \subset M$ for which every point $p \in P$ has a neighbouhood U such that $P \cap U$ is the zero locus of a holomorphic function $f\colon U \to \mathbb{C}^k$. Any such function f is a *local defining function for* P.

Definition 6.12. Given a singular submanifold $P \hookrightarrow M$ a point $p \in P$ is *smooth* or *regular* if there is a local defining function for P which has p as a regular point. A point in P is *singular* if it is not smooth.

Definition 6.13 (Desingularization within an ambient space)**.** Let $P \hookrightarrow M$ be singular submanifold of a complex manifold. A *desingularization* of P is a triple consisting of a complex manifold $\widetilde{M}$ a complex submanifold $\widetilde{P}$ and a holomorphic map $\pi\colon \widetilde{M} \to M$ such that

- $\widetilde{M}$ has the same dimension as M,
- π is surjective, and proper,
- $\pi(\widetilde{P}) = P$ and
- $\pi\colon \widetilde{P}\backslash\pi^{-1}(\Delta) \to P\backslash\Delta$ is a biholomorphism, where Δ is the set of singular points of P.

It is worth pausing and convincing yourself that in Example 6.10 Σ_0 was a singular submanifold and the construction provided gives a desingularization of Σ_0 according to the definitions above.

Exercise 6.4. [Cusp singularity] Find a desingularisation of the singular space given by

$$\Sigma_0 = \{(z, w) \in \mathbb{C}^2 \colon z^3 - w^2 = 0\}$$

The candidate for desingularisation proposed in Example 6.10 deserves a name.

Definition 6.14. Let $N \hookrightarrow M$ be a complex submanifold of a complex manifold and let $\widetilde{M}$ be the blow-up of M along N. Let $P \subset M$ be a singular submanifold. The *proper transform of* P in $\widetilde{M}$ is the subset

$$\widetilde{P} = \overline{\pi^{-1}(P\backslash N)}.$$

Exercise 6.5. Show that the proper transform of a singular submanifold is a singular submanifold.

In the context of desingularising submanifolds using proper transforms, the usual situation is when N is either made up of singular points of P or at least is related to them in some way.

Notice that the proper transform of P may or may not be a smooth submanifold. If it is not, one may still be able to identify its singularities and try to blow them up again in the hopes that after repeating this process finitely many times one will arrive at a smooth submanifold which would then deserve the name ‘desingularization of P’.

Example 6.15. Consider the map $f\colon \mathbb{C}^2 \to \mathbb{C}$ given by $f(z, w) = z^5 - w^2$ and let $\Sigma_\epsilon = f^{-1}(\epsilon)$. Once again f is holomorphic and $df = 5zdz - 2wdw$, which is nonzero as long as either z or w is not zero. Therefore $(0, 0)$ is the only critical point of f, 0 is the only critical value and Σ_0 is a singular submanifold of $\mathbb{C}^2$.

As before we can blow-up $\mathbb{C}^2$ at the origin and form the proper transform of Σ_0. Notice that since Σ_0 approaches the origin tangent to the z-axis, we expect that the relevant behaviour, as far as desingularization goes, will be captured by the parametrisation that contains the point $[1,0] \in \mathbb{C}P^1$. As in Example 6.10 we compute

$$\begin{aligned}\psi_1^{-1} \circ \pi^{-1}(\Sigma_0 \backslash \{(0,0)\}) &= \psi_1^{-1} \circ \pi^{-1}\{(z,w)\colon z^5 - w^2 = 0, z \neq 0\}\\ &= \{(u_1,v_1)\colon u_1^5 - (u_1 v_1)^2, u_1 \neq 0\}\\ &= \{(u_1,v_1)\colon u_1^3 - v_1^2, u_1 \neq 0\}\end{aligned}$$

Therefore we see that the part of the proper transform, $\widetilde{\Sigma_0}$ of Σ_0, covered by this chart is a cusp (c.f. Exercise 6.4). In particular it is still a singular submanifold of $\widetilde{\mathbb{C}P^2}$. Following Exercise 6.4, we have that after doing one further blow-up of the origin in the (u_1, v_1)-chart and taking one further proper transform desingularises Σ_0. ■

From this example we see that the blow-up procedure seems to reduce the degree of degeneracy of the defining function around a singularity and one might expect that by blowing-up enough times a resolution can always be found. While we are dealing with isolated singularities (as in Examples 6.10, 6.15 and Exercise 6.4) this strategy seems to be rather obvious, but once the singularities themselves form singular submanifolds it may be unclear how to proceed.

Adding one level of difficulty, one sometimes has spaces that clearly deserve the qualification 'singular', but do not appear embedded in a smooth space, so the theory developed so far breaks down a little.

Definition 6.16 (Abstract singular space)**.** A *singular complex manifold* is a second countable, Haudorff topological space for which every point has a neighbourhood which is homeomorphic to an embedded singular submanifold of $\mathbb{C}^n$, for some n, and the change of coordinates restricted to the smooth locus are holomorphic maps.

From this definition it is not clear that every singular complex manifold can be realised as a singular submanifold of some smooth complex manifold. It is not even clear that a neighbourhood of a connected component of the singular locus can be realised as a singular submanifold, but in the cases when it can, we can once again try to desingularise it by using repeated blow-ups.

Example 6.17. For $n \in \mathbb{N}$, $n > 1$, let ξ be a primitive n^{th} root of 1 and define an action of $\mathbb{Z}_n = \langle a\colon a^n = 1\rangle$ on $\mathbb{C}^2$ by

$$a \cdot (z_1, z_2) = (\xi z_1, \xi z_2).$$

The action of $\mathbb{Z}_n$ on $\mathbb{C}^2 \backslash (0,0)$ is free and proper, but the origin is a fixed point of this action and hence the quotient $X = \mathbb{C}^2/\mathbb{Z}_n$ is potentially a singular complex manifold. This is indeed the case and to see it we consider the map[3]

$$\Phi\colon \mathbb{C}^2 \to \mathbb{C}^3, \qquad \Phi(z_1, z_2) = (z_1^n, z_1^{n-1} z_2, z_2^n).$$

[3]Depending on your upbringing, the map we chose may be deeply disturbing, as you might want to write instead a generating set for the ring of invariant functions, for example,

$$\Phi\colon \mathbb{C}^2 \to \mathbb{C}^{n+1}, \qquad \Phi(z_1, z_2) = (z_1^n, z_1^{n-1} z_2, \ldots, z_1 z_2^{n-1}, z_2^n).$$

This is algebraically much more satisfying, but the effect is that this provides an embedding into an affine space of very high dimension and one would have to figure out many more relations later to cut out the correct subspace.

It is clear that if two points in $\mathbb{C}^2$ lie in the same orbit of the $\mathbb{Z}_n$ action, they have the same image under Φ. Further, if two points have the same image, they lie in the same orbit. Indeed, if $\Phi(z_1, z_2) = \Phi(w_1, w_2)$, then $z_i^n = w_i^n$ and hence $z_i = \xi^{l_i} w_i$ for some l_i. Comparing the middle components, $z_1^{n-1} z_2 = w_1^{n-1} w_2$ then gives that $\xi^{-l_1+l_2} = 0$, showing that $l_1 = l_2 \bmod n$ and hence (z_1, z_2) is in the orbit of (w_1, w_2). That is, Φ induces an injection $\hat{\Phi}\colon X \to \mathbb{C}^3$ identifying X with the image of Φ

Note that we can also describe the image of Φ as

$$\hat{\Phi}(X) = \{(w_1, w_2, w_3) \in \mathbb{C}^3 \colon w_2^n = w_1^{n-1} w_3\},$$

which is a singular submanifold of $\mathbb{C}^3$ as it is the zero level set of the function $f(w_1, w_2, w_3) = w_1^{n-1} w_3 - w_2^n$. ■

Exercise 6.6. Resolve the singularity of the manifold $\hat{\Phi}(X)$ from the previous example.

Every singularity modelled on the quotient of $\mathbb{C}^n$ by the action of a finite group acting linearly is modelled locally by the zeros of polynomials as in Example 6.17 (and I will find a proper citation for this claim, but this seems to go back to work of Hilbert [2] and Noether [6]). These appear frequently enough that they deserve a name:

Definition 6.18. An *(Abelian) quotient singularity* is a singularity locally of the form $\mathbb{C}^n/G$, where G is a finite (Abelian) group.

For the case of quotients singularities of the form $\mathbb{C}^n/G$ which only have the origin as fixed point it may be easier to find a resolution by first blowing up the origin in $\mathbb{C}^n$ enough times so that the group action is free or at least admits a smooth quotient, and then take the quotient $\widetilde{\mathbb{C}^n}/G$ to obtain a resolution of the singularity.

Example 6.19. Consider the action of $\mathbb{Z}_n = \langle a \colon a^n = e\rangle$ on $\mathbb{C}^2$ from Example 6.17 and blow up the origin in $\mathbb{C}^2$. Notice that by continuity, the action of $\mathbb{Z}_n$ induces an action on $\widetilde{\mathbb{C}^2}$ which can be described in local charts:

$$a \cdot (u_1, w_1) = a \cdot (z_1, z_2/z_1) = (\xi z_1, z_2/z_1) = (\xi u_1, u_2).$$

Since the coordinate u_1 in this chart for the tautological bundle $\tau = \widetilde{\mathbb{C}^2}$ corresponds to the fiber direction, we see that the action of $\mathbb{Z}_n$ on $\widetilde{\mathbb{C}^2}$ amounts to rotation on each fiber by $2\pi/n$, which fixes the zero section. This description allows us to identify the quotient space more easily using the following lemma. ■

Lemma 6.20. *Let $E \to M$ be a complex line bundle and let $\mathbb{Z}_n$ act on E by fiberwise rotation by $2\pi/n$. Then $E/\mathbb{Z}_n$ is isomorphic to E^n.*

Proof. Let $\mathcal{U} = \{U_\alpha \colon \alpha \in I\}$ be an open cover of M over which E is trivialisable. Pick trivialisations $\Phi_\alpha \colon E|_{U_\alpha} \to U_\alpha \times \mathbb{C}$ and let $g^\alpha_\beta \colon U_{\alpha\beta} \to \mathbb{C}^*$ be the corresponding transition functions. Then $(g^\alpha_\beta)^n$ are the transition functions for E^n. Since the action preserves $E|_{U\alpha}$ we can describe the quotient $E/\mathbb{Z}_n$ by taking the quotient of each $E|_{U\alpha}$ and then gluing them back together, but the action of $\mathbb{Z}_n$ expressed in the trivialisation Φ_α is just

$$(p, z) \in U_\alpha \times \mathbb{C} \overset{a^i}{\mapsto} (p, \xi^i z) \in U_\alpha \times \mathbb{C},$$

and we see that the quotient map in this trivialisation is just

$$(p, z) \in U_\alpha \times \mathbb{C} \to (p, z^n).$$

For the transition between U_α and U_β we have

$$\Phi_\beta \circ (\Phi_\alpha)^{-1}(p, z) = (p, g^\alpha_\beta(p)z),$$

which yields the relation

$$(p, z^n) \cong (p, (g^\alpha_\beta(p))^n z^n),$$

showing that the quotient $E/\mathbb{Z}_n$ is obtained from the open sets $\{U_\alpha \times \mathbb{C} \colon \alpha \in I\}$ glued together using the transition functions $(g^\alpha_\beta)^n$, which establishes $E/\mathbb{Z}_n \cong E^n$. □

Example 6.21. Let M^n be a complex manifold and form the space

$$\mathcal{M}_2 = (M \times M)/\mathbb{Z}_2,$$

where the action of the generator of $\mathbb{Z}_2$ is $(x, y) \mapsto (y, x)$. We call $\mathcal{M}_2$ the space the space of two unordered points in M.

The fixed points for this action are the diagonal embedding of M in $M \times M$ and in local coordinates, this action is given by

$$(z, w) \mapsto (w, z),$$

To describe the quotient process it is useful to take a different parametrization of $\mathbb{C}^2$. We use instead

$$x = z + w, \quad y = z - w.$$

Then the $\mathbb{Z}_2$ action in these coordinates is

$$(x, y) \mapsto (x, -y).$$

We recognise that this is nothing but the product of (the diagonal embedding of) $\mathbb{C}^n$ with the trivial action and the previous example (with the cyclic group $\mathbb{Z}_2$).

Therefore, combining Example 6.19 with the description of the blow-up along a submanifold we see that

- $\mathbb{Z}_2$ acts on $\widetilde{M \times M}$, the blow-up of $M \times M$ along the diagonal,
- this action has the exceptional divisor as fixed points,
- the quotient $\widetilde{\mathcal{M}}_2 = (\widetilde{M \times M})/\mathbb{Z}_2$ is a smooth complex manifold.

That is $\widetilde{\mathcal{M}}_2$ is a resolution of the singularities of $\mathcal{M}_2$. ■

The big result on this topic is Hironaka's Theorem on resolution of singularities of algebraic varieties. A version of this theorem for projective varieties can be stated with the language we have introduced so far.

Theorem 6.22 (Hironaka [3]). *Let $X \subset \mathbb{C}P^n$ be a singular complex submanifold. Then X has has a resolution obtained by a sequence of blow-ups of $\mathbb{C}P^n$.*

Chapter 7

Sheaves and cohomology

In Chapters 3 and 5 we introduced Dolbeault cohomology, first just for forms and then with coefficients in a holomorphic vector bundle, which somehow is the complex analogue of flat vector bundles for smooth manifolds. It turns out that singular cohomology, de Rham cohomology and Dolbeault cohomology are all examples of the more general theory of sheaf cohomology. Not only does sheaf theory provides a unified framework for all these cohomology theories, but the presence of relations between different sets of coefficients induces relations between the corresponding cohomology theories.

In this chapter we will introduce sheaf theory, sheaf cohomology, Čech cohomology and prove the Čech-to-de Rham theorem. One particularly nice application of sheaf cohomology is the proof of the Riemann–Roch theorem for complex curves that we give in the final section of this chapter.

7.1 Sheaves and presheaves

Coming from differential geometry, it is useful to think of a sheaf over a manifold M as an abstraction to be used in a similar way as we use local smooth functions. To each open set $U \subset M$ we can assign a 'space over U', namely, the smooth functions on U, $C^\infty(U)$. There are three basic properties of this space that we want to axiomatise: one can restrict functions defined on U to any open subset $V \subset U$, if we have a collection of functions $f_i \in C^\infty(U_i)$ which agree of double overlaps, then there is a function $f \in C^\infty(\cup_i U_i)$ which is equal to f_i on each U_i and if two functions f and g agree on all open subsets of a cover of M, then they agree on M. One final property of local smooth functions that is not required for the definition but is required for the development of the theory is that they have the structure of an Abelian group given by pointwise addition of functions.

Definition 7.1 (Presheaf)**.** A *presheaf*, $\mathcal{F}$, over a topological space M is

- an assignment of a set $\Gamma(U;\mathcal{F})$ to each open set $U \subset M$ called the set of *sections of* $\mathcal{F}$ *over* U,
- a collection of *restriction maps* $\{r_V^U \colon \Gamma(U;\mathcal{F}) \to \Gamma(V;\mathcal{F}) \colon V \subset U\}$,

The restriction maps are subject to the following conditions

- $r_U^U = \mathrm{Id}$ for all U,
- For $W \subset V \subset U$, $r_W^U = r_W^V \circ r_V^U$,

Definition 7.2 (Sheaf)**.** A *sheaf*, $\mathcal{F}$, over a topological space M is a presheaf over M for which whenever an open set U is the union of open sets, $U = \cup_{i\in I} U_i$ then

- if a collection of sections, $f_i \in \Gamma(U_i; \mathcal{F})$, $i \in I$ is such that
$$r_{U_{ij}}^{U_i} f_i = r_{U_{ij}}^{U_j} f_j \quad \text{for all } i, j \in I,$$
then there is $f \in \Gamma(U; \mathcal{F})$ such that $r_{U_i}^U f = f_i$.
- If $f, g \in \Gamma(U; \mathcal{F})$ satisfy $r_{U_i}^U f = r_{U_i}^U g$ for all $i \in I$ then $f = g$.

In words, the first condition says that we can patch sections that agree on overlaps to produce a section on the bigger open set and if two sections agree on all open sets of a cover, then they agree.

We have encountered several sheaves already in this course: smooth functions, holomorphic functions (on a complex manifold), smooth or holomorphic sections of bundles and in particular forms. In all these cases, sections of the sheaf over U agree with the usual notion of sections of over U (or functions defined on U). Other examples include locally constant functions with values in any fixed set, nonvanishing functions with values on $\mathbb{R}$ or $\mathbb{C}$ and maps to $\mathrm{GL}\,(k; \mathbb{C})$.

Notation. To avoid confusion, we will denote the sheaf of locally constant functions with values in a ring R by $\underline{R}$, while the sheaf of smooth functions with values in R will be denoted by $C^\infty(R)$. The sheaf of holomorphic functions will be denoted by $\mathcal{O}$.

Example 7.3 (Bounded functions)**.** An example of pre-sheaf which is not a sheaf is given by *bounded functions*. Indeed, one can easily produce a collection of functions which are bounded on open sets U_i, agree on overlaps but fail to produce a bounded function in the union. A concrete example would be the identity function, $\mathrm{Id}\colon \mathbb{R} \to \mathbb{R}$, whose restriction to $(-n, n)$ is bounded for all $n \in \mathbb{N}$, but these to not patch to produce a bounded function in $\mathbb{R}$.

One way to think of this example is that a sheaf should be defined by local properties (e.g., smoothness, holomorphicity), not global ones (e.g. bounded or specific decay at infinity). ■

Example 7.4. Consider $M = \{0, 1\}$ with the discrete topology and define a presheaf on M as follows:

$$\begin{aligned} \Gamma(\{0,1\}; \mathcal{F}) &= \{1, 0\}, \\ \Gamma(\{1\}; \mathcal{F}) = \Gamma(\{0\}; \mathcal{F}) &= \{0\}, \\ \Gamma(\emptyset; \mathcal{F}) &= \{\emptyset\}, \end{aligned}$$

and the restrictions are the only possible maps. One can check directly that this defines a presheaf for which the second sheaf condition fails.

One way to phrase the failure of the second condition (for Abelian sheaves, see below) is by saying that there are sections which restrict to zero for all open sets in a cover, but which are globally nonzero. ■

A recurring example that departs from differential geometry is the skyscraper sheaf.

Exercise 7.1 (Skyscraper sheaf)**.** Let $\iota\colon N \to M$ be an embedded submanifold and let $\mathcal{F}_N$ be a sheaf over N. Define a sheaf $\mathcal{F}$ over M as follows

- For $U \subset M$, $\Gamma(U;\mathcal{F}) = \Gamma(U \cap N, \mathcal{F}_N)$, in particular $\Gamma(U;\mathcal{F}) = \emptyset$ if $U \cap N = \emptyset$.
- For $V \subset U$ and $s \in \Gamma(U;\mathcal{F}|_N)$, let $r^U_V s = r^{U\cap N}_{V\cap N} s$.

Check that this defines a sheaf.

It is necessary to understand the difference between a presheaf and a sheaf and how to turn a presheaf into a sheaf. To do that we need the notion of germs and stalks.

Definition 7.5. Let $\mathcal{F}$ be a sheaf over M, $x \in M$. The *stalk* of $\mathcal{F}$ over x, is the direct limit

$$\mathcal{F}_x := \varinjlim_{x\in U} \Gamma(U;\mathcal{F}),$$

that is, $\mathcal{F}_x$ is a quotient space and two sections $s_i \in \Gamma(U_i;\mathcal{F})$, $i = 1, 2$, are equivalent if there is an open neighbourhood $U \subset U_1 \cap U_2$ of x such that $r^{U_1}_U s_1 = r^{U_2}_U s_2$.

Another way of phrasing this is that two sections have the same *germ* at x if they agree on a neighbourhood of x and the stalk $\mathcal{F}_x$ is made of the germs of sections defined on neighbourhoods of x.

The notion of stalk is what allows one to pass from a presheaf to a sheaf.

Proposition 7.6. *Given any presheaf, $\mathcal{F}$, there is a unique sheaf, $\overline{\mathcal{F}}$, up to isomorphism whose stalks agree with those of $\mathcal{F}$. If $\mathcal{F}$ is a sheaf, then $\overline{\mathcal{F}}$ is naturally isomorphic to $\mathcal{F}$.*

We will not provide a formal proof of this proposition, but just observce that roughly the statement is the proof. Namely, we define $\overline{\mathcal{F}}$ to be the sheaf of sections of the stalks of $\mathcal{F}$. The effect of forming $\overline{\mathcal{F}}$ is that we at the same time

- add sections to $\mathcal{F}$ by patching local sections that agree on overlaps but which might not be patchable on $\mathcal{F}$
- remove sections from $\mathcal{F}$ by identifying sections which are different on $\mathcal{F}$ but agree on all opens sets of a cover.

Moving on, we start putting more structure in our sheaves.

Definition 7.7. A sheaf $\mathcal{F}$ is *Abelian* if for every U, $\Gamma(U;\mathcal{F})$ has the structure of an Abelian group and this is structure is compatible with restrictions, i.e.,

$$r^U_V(s_1 + s_2) = r^U_V(s_1) + r^U_V(s_2).$$

For example, sections of vector bundles with pointwise addition and nonvanishing $\mathbb{R}$ or $\mathbb{C}$-valued functions with pointwise multiplication are Abelian, while functions with values in $\mathrm{GL}\,(k;\mathbb{R})$ or $\mathrm{GL}\,(k;\mathbb{C})$ do not form an Abelian sheaf for $k > 1$.

Similarly to the notion of Abelian sheaf, there is the notion of a sheaf of commutative rings, R-modules, for a ring R and more, obtained by requiring that $\Gamma(U;\mathcal{F})$ are commutative rings, R-modules for every U, etc. Also given two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ we can form the sheaf $\mathcal{F}_1 \oplus \mathcal{F}_2$ whose sections are pairs of sections of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Definition 7.8. Given two sheaves, $\mathcal{F}$ and $\mathcal{G}$ over M, a *sheaf morphism* between $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{A}\colon \mathcal{F} \to \mathcal{G}$ is a collection of maps

$$\mathcal{A}|_U\colon \Gamma(U;\mathcal{F}) \to \Gamma(U;\mathcal{G})$$

defined for every open set $U \subset M$ which is compatible with restrictions, that is

$$r_V^U \circ \mathcal{A}|_U = \mathcal{A}|_V \circ r_V^U.$$

If the sheaves $\mathcal{F}$ and $\mathcal{G}$ have further structure (e.g., are Abelian, R-modules, etc) we require A to be compatible with that structure.

Definition 7.9. Let $\mathcal{A}\colon \mathcal{F} \to \mathcal{G}$ be a sheaf morphism between Abelian sheaves,

- the *image* of $\mathcal{A}$ is the sheaf generated by the images of stalks of $\mathcal{F}$ via the map $\mathcal{A}$,
- the *kernel* of $\mathcal{A}$ is the sheaf generated by elements of the stalks that are mapped to 0 via A.

Definition 7.10. Let $\mathcal{H} \subset \mathcal{F}$ be a subsheaf of an Abelian sheaf. The *quotient sheaf* $\mathcal{Q} = \mathcal{F}/\mathcal{H}$ is the sheaf whose stalks are the quotients of the stalks of $\mathcal{F}$ by the stalks of $\mathcal{H}$.

Example 7.11. Let M be a manifold together with the sheaf of smooth functions. Let $\lambda \in \mathbb{C}^\infty(M)$ be a smooth function which has 0 as a regular value and define a map of sheaves by

$$\mathcal{A}\colon f \in \Gamma(U;\mathbb{C}^\infty) \mapsto \lambda f \in \Gamma(U;\mathbb{C}^\infty).$$

If we let $N = \lambda^{-1}(0)$, we see that the image of $\mathcal{A}$ consists of functions which vanish along N while the kernel is trivial because if $\lambda f = 0$ in U, since $\lambda \neq 0$ in the complement of N this relation only holds if $f = 0$ in the complement of N and by continuity $f = 0$.

Notice that even though $\mathcal{A}$ is injective, it is not surjective, since its image is made of functions which vanish on N. We can therefore form the quotient sheaf $\mathcal{Q} = \Gamma(U;\mathbb{C}^\infty)/\mathrm{Im}(\mathcal{A})$. For any open set U that does not intersect N, $\mathrm{Im}(\mathcal{A}) = \Gamma(U;\mathbb{C}^\infty)$ and therefore the quotient is trivial. If an open set U intersects N, then the class a section $f \in \Gamma(U;\mathbb{C}^\infty)$ determines in the quotient is completely determined by the values f takes on N. That is $\mathcal{Q}$ is the skyscraper sheaf on M obtained from the smooth functions on N. ■

Exercise 7.2. Let $\lambda\colon \mathbb{C}^2 \to \mathbb{C}$ be given by $\lambda(z_1, z_2) = z_1 z_2$ and consider the map of sheaves

$$\mathcal{A}\colon \mathcal{O} \to \mathcal{O}, \qquad \mathcal{A}(f) = \lambda f.$$

Determine the kernel, the image and the quotient $\mathcal{O}/\mathrm{Im}(\mathcal{A})$.

Consider the morphism

$$\hat{\mathcal{A}}\colon \mathcal{O} \to \mathcal{O}, \qquad \hat{\mathcal{A}}(f) = \lambda^2 f,$$

and determine again the kernel, the image and the quotient $\mathcal{O}/\mathrm{Im}(\hat{\mathcal{A}})$.

Definition 7.12. A sheaf of R-modules, $\mathcal{F}$ is *free* if $\Gamma(U;\mathcal{F})$ is isomorphic to $\Gamma(U;C^\infty(R)) \oplus \cdots \oplus \Gamma(U;C^\infty(R))$.

A sheaf of R-modules, $\mathcal{F}$ is *locally free* if every point $x \in M$ has a neighbourhood U in which $\Gamma(U;\mathcal{F})$ is isomorphic to $\Gamma(U;C^\infty(R)) \oplus \cdots \oplus \Gamma(U;C^\infty(R))$.

An example of a free sheaf is that of smooth functions with values in $\mathbb{R}^k$ where R above is taken to be the sheaf of smooth real valued functions. An example of a locally free sheaf is that of sections of a given vector bundle $E \to M$, due to the fact that vector bundles are locally trivial.

The notion of locally free sheaf completely encodes the information about vector bundles. Below we let $\mathcal{S}$ denote either of the following sheaves: continuous, smooth or holomorphic functions (the latter, on a complex manifold)

Theorem 7.13. *Let M be a connected manifold and let $\mathcal{F}$ be a locally free sheaf of $\mathcal{S}$-modules. Then $\mathcal{F}$ is the sheaf of sections of some $\mathcal{S}$-vector bundle over M.*

Proof. The map from vector bundles to locally free sheaves is just the one that to each vector bundle associates the sheaf of sections of E. Conversely, given a locally free sheaf, consider an open cover of M over which the sheaf is free, $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ and fix isomorphisms $\Phi_\alpha \colon \Gamma(U;\mathcal{F}) \to \mathcal{S} \oplus \cdots \oplus \mathcal{S}$. For $\alpha, \beta \in A$ with $U_{\alpha\beta} \neq \emptyset$ we can compare the corresponding isomorphisms by considering $\Phi_\beta \circ \Phi_\alpha^{-1}$. Since this must preserve the module structure over $\mathcal{S}$ (which is given by fiberwise vector space structure of $\mathcal{S} \oplus \cdots \oplus \mathcal{S}$), we conclude that there is $g_\beta^\alpha \colon U_{\alpha\beta} \to \mathrm{GL}\,(k)$ such that $\Phi_\beta \circ \Phi_\alpha^{-1}(s(x)) = g_\beta^\alpha(x)s(x)$. By cycling back to the open set U_α we see that $\breve{g} = \{g_\beta^\alpha \colon U_{\alpha\beta} \to \mathrm{GL}\,(k) \colon \alpha, \beta \in A\}$ is a skew symmetric cocycle which therefore determines a vector bundle. The regularity of the vector bundle depends on the regularity of the functions g_β^α which in turn agrees with that of the sheaf $\mathcal{S}$ we started with. □

Before we continue with the meat of the theory, we need to take a step back and introduce a few basic concepts, such as maps between sheave, kernels and images.

Definition 7.14. Given two sheaves, $\mathcal{F}$ and $\mathcal{G}$ over M, a *sheaf morphism* between $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{A} \colon \mathcal{F} \to \mathcal{G}$ is a collection of maps

$$\mathcal{A}|_U \colon \Gamma(U;\mathcal{F}) \to \Gamma(U;\mathcal{G})$$

defined for every open set $U \subset M$ which is compatible with restrictions, that is

$$r_V^U \circ \mathcal{A}|_U = \mathcal{A}|_V \circ r_V^U.$$

If the sheaves $\mathcal{F}$ and $\mathcal{G}$ have further structure (e.g., are Abelian, R-modules, etc) we require A to be compatible with that structure.

The definition of image and kernel of a sheaf morphism requires a little detour via germs of sections, or, since the name is *sheaf* theory, *stalks* of sections ;-)

Definition 7.15. Let $\mathcal{F}$ be a sheaf over M, $x \in M$. The *stalk* of $\mathcal{F}$ over x, is the direct limit

$$\mathcal{F}_x := \varinjlim_{x \in U} \Gamma(U;\mathcal{F}),$$

that is, $\mathcal{F}_x$ is a quotient space and two sections $s_i \in \Gamma(U_i;\mathcal{F})$, $i = 1, 2$, are equivalent if there is an open neighbourhood $U \subset U_1 \cap U_2$ of x such that $r_U^{U_1} s_1 = r_U^{U_2} s_2$.

Another way of phrasing this is that two sections have the same *germ* at x if they agree on a neighbourhood of x and the stalk $\mathcal{F}_x$ is made of the germs of sections defined on neighbourhoods of x.

One very important property of smooth functions on manifolds is that they admit partitions of unity. That property is key to so many results that we should have a corresponding notion for sheaves:

Definition 7.16. A *partition of unit* of an Abelian sheaf $\mathcal{F}$ subbordinate to a locally finite open cover $\mathcal{U} = \{U_i : i \in I\}$ of M is a collection of sheaf morphisms $\eta_i : \mathcal{F} \to \mathcal{F}$, $i \in I$ such that

- $\sum_{i\in I} \eta_i = 1$,
- $\eta_i(\mathcal{F}_x) = 0$ if x has a neighbourhood that does not intersect U_i.

Of all types of Abelian sheaves, those that admit partitions of unity play a special role:

Definition 7.17. An Abelian sheaf $\mathcal{F}$ over a manifold M is *fine* if for every locally finite open cover $\mathcal{U}$ of M there is a partition of unity of $\mathcal{F}$ subbordinate to $\mathcal{U}$

For example smooth functions or smooth sections of a vector bundle are fine sheaves because those are modules over C^∞ and smooth functions admit partitions of unity. Locally constant functions, nonvanishing functions or holomomorphic functions are not fine sheaves (if you multiply a locally constant function by a partition of unity it ceases to be locally constant).

Finally, sometimes sheaves appear in families, in the form of a complex:

Definition 7.18. A *complex of sheaves* over a manifold M is a collection of Abelian sheaves, $\{\mathcal{F}_i : i \in \mathcal{N}\}$ and maps of groups d_i such that $d_i \circ d_{i-1} = 0$. A complex is *exact* if $\ker(d_i) = \mathrm{Im}(d_{i-1})$ as sheaves for all i, that is, there is no local cohomology.

Definition 7.19. A *resolution* of aa Abelian sheaf $\mathcal{F}$ over M is an exact complex of sheaves $\{(\mathcal{F}_i, d_i) : i \geq -1\}$ with $F_{-1} = \mathcal{F}$ and $\mathcal{F}_i$ fine for $i \geq 0$:

$$0 \to \mathcal{F} \xrightarrow{d_{-1}} \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \cdots$$

Example 7.20. Consider the sheaf $\underline{\mathbb{R}}$ of locally constant real functions on M. Since each locally constant function is smooth, we have a natural inclusion of sheaves $\iota : \underline{\mathbb{R}} \to C^\infty(\mathbb{R})$. The exterior derivative provides a continuation of this inclusion into a complex.

$$0 \to \underline{\mathbb{R}} \to C^\infty(\mathbb{R}) \xrightarrow{d} \Omega^1 \to \xrightarrow{d} \Omega^2 \to \ldots$$

By the Poincaré Lemma, this is an exact sequence of sheaves. Since smooth functions and smooth sections of vector bundles admit partitions of unity, this is a resolution of the sheaf of locally constant functions on M. ■

Example 7.21. Consider the sheaf $\mathcal{O}$ of holomorphic functions over a complex manifold M. Since each holomorphic function constant function is smooth, we have a natural inclusion of sheaves $\iota : \mathcal{O} \to C^\infty$. The $\overline{\partial}$ operator provides a continuation of this inclusion into a complex.

$$0 \to \mathcal{O} \to C^\infty(\mathbb{C}) \xrightarrow{\overline{\partial}} \Omega^{0,1} \to \xrightarrow{\overline{\partial}} \Omega^{0,2} \to \ldots$$

By the Poincaré holomorphic Lemma, this is an exact sequence of sheaves. Since smooth functions and smooth sections of vector bundles admit partitions of unity, this is a resolution of the sheaf of holomorphic functions on M. ■

Exercise 7.3. Find a sheaf resolution for the sheaf of sections of a holomorphic vector bundle E over a complex manifold M.

Example 7.22. Let $\mathcal{F}$ be a fine sheaf over M, then

$$0 \to \mathcal{F} \overset{\mathrm{Id}}{\to} \mathcal{F} \to 0$$

is a sheaf resolution of $\mathcal{F}$. ■

Definition 7.23. Given an Abelian sheaf $\mathcal{F}$ over M we define the cohomology of $\mathcal{F}$ to be the cohomology of the complex

$$0 \to \Gamma(M; \mathcal{F}_0) \overset{d_0}{\to} \Gamma(M; \mathcal{F}_1) \overset{d_1}{\to} \Gamma(M; \mathcal{F}_2) \overset{d_2}{\to} \dots,$$

where $\{(\mathcal{F}_i, d_i) \colon i \geq -1\}$ is a sheaf resolution of $\mathcal{F}$.

This definition is very "green" at this stage since we did not establish the existence of resolutions and even if those exist, it is not clear at all that the corresponding cohomology groups are independent of the chosen resolution. Regarding the first of these problem we will just state the relevant result:

Theorem 7.24. *Every Abelian sheaf admits a canonical sheaf resolution.*

This result seems excellent: it has the word "canonical" in it, and the proof is constructive. It is a big bummer that it turns out to be useless as well, so we will not present the proof here. Steps towards the proof of the independence of chosen sheaf resolution will be done later, but for now it is interesting to look back at the examples of resolutions we encountered before and see what the corresponding sheaf cohomology becomes.

Example 7.25. Returning to Example 7.20, we see that for the resolution presented there we have that the cohomology of the sheaf of locally constant functions is the cohomology of the complex

$$0 \to \Omega^0(M; \mathbb{R}) \overset{d}{\to} \Omega^1(M; \mathbb{R}) \to\overset{d}{\to} \Omega^2(M; \mathbb{R}) \to \dots$$

That is, the cohomology of the sheaf of locally constant functions is the de Rham cohomology of M:

$$H^k(M; \underline{\mathbb{R}}) \cong H^k(M; \mathbb{R}).$$

■

Exercise 7.4. Show that the sheaf cohomology of the sheaf of holomorphic sections of a holomorphic vector bundle E over a complex manifold M is the Dolbeault cohomology of M with coefficients on E:

$$H^k(M; \Gamma(E)) = H^k(M; E).$$

Exercise 7.5. Show that if $\mathcal{F}$ is a fine sheaf, then the sheaf cohomology of $\mathcal{F}$ is

$$\begin{cases} H^0(M; \mathcal{F}) & = \Gamma(M; \mathcal{F}); \\ H^i(M; \mathcal{F}) & = \{0\}, \quad \text{for } i > 0 \end{cases}$$

7.2 Čech cohomology

One way to think of Čech cohomology is that it attempts to recover information about the shape of a manifold from combinatorial data regarding an open cover. The point of view being that if we knew how many open sets are in a (good) cover as well as how they overlap each other then we might be able to recover information about the topology of the manifold. Taking a step further, if we knew information about sections of a specific sheaf over the open sets of a (good) cover, we might obtain global information about that sheaf. So without further ado, let's introduce the main concepts.

Definition 7.26. Let $\mathcal{F}$ be a sheaf of Abelian groups over a manifold M and let $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ be a locally finite open cover of M, then a *k-Čech cochain on M with coefficients in $\mathcal{F}$ with respect to the cover $\mathcal{U}$* is

$$\check{f} = \{f_{\alpha_0,\dots,\alpha_k} \in \Gamma(U_{\alpha_0,\dots,\alpha_k}) \colon \alpha_i \in A, f_{\alpha_0,\dots,\alpha_i,\alpha_{i+1},\dots,\alpha_k} = -f_{\alpha_0,\dots,\alpha_{i+1},\alpha_i,\dots,\alpha_k}\},$$

where, $U_{\alpha_0,\dots,\alpha_k} = U_{\alpha_0} \cap \dots \cap U_{\alpha_k}$.

The *space of k-Čech cochains on M with coefficients in $\mathcal{F}$ with respect to the cover $\mathcal{U}$* is

$$\check{C}^k(M;\mathcal{F},\mathcal{U}) = \{\check{f} \colon \check{f} \text{ is a } k\text{-Čech cochain on } M \text{ w.r.t. } \mathcal{U}\}.$$

To be clear, each Čech cochain is a set, namely a family of sections defined on multiple overlaps and the space of Čech cochains is the set containing such sets.

Example 7.27. Consider the sheaf of locally constant real valued functions over the circle, $S^1 = \mathbb{R}/\mathbb{Z}$ and take $\mathcal{U}$ to be the open cover of S^1 given by open sets

$$U_a = (0, 2/3), \quad U_b = (1/3, 1), \quad U_c = (2/3, 1/3).$$

Then an element $\check{f} \in \check{C}^0 = \check{C}^0(S^1;\underline{\mathbb{R}},\mathcal{U})$ consists of three locally constant functions, one for each U_i. Since each U_i is connected, each locally constant function making up $\check{f}$ is in fact constant so $\check{f}$ is determined by three real numbers and $\check{C}^0 \cong \mathbb{R}^3$.

Similarly, there are three double overlaps of sets in $\mathcal{U}$ and an element in $\check{C}^1 = \check{C}^1(S^1;\underline{\mathbb{R}},\mathcal{U})$ corresponds to a collection of three locally contant functions defined on these (connected) double overlaps, so we have also that $\check{C}^1 \cong \mathbb{R}^3$.

The triple intersection of the open sets in $\mathcal{U}$ is empty and, since $\mathcal{U}$ has only three sets and Čech cocycles are skew-symmetric, the higher degree Čech cochains are trivial.

If replace $\underline{\mathbb{R}}$ by $C^\infty(\mathbb{R})$ the space of Čech cochains changes drastically. For example, $\check{C}^0(S^1;C^\infty(\mathbb{R}),\mathcal{U})$ would correspond all collections of three smooth functions defined on an interval. ■

Notice that the space of degree k-Čech cochains also has the structure of a group with multiplication is given by

$$(\check{f} + \check{g})_{\alpha_0\dots\alpha_k} = (\check{f})_{\alpha_0\dots\alpha_k} + (\check{g})_{\alpha_0\dots\alpha_k}.$$

Definition 7.28. Let $\mathcal{F}$ be a sheaf of Abelian groups over a manifold M and let $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ be a locally finite open cover of M. We define the *Čech differential* as the map

$$\delta \colon \check{C}^k(M;\mathcal{F},\mathcal{U}) \to \check{C}^{k+1}(M;\mathcal{F},\mathcal{U}), \qquad \delta(\check{f})_{\alpha_0,\dots\alpha_{k+1}} = \Sigma_{i=0}^{k+1}(-1)^i \check{f}_{\hat{\alpha_i}},$$

where $f_{\hat{\alpha_i}} = f_{\alpha_0,\dots\alpha_{i-1},\alpha_{i+1}\dots\alpha_{k+1}}$.

Exercise 7.6. Check that $\delta \check{f}$ defined above is indeed a Čech cochain and that δ is a group homomorphism.

Proposition 7.29. $\delta^2 = 0$.

Proof. This is a direct computation from the definition

$$\begin{aligned}(\delta^2 \check{f})_{\alpha_0,\ldots\alpha_{k+2}} &= \Sigma_i (-1)^i (\delta \check{f})_{\hat{\alpha}_i} \\ &= \left(\Sigma_{j<i} (-1)^{j+i} (\delta \check{f})_{\hat{\alpha}_j \hat{\alpha}_i}\right) \cdot \left(\Sigma_{i<j} (-1)^{j+i-1} (\delta \check{f})_{\hat{\alpha}_i \hat{\alpha}_j}\right) \\ &= 0\end{aligned}$$

□

This allows us to introduce the cohomology language to the Čech world.

Definition 7.30. A Čech cochain, $\check{f}$, is

- a *(Čech) cocycle* if $\delta \check{f} = 0$,
- a *(Čech) coboundary* if $\check{f} \in \mathrm{Im}(\delta)$.

Definition 7.31. Let $\mathcal{U}$ be an open cover of a manifold M and let $\mathcal{F}$ be a sheaf of Abelian groups over M. The *Čech cohomology of M with values in $\mathcal{F}$ with respect to the cover $\mathcal{U}$* is

$$\check{H}^k(M;\mathcal{F},\mathcal{U}) = \frac{\ker(\delta\colon \check{C}^k(M;\mathcal{F},\mathcal{U}) \to \check{C}^{k+1}(M;\mathcal{F},\mathcal{U})}{\mathrm{Im}(\delta\colon \check{C}^{k-1}(M;\mathcal{F},\mathcal{U}) \to \check{C}^{k}(M;\mathcal{F},\mathcal{U})}.$$

Even at this stage where Čech cohomology seems to depend on the cover we can already see something interesting happening.

Lemma 7.32. *For any cover $\mathcal{U}$,*

$$\check{H}^0(M;\mathcal{F},\mathcal{U}) = \Gamma(M;\mathcal{F}).$$

Proof. Since there are no cochains of degree -1, we only need to describe the the kernel of δ. This can be done directly:

$$\begin{aligned}\delta \check{f} = 0 &\Leftrightarrow \check{f}_\alpha = \check{f}_\beta \quad \forall \alpha, \beta \\ &\Leftrightarrow \exists f \in \Gamma(M;\mathcal{F}) \text{ such that } f|_{U_\alpha} = \check{f}_\alpha, \quad \forall \alpha.\end{aligned}$$

□

Higher degree Čech cohomology is mysterious, as are all higher degree cohomologies, yet we have already encountered one of them before.

Theorem 7.33. *Complex (resp. real) line bundles over M are in one-to-one correspondence with Čech cohomology classes in $\check{H}^1(M; C^\infty(\mathbb{C}^*),\mathcal{U})$ (resp. $\check{H}^1(M; C^\infty(\mathbb{R}^*),\mathcal{U})$) for any cover $\mathcal{U}$ of M by open sets diffeomorphic to discs.*

Proof. Consider the sheaf $C^\infty(\mathbb{C}^*)$, of nonvanishing complex valued functions, on a manifold M and let $\mathcal{U}$ be an open cover of M. Then a 1-Čech cochain for this cover is

$$\check{g} = \{(\check{g})_{\alpha\beta}\colon U_{\alpha\beta} \to \mathbb{C}^*\colon (\check{g})_{\alpha\beta} = (\check{g})^{-1}_{\beta\alpha}, \}$$

where the $\cdot^{-1}$ appears because the Abelian group in question is expressed multiplicatively, instead of additively, as we have been doing so far. Such a cochain is a cocycle if

$$1 = (\delta\check{g})_{\alpha\beta\gamma} = g_{\beta\gamma}\cdot (g_{\alpha\gamma})^{-1}\cdot g_{\alpha\beta} = g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha}.$$

That is, Čech cocycles classify complex line bundles over M with trivialisations over the cover $\mathcal{U}$ (c.f. Lemma 5.3).

Further, two cocycles $\check{g}$ and $\check{g}'$ are cohomologous if there is a degree zero Čech cochain, $\check{f}$, for which

$$(\check{g}')_{\alpha\beta} = (\check{g})_\alpha\beta\cdot(\delta\check{f})_{\alpha\beta} = \check{f}_\beta\cdot(\check{g})_{\alpha\beta}\cdot\check{f}_\alpha^{-1}.$$

That is, $\check{H}^1(M; C^\infty(\mathbb{C}^*),\mathcal{U})$ classifies complex lines bundles that can be trivialised over the cover $\mathcal{U}$. (c.f. Theorem 5.4).

The theorem then holds because every vector bundle over a set diffeomorphic to a disc is trivialisable. $\square$

Notice that to extend the result from Theorem 7.33 to holomorphic line bundles over a complex manifold the only missing ingredient is the proof that such bundles are trivialisable over specific open covers (those whose open sets are biholomorphic to polydiscs).

Interestingly in the two cases above we cound find conditions on the cover that guaranteed the Čech cohomology under consideration was in fact independent of the open cover used. This is not the case in general and a limit must be taken.

So, continuing with Čech cohomology theory we need to take the limit as we refine the cover. To do so, we observe that if $\mathcal{V}$ is a refinement from a cover $\mathcal{U}$, for each $V_\beta \in \mathcal{V}$ we can choose a $U_\alpha \in U$ such that $V_\beta \subset U_\alpha$. That is, we can find a map $\tau\colon B \to A$, from the index set of $\mathcal{V}$ to the index set of $\mathcal{U}$, such that $V_\beta \subset U_{\tau(\beta)}$ for all β. This map gives rise to a correponding map of Čech cochains via restrictions:

$$\tau^*\colon \check{C}^k(M;\mathcal{F},\mathcal{U}) \to \check{C}^k(M;\mathcal{F},\mathcal{V}), \quad (\tau\check{f})_{\beta_0\dots\beta_k} = (\check{f})_{\tau(\beta_0)\dots\tau(\beta_k)}|_{V_{\beta_0\dots\beta_k}}.$$

Even though the map at the cochain level clearly depends on the choice of τ, the induced map in cohomology does not.

Lemma 7.34. *For any choice of map between index sets $\tau\colon B \to A$ as above, $\tau^*\delta = \delta\tau^*$, hence τ^* descend to a map in cohomology. Further if $\tilde{\tau}$ is another choice of map for which $V_\beta \subset U_{\tilde{\tau}(\beta)}$ for all β, then τ^* and $\tilde{\tau}^*$ induce the same map in Čech cohomology.*

Proof. The first claim is clear so we only need to check that the map induced in cohomology by τ and $\tilde{\tau}$ agree. $\square$

We call the induced map in cohomology the *refinement map.*

Definition 7.35. The *Čech cohomology of M with values in $\mathcal{F}$*, $\check{H}^k(M;\mathcal{F})$ is the direct limit of $\check{H}^k(M;\mathcal{F},\mathcal{U})$ under refinement of covers.

In words, $\check{H}^k(M;\mathcal{F})$ is made of cohomology classes in $\check{H}^k(M;\mathcal{F},\mathcal{U})$ for any open cover $\mathcal{U}$ and two classes $[\check{g}_i] \in \check{H}^k(M;\mathcal{F},\mathcal{U}_i)$, $i = 1, 2$, are equivalent classes in $\check{H}^k(M;\mathcal{F})$ if there is a common refinement $\mathcal{V}$ of $\mathcal{U}_1$ and U_2 for which the images of $[\check{g}_i]$ under the corresponding refinement maps agree.

It will be awesome if we can find enough cases for which the process of taking the limit is not necessary.

Proposition 7.36. *Let $\mathcal{F}$ be a fine sheaf over M. Then*

$$\begin{cases} \check{H}^0(M;\mathcal{F},\mathcal{U}) & = \Gamma(M;\mathcal{F}); \\ \check{H}^i(M;\mathcal{F},\mathcal{U}) & = \{0\}, \quad \textit{for } i > 0. \end{cases}$$

In particular the Čech cohomology of a fine sheaf does not depend on the cover.

Proof. The computation of $\check{H}^0$ was done in Lemma 7.32, so we only need to do the hard part now.

Let $\{\psi_\alpha \colon \alpha \in A\}$ be a partition of unit subordinate to the cover $\mathcal{U}$ and let $\check{g} \in \check{C}^k(M;\mathcal{F},\mathcal{U})$ be a k-cocycle. We can write the cocycle condition as

$$(\check{g})_{\alpha_1\dots\alpha_{k+1}} = -\sum_{i=0}^{k}(-1)^i(\check{g})_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_{k+1}}$$

on $U_{\alpha_1\dots\alpha_{k+1}}$.

The way the partition of unit comes in is that $\psi_{\alpha_0} g_{\alpha_0\dots\alpha_k}$ can be extended to $\Gamma(U_{\alpha_1\dots\alpha_k}, \mathcal{F})$ by declaring that vanishes outside U_{α_0}. Since the cover is locally finite we can define a $k-1$-cochain, $\check{f}$, by

$$(\check{f})_{\alpha_0,\dots,\alpha_{k-1}} = \sum_\alpha \psi_\alpha g_{\alpha,\alpha_0,\dots\alpha_{k-1}}.$$

And, by what can only be described as a miracle, we have that $\delta\check{f} = \check{g}$:

$$\begin{aligned}
(\delta\check{f})_{\alpha_0\dots\alpha_k} &= \sum_{i=0}^{k}(-1)^i\sum_\alpha \psi_\alpha \check{g}_{\alpha,\alpha_0,\dots\hat{\alpha}_i\dots\alpha_k} \\
&= \sum_\alpha \psi_\alpha \sum_{i=0}^{k}(-1)^i \check{g}_{\alpha,\alpha_0,\dots\hat{\alpha}_i\dots\alpha_k} \\
&= \sum_\alpha \psi_\alpha \check{g}_{\alpha_0,\dots\alpha_k} \\
&= \check{g}_{\alpha_0,\dots\alpha_k}.
\end{aligned}$$

□

Exercise 7.7. Let M be a manifold, $p \in M$ and let $\mathbb{C}_p$ be the skyscraper sheaf on M supported on p with fiber $\mathbb{C}$ over p. Compute the Čech cohomology of M with coefficients in $\mathbb{C}_p$.

Finally, when there are relations between the sheaves under consideration, we get corresponding relations between the Čech cohomologies. Indeed, since the Čech differential is purely combinatorial, if $\mathcal{A}\colon \mathcal{F} \to \mathcal{G}$ is a morphism of Abelian sheaves, then it induces a corresponding map between Čech cochains, $\mathcal{A}^*\colon \check{C}^k(M;\mathcal{F},\mathcal{U}) \to \check{C}^k(M;\mathcal{G},\mathcal{U})$ which commutes with differentials and hence induces a map in cohomology,

$$\mathcal{A}^*\colon \check{H}^k(M;\mathcal{F},\mathcal{U}) \to\to \check{H}^k(M;\mathcal{G},\mathcal{U}).$$

If $\mathcal{A}$ is injective or surjective (and the cover is fine enough) so is the map $\mathcal{A}^*$ at the cochain level. In particular, if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of Abelian sheaves, we obtain a short exact sequence of differential complexes

$$0 \to \check{C}^\bullet(M;\mathcal{F},\mathcal{U}) \to \check{C}^\bullet(M;\mathcal{G},\mathcal{U}) \to \check{C}^\bullet(M;\mathcal{H},\mathcal{U}) \to 0.$$

And any such sequence gives rise to a long exact sequence in cohomology:

$$\cdots \check{H}^k(M;\mathcal{F},\mathcal{U}) \to \check{H}^k(M;\mathcal{G},\mathcal{U}) \to \check{H}^k(M;\mathcal{H},\mathcal{U}) \to \check{H}^{k+1}(M;\mathcal{F},\mathcal{U}) \to \cdots$$

We sumarise this discussion in the following theorem:

Theorem 7.37. *A short exact sequence of Abelian sheaves*

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{q} \mathcal{H} \to 0$$

gives rise to a long exact sequence in cohomology

$$\cdots \check{H}^k(M;\mathcal{F},\mathcal{U}) \xrightarrow{i^*} \check{H}^k(M;\mathcal{G},\mathcal{U}) \xrightarrow{q^*} \check{H}^k(M;\mathcal{H},\mathcal{U}) \longrightarrow \check{H}^{k+1}(M;\mathcal{F},\mathcal{U}) \to \cdots$$

Exercise 7.8. Show that the following is an exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} C^\infty(\mathbb{C}) \xrightarrow{\exp} C^\infty(\mathbb{C}^*) \to 0.$$

Conclude that complex line bundles are classified by $\check{H}^2(M;\mathbb{Z},\mathcal{U})$ for any open cover $\mathcal{U}$ for which each element of $\mathcal{U}$ is diffeomorphic to a disc and and each double overlap is either empty or diffeomorphic to a disc.

Definition 7.38. The *first Chern class* of a complex line bundle $L \to M$, is the class in $\check{H}^2(M;\mathbb{Z},\mathcal{U})$ corresponding to L.

Exercise 7.9. Let Σ be a Riemann surface, let $p \in \Sigma$ and let $L_p \to \Sigma$ be the holomorphic line bundle which has a section s which vanishes only at p with a simple zero (c.f. Exercise 5.7). Let $E \to \Sigma$ be a holomorphic line bundle. Show that we have an exact sequence of sheaves

$$0 \to \Gamma(E) \xrightarrow{\otimes s} \Gamma(E \otimes L_p) \xrightarrow{\mathrm{ev}_p} E_p \to 0,$$

where ev_p is the evaluation of a section at p and E_p is the skyscraper sheaf corresponding to the bundle $E_p \to \{p\}$ together with the inclusion $\{p\} \to \Sigma$.

7.3 Čech-to-de Rham

So far we have introduced two cohomology theories associate to a sheaf $\mathcal{F}$ over a manifold M: sheaf cohomology, which in principle depends on a choice of sheaf resolution, and Čech cohomology, which, before taking a limit, depends on a choice of cover. In this section we study a general situation in which these dependencies do not hold.

Definition 7.39. Given a sheaf $\mathcal{F}$ over M and a resolution of $\mathcal{F}$,

$$0 \to \mathcal{F} \xrightarrow{d} \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1 \xrightarrow{d} \cdots$$

a *fine cover of M for the resolution of $\mathcal{F}$* is a locally finite open cover $\mathcal{U} = \{U_\alpha \colon \alpha \in A\}$ such that $H^i(M; \mathcal{F}, U_{\alpha_0, \ldots \alpha_k}) = \{0\}$ for $i > 0$.

Remark. The question of course is how to decide, in concrete situations, if a given cover is fine for a given resolution. As we saw in Examples 7.20 and 7.21, the Poincaré Lemma and the holomorphic Poincaré Lemma seem to be the tools we need to establish the existence of fine covers for the sheaves $\underline{\mathbb{R}}$ and $\mathcal{O}$. If we could establish that for any (complex) manifold there is a cover whose open sets are diffeomorphic to (poly-)discs and whose multiple overlaps are either diffeomorphic to (poly)-discs or empty, then we would have found our fine open cover for those resolutions. This is probably wishful, but gives you the correct gist of how to go about finding a fine cover for those resolutions.

To understand the point that needs attention, let's first consider the case of smooth manifolds and the sheaf $\underline{\mathbb{R}}$. One way to produce a fine cover for the resolution of $\underline{\mathbb{R}}$ given by differential forms is to take small geodesic balls. One then argues that if these balls are taken small enough they are *geodesically convex*, that is, the curve of minimal length connecting points in each ball lies in that ball. Hence the intersection of two such balls is also geodesically convex and therefore the multiple intersections are *star shaped domains* and the Poincaré lemma holds for any such domain. Notice that by dealing with star shaped domains we dodged the question of whether these domains are diffeomorphic to balls.

One can similarly argue that if M is a complex manifold, the cover described above is also fine for the resolution of $\mathcal{O}$ given in Example 7.21. Since the intersection of discs (or of polydiscs) may not be biholomorphic to a disc (or a polydisc), the vanishing of the cohomology of the multiple intersections for such a cover requires a more refined version of the holomorphic Poincaré Lemma. Such a version also exists, but must take into account convexity properties of the boundary of such intersections.

Theorem 7.40 (Čech-to-de Rham)**.** *Let $\mathcal{F}$ be an Abelian sheaf over M, let*

$$0 \to \mathcal{F} \xrightarrow{d} \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1 \xrightarrow{d} \cdots$$

be a sheaf resolution of $\mathcal{F}$ and let $\mathcal{U}$ be a fine cover of M for the given resolution of $\mathcal{F}$. Then

$$H^k(M; \mathcal{F}) \cong \check{H}^k(M; \mathcal{F}, U).$$

Proof. To relate these two cohomology theories we create a larger double complex that includes information about the resolution and Čech cochains. The proof amounts to showing that the cohomology of this larger complex is isomorphic to the Čech and to the sheaf cohomology.

We form the spaces

$$E_0^{p,q} = \check{C}^q(M; \mathcal{F}_p, \mathcal{U}), \qquad p, q \geq 0.$$

There are two differentials we can consider here, the one coming from the resolution of $\mathcal{F}$ and one from Čech cohomology theory. Since the Čech differential is purely combinatorial and the sheaf differential is a group homomorphism we see that these two differentials commute:

$$d\colon E_0^{p,q} \to E^{p+1,q}, \quad \delta\colon E_0^{p,q} \to E^{p,q+1}, \quad d\delta - \delta d = 0.$$

We combine the (p,q)-grading into a total degree by declaring that elements in $E_0^{p,q}$ have degree $p+q$. Similarly we can combine the two differentials into a single degree-one differential:

$$D\colon \oplus_{p+q=k} E_0^{p,q} \to \oplus_{p+q=k+1} E_0^{p,q}, \quad D = d + (-1)^p \delta.$$

We compute D^2 by applying it to an element $\rho \in E_0^{p,q}$:

$$D^2\rho = D(d\rho + (-1)^p\delta\rho) = d^2\rho + (-1)^{p+1}\delta d\rho + (-1)^p d\delta\rho + \delta^2\rho = 0.$$

This allows to define the cohomology of $(E_0^{\bullet,\bullet}, D)$:

$$\mathbb{H}^k = \frac{\ker(D\colon \oplus_{p+q=k} E_0^{p,q} \to \oplus_{p+q=k+1} E_0^{p,q})}{\operatorname{Im}(D\colon \oplus_{p+q=k-1} E_0^{p,q} \to \oplus_{p+q=k} E_0^{p,q})}$$

Notice that we have natural inclusions of differential complexes:

$$i\colon (\Gamma(M; \mathcal{F}_\bullet), d) \hookrightarrow (E_0^{\bullet,0}, D) \subset (E_0^{\bullet,\bullet}, D), \qquad j\colon (\check{C}^\bullet(M; \mathcal{F}), \delta) \hookrightarrow (E_0^{0,\bullet}, D) \subset (E_0^{\bullet,\bullet}, D),$$

and therefore we have corresponding maps, i^* and j^* in cohomology. The theorem is proved by showing that i^* and j^* are isomorphisms.

The proof relies on the repeated use of two observations

Lemma 7.41. *Given an element $\rho \in \oplus_{q=0}^k E_0^{k-q,q}$, let q_0 be such that $\rho^{k-q,q} = 0$ for $q > q_0$. If $q_0 > 0$ and $\delta\rho^{k-q_0,q_0} = 0$, then there is $\tau \in E^{k-q_0,q_0-1}$ for which the $(k-q,q)$ component of $\rho - (-1)^{k-q_0} D\tau$ vanishes for $q \geq q_0$.*

Lemma 7.42. *Given an element $\rho \in \oplus_{q=0}^k E_0^{p,k-p}$, let p_0 be such that $\rho^{p,k-p} = 0$ for $p > p_0$. If $p_0 > 0$ and $d\rho^{p_0,k-p_0} = 0$, then there is $\tau \in E^{p_0-1,k-p_0}$ for which the $(p,k-p)$ component of $\rho - D\tau$ vanishes for $p \geq p_0$.*

Proof of Lemma 7.41. Since $\mathcal{F}_{k-q_0}$ is a fine sheaf, it has no Čech cohomology in positive degree by Proposition 7.36, that is, there is $\tau \in E^{k-q_0,q_0-1}$ such that $\delta\tau = \rho^{k-q_0,q_0}$. A direct computation shows that $\rho - (-1)^{k-q_0} D\tau$ has the desired property. □

Proof of Lemma 7.42. Since the cover $\mathcal{U}$ is fine for the given resolution, $\mathcal{F}_{p_0}$ has no sheaf cohomology in positive degree, that is, there is $\tau \in E^{p_0-1,k-p_0}$ such that $d\tau = \rho^{p_0,k-p_0}$. A direct computation shows that $\rho - D\tau$ has the desired property. □

$$\begin{tikzcd}
0 & & & \\
0 \arrow[u, "(-1)^{k-q_0-1}\delta"] \arrow[r, "d"] & 0 & & \\
 & \rho^{k-q_0,q_0} \arrow[u, "(-1)^{k-q_0}\delta"] \arrow[r, "d"] & ? & \\
 & \tau \arrow[u, "(-1)^{k-q_0}\delta"] \arrow[r, "d"] & \rho^{k-q_0+1,q_0-1} \arrow[u, "(-1)^{k-q_0}\delta"] \arrow[r, "d"] & ?
\end{tikzcd}$$

Figure 7.1: If $\delta\rho^{k-q_0,q_0} = 0$, we can 'kill' the $(k-q_0, q_0)$-component of ρ by adding an exact term without adding terms whose q-degree is higher than q_0.

Continuation of the proof of Theorem 7.40.

i^* **is surjective.** To prove that this is the case we must show that every degree-k element $\rho \in E_0^{\bullet,\bullet}$ is cohomologous to one in the image of i. To do this, we write $\rho = \sum_{q=0}^{k} \rho^{k-q,q}$ and let q_0 be the biggest value of q for which $\rho^{k-q,q} \neq 0$. The condition $D\rho = 0$ implies that $\delta\rho^{k-q_0,q_0} = 0$. If $q_0 > 0$, by Lemma 7.41 we can change ρ by a D-exact element, $\tilde{\rho} = \rho - (-1)^{k-q_0} D\tau$, so that the biggest value of q for which $\tilde{\rho}^{k-q,q} \neq 0$ at most $q_0 - 1$. Repeating this process inductively we show that ρ is cohomologous to an element $\rho^{k,0}$ and the condition that $D\rho^{k,0} = 0$ implies that $\delta\rho^{k,0} = 0$, that is, $\rho^{k,0}$ is induced by a global section $\rho^{k,0} \in \Gamma(M; \mathcal{F}_k)$ and therefore is in the image of the map i^*.

i^* **is injective.** Say a closed element $\phi \in \Gamma(M; \mathcal{F}_k)$ becomes exact in $E_0^{\bullet,\bullet}$, that is there is an element $\rho \in E_0^{\bullet,\bullet}$ such that $D\rho = \phi$. As in the previous case, let q_0 be the biggest value for which $\rho^{k-q_0-1,q_0} \neq 0$. If $q_0 > 0$, the equation $D\rho = \phi$ gives that $\delta\rho^{k-q_0-1,q_0} = 0$ and, by Lemma 7.41, there is τ such that $\tilde{\rho} = \rho - (-1)^{k-q_0-1} D\tau$ also has vanishing $(k-q_0, q_0)$ component and $D\tilde{\rho} = D\rho = \phi$. By induction, we conclude that $\phi = D\rho^{k-1,0}$ which shows that $\rho^{k-1,0}$ is induced by a global section of $\mathcal{F}_{k-1}$ and ϕ is globally exact.

j^* **is surjective.** To prove that this is the case we must show that every degree-k element $\rho \in E_0^{\bullet,\bullet}$ is cohomologous to one in the image of j. To do this, we once again write $\rho = \sum_{p=0}^{k} \rho^{p,k-p}$ and let p_0 be the biggest value of p for which $\rho^{p,k-p} \neq 0$. The condition $D\rho = 0$ implies that $d\rho^{p_0,k-p_0} = 0$. If $p_0 > 0$, by Lemma 7.42 we can change ρ by a D-exact element, $\tilde{\rho} = \rho - D\tau$, so that the biggest value of p for which $\tilde{\rho}^{p,k-p} \neq 0$ at most $p_0 - 1$. Repeating this process inductively we show that ρ is cohomologous to an element $\rho^{0,k}$ and the condition that $D\rho^{0,k} = 0$ implies that $d\rho^{k,0} = 0$, that is, $\rho^{0,k}$ is induced by an element $\rho^{0,k} \in \check{C}^k(M; \mathcal{F})$ and therefore is in the image of the map j^*.

j^* **is injective.** Say a closed element $\phi \in \check{C}^k(M; \mathcal{F})$ becomes exact in $E_0^{\bullet,\bullet}$, that is there is an element $\rho \in E_0^{\bullet,\bullet}$ such that $D\rho = \phi$. As in the previous case, let p_0 be the biggest value for which $\rho^{p_0,k-p_0-1} \neq 0$. If $p_0 > 0$, the equation $D\rho = \phi$ gives that $d\rho^{p_0,k-p_0-1} = 0$ and, by Lemma 7.42, there is τ such that $\tilde{\rho} = \rho - D\tau$ also has vanishing $(p_0, k-p_0)$ component and $D\tilde{\rho} = D\rho = \phi$. By induction, we conclude that $\phi = D\rho^{0,k-1}$ which shows that $\rho^{0,k-1}$ is induced by an element $\rho^{0,k-1} \in \check{C}^{k-1}(M; \mathcal{F})$ and ϕ is exact in the $\mathcal{F}$-Čech cohomology. $\square$

Remark. One result I would like to include here, but for which I do not have a proof, is

Wishful Theorem. *Given any Abelian sheaf $\mathcal{F}$, a resolution of $\mathcal{F}$*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0,$$

and a locally finite open cover $\mathcal{U}$, there is a locally finite refinement $\tilde{\mathcal{U}}$ of $\mathcal{U}$ which is fine for the resolution of $\mathcal{F}$.

If this held we would be able to conclude from the theorem above that sheaf cohomology is isomorphic to Čech cohomology (in the limit under refinement of covers) and hence independent of the chosen resolution. And similarly, for any fine cover $\mathcal{U}$ for the resolution of $\mathcal{F}$, Čech cohomology of $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is isomorphic to the sheaf cohomology which is isomorphic to the Čech cohomology. That is once we have a fine cover, we already "arrived at the limit" for Čech cohomology.

7.4 Consequences of the Čech-to-de Rham theorem

Theorem 7.40 has several consequences which we analyse now.

First we observe that there is an agreement of definitions:

Definition 7.43. The *Čech dimension* of a locally finite open cover $\mathcal{U}$ is the largest integer n for which there is a nonempty intersection $U_{\alpha_0 \ldots \alpha_n}$.

Proposition 7.44. *Given an Abelian sheaf $\mathcal{F}$, if $\mathcal{F}$ admits a finite resolution*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$$

then $H^k(M;\mathcal{F}) = 0$ for $k > n$.

Further, M admits a fine cover $\mathcal{U}$ for $\mathcal{F}$ whose Čech dimension is m, then $H^k(M;\mathcal{F}) = 0$ for $k > m$.

Proof. The first claim follows from the definition of sheaf cohomology and the second from the fact that $\check{H}^k(M;\mathcal{F},\mathcal{U}) = 0$ for $k > m$ plus Theorem 7.40. $\square$

Theorem 7.45. *A short exact sequence of Abelian sheaves*

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{q} \mathcal{H} \to 0$$

gives rise to a long exact sequence in cohomology

$$\cdots H^k(M;\mathcal{F},\mathcal{U}) \xrightarrow{i^*} H^k(M;\mathcal{G},\mathcal{U}) \xrightarrow{q^*} H^k(M;\mathcal{H},\mathcal{U}) \longrightarrow H^{k+1}(M;\mathcal{F},\mathcal{U}) \to \cdots$$

Corollary 7.46 (Baby Riemann–Roch)**.** *Let $E \to \Sigma$ be a line bundle over a Riemann surface and let L_p be the line bundle with a section which vanishes transversely at p (and nowhere else). Let $h^i(\Sigma; L) = \dim(H^i(\Sigma;\Gamma(L))$ for any holomorphic line bundle L. Then*

$$h^0(\Sigma; E \otimes L_p) - h^1(\Sigma; E \otimes L_p) = h^0(\Sigma; E) - h^1(\Sigma; E) + 1. \tag{7.4.1}$$

In particular if $E = L_{p_1} \otimes \ldots \otimes L_{p_n} \otimes (L_{q_1})^* \otimes \ldots \otimes (L_{q_m})^*$ and we set $d = n - m$, then

$$h^0(\Sigma; E) - h^1(\Sigma; E) = h^{0,0}(\Sigma) - h^{0,1}(\Sigma) + d.$$

Proof. We start with the first claim. From Exercise 7.9, we have a short exact sequence

$$0 \to \Gamma(E) \xrightarrow{\otimes s} \Gamma(E \otimes L_p) \xrightarrow{\mathrm{ev}_p} E_p \to 0, \tag{7.4.2}$$

which gives rise to a long exact sequence

$$\begin{aligned} 0 \to H^0(\Sigma; E) &\xrightarrow{\otimes s} H^0(\Sigma; E \otimes L_p) \xrightarrow{\mathrm{ev}_p} H^0(\Sigma; \Gamma(E_p)) \to \\ \to H^1(\Sigma; E) &\xrightarrow{\otimes s} H^1(\Sigma; E \otimes L_p) \xrightarrow{\mathrm{ev}_p} H^1(\Sigma; \Gamma(E_p)) \to 0. \end{aligned}$$

Notice that because the resolution of the skyscraper sheaf E_p finishes in degree 0 and the resolution of holomorphic sections of L finishes at $\Omega^{0,1}(M; L)$, there are no further terms in the long exact sequence. Since the sequence above is exact, the alternating sum of the dimensions of the spaces involved vanishes, and, by Exercise 7.7, $\dim H^0(\Sigma; \Gamma(E_p)) = 1$, so we have

$$h^0(\Sigma; E) - h^0(\Sigma; E \otimes L_p) + 1 - h^1(\Sigma; E) + h^1(\Sigma; E \otimes L_p) = 0,$$

which can be rearranged to (7.4.1).

The second claim is proved inductively, starting with the trivial bundle. Each time we tensor with L_p the alternating sum of the cohomologies increases by 1, by the previous part and each time we tensor with L_q^*, it decreases by 1 because of the sequence

$$0 \to \Gamma(E \otimes L_q^*) \xrightarrow{\otimes s} \underbrace{\Gamma(E \otimes L_q^* \otimes L_q)}_{\cong \Gamma(E)} \xrightarrow{\mathrm{ev}_q} E_q \to 0,$$

$\square$

Chapter 8

Vector bundles – Part II

8.1 Connections and curvature

This section we give a quick review of connections on Hermitian vector bundles. The notion of a Hermitian metric on a complex vector space was introduced in Chapter 1. The extension to complex vector bundles is immediate:

Definition 8.1. Given a complex vector bundle $E \to M$, a Hermitian metric on E is a section $h \in \Gamma(E^* \otimes \overline{E}^*)$ such that

$$h(v,w) = \overline{h(w,v)}, \qquad \text{for } v, w \in E,$$

$$h(v,v) > 0, \qquad \text{for } v \in E \backslash 0.$$

The condition $h \in \Gamma(E^* \otimes \overline{E}^*)$ is often also phrased by saying that h is a symmetric sesquilinear map from $E \times E$ to $\mathbb{C}$, which means it is complex linear in the first entry and complex antilinear in the second:

$$h(\lambda_1 v_1, \lambda_2 v_2) = \lambda_1 \overline{\lambda_2} h(v_1, v_2).$$

One can always decompose a Hermitian metric into its real and imaginary parts: $h = g + i\omega$

Exercise 8.1. Let $h = g + i\omega$ be a Hermitian metric. Then g is a Riemannian metric compatible with the complex structure, ω is a nondegenerate 2-form compatible with complex structure and these two are related by

$$g(v, Iw) = \omega(v, w).$$

Similarly, given a Riemannian metric, g, compatible with the complex structure, let $\omega(v,w) = g(v, Iw)$, then ω is a nondegenerate 2-form compatible with the complex structure and $h = g + i\omega$ is a Hermitian metric.

Finally, given a nondegenerate 2-form ω compatible with the complex structure, let $g(v,w) = -\omega(v, Iw)$, then g is a Riemannian metric compatible with the complex structure and $h = g + i\omega$ is a Hermitian metric.

Given a collection of Hermitian metrics h_i on E, any (fiberwise) convex combination of them is still a Hermitian metric and this is the key fact that goes into the proof of the following lemma:

Lemma 8.2. *Every complex vector bundle $E \to M$ admits a Hermitian metric.*

Proof. Indeed, pick a locally finite open cover of M, $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$, such that E is trivialisable over each U_α and pick local frames $\{s^1_\alpha, \dots, s^k_\alpha\}$. Define a Hermitian metric h_α on E_{U_α} by declaring that the local frame $\{s^1_\alpha, \dots, s^k_\alpha\}$ is orthonormal. Let $\{\psi_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to the cover $\mathcal{U}$ and define $h = \sum \psi_\alpha h_\alpha$, where $\psi_\alpha h_\alpha$ is smoothly extended as zero outside of U_α.

Then $h \in \Gamma(E^* \otimes \overline{E}^*)$, since this is the case for each h_α. Also

$$h(v,w) = \sum \psi_\alpha h_\alpha(v,w) = \sum \psi_\alpha \overline{h_\alpha(w,v)} = \overline{h(w,v)}.$$

And finally, for $v \neq 0$

$$h(v,v) = \sum \psi_\alpha h_\alpha(v,v) > 0,$$

because it is a convex sum of positive numbers. □

Once we have enough geometric structure on a vector bundle, the next step is to introduce the notion of derivatives. Here connections replace the exterior derivative of functions.

Definition 8.3. Given a vector bundle $E \to M$, a connection on E is a linear differential operator

$$\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E),$$

such that for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$\nabla(fs) = df \otimes s + f\nabla s.$$

Example 8.4 (The trivial connection)**.** If $E = \mathbb{C}^k \times M \to M$ is the trivial bundle, then each section of E is just a collection of k complex valued functions and the exterior derivative is a connection on E. ■

Exercise 8.2. Let be a connection on $E \to M$ for $i = 1, \dots, l$ let ∇_i. Let $\{f_i\}_{\{i=1,\dots,l\}}$ be smooth functions such that $\sum_{i=1}^l f_i = 1$. Show that $\sum f_i \nabla_i$ is a connection on E. Using partitions of unity, conclude that every vector bundle admits a connection.

Exercise 8.3. Let ∇_0 and ∇_1 be two connections on $E \to M$. Show that

$$\nabla_0 - \nabla_1 \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$$

is C^∞-linear. Conclude that there is $A \in \Gamma(T^*M \otimes \mathrm{End}\,(E))$ such that

$$\nabla_0 - \nabla_1 = A.$$

Conversely, given $A \in \Gamma(T^*M \otimes \mathrm{End}\,(E))$, $\nabla_0 + A$ is also a connection on E. Conclude that the space of connections on E is an affine space modelled on $\Gamma(T^*M \otimes \mathrm{End}\,(E))$.

Definition 8.5. Given a Hermitian vector bundle $(E, h) \to M$, a connection ∇ on E is compatible with the Hermitian structure if for all $v, w \in \Gamma(E)$,

$$d(h(v,w)) = h(\nabla v, w) + h(v, \nabla w).$$

Exercise 8.4. Show that every Hermitian vector bundle admits a connection compatible with the Hermitian metric. Further, show that if ∇_0 is a connection on E compatible with the Hermitian metric another connection, ∇_1, on E is compactible with the Hermitian metric if and only if $\nabla_1 = \nabla_0 + A$, with $A \in \Gamma(T^*M \otimes \mathrm{End}\,(E))$ satisfying $A^* = -A$.

These results are useful to give us local expressions for connections. Indeed, if $E \to M$ is trivialisable over an open cover $\mathcal{U}$ and we choose trivialisations $\phi_\alpha \colon E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^k$ (if E is Hermitian, by Gram–Schmidt we can ensure that the trivialisation is by an orthonormal frame), then we can compare a connection ∇ on E with the trivial connection on $E|_{U_\alpha}$:

$$\phi_\alpha(\nabla s(x)) := (x, d\vec{f} + A_\alpha \vec{f}),$$

where A_α is a $k \times k$-matrix of 1-forms and $\vec{f}$ is the column vector whose entries are the components of s in the trivialisation Φ_α. The 1-form matrices A_α are the *local connection 1-forms.*

Proposition 8.6. *Let $(E, \nabla) \to M$ be a vector bundle with connection, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M over which E is trivialisable and let $\{\Phi_\alpha\}_{\alpha \in A}$ be a collection of trivialisations of E. Let $\{g^\alpha_\beta\}_{\alpha,\beta \in A}$ be the transition functions for the trivialisations.*

Denoting by $\{A_\alpha\}_{\alpha \in A}$ the local connection 1-forms of ∇. Then

$$A_\alpha = g^\beta_\alpha A_\beta g^\alpha_\beta + g^\beta_\alpha d g^\alpha_\beta. \tag{8.1.1}$$

Conversely, given a collection of 1-forms matrices $\{A_\alpha\}_{\alpha \in A}$ satisfying 8.1.1, there is a unique connection ∇ on E which on the trivialisation Φ_α has connection 1-form A_α.

We can consider the spaces of higher degree forms on E, $\Omega^k(M; E) = \Gamma(\wedge^k T^*M \otimes E)$ and we can extend any connection, ∇ to a map

$$\nabla \colon \Omega^k(M; E) \to \Omega^{k+1}(M; E),$$

by requiring that it satisfies the Leibniz identity:

$$\nabla(\phi s) = d\phi \otimes s + (-1)^k \phi \wedge \nabla s, \qquad \forall \phi \in \Omega^k(M; \mathbb{C}), s \in \Gamma(E).$$

The map above is often also denoted by d_∇ since it mimics the exterior derivative. Yet, differently from the exterior derivative, ∇^2 is frequently not zero.

Proposition 8.7. *Let $(E, \nabla) \to M$ be a vector bundle with connection over M, let $\phi \in \Omega^k(M; \mathbb{C})$ and $s \in \Gamma(E)$, then*

$$\nabla^2(\phi \otimes s) = \phi \otimes \nabla^2 s,$$

That is $\nabla^2 \colon \Omega^\bullet(M; E) \to \Omega^\bullet(M; E)$ is given by a section $F_\nabla \in \Omega^2(M; \mathrm{End}\,(E))$.

Definition 8.8. The *curvature* of a connection ∇ is the tensor $F_\nabla = \nabla^2 \in \Omega^2(M; \mathrm{End}\,(E))$.

Exercise 8.5. Let ∇ be a connection on E and let $A \in \Omega^1(M; \mathrm{End}\,(E))$. Then the curvatures of ∇ and $\nabla + A$ are related by

$$F_{\nabla + A} = F_\nabla + \nabla A + A \wedge A,$$

where the wedge indicates matrix multiplication and exterior product.

Conclude that given a (local) trivialisation of a bundle E, if A is the local connection 1-form of the connection ∇, then the curvature of ∇ is given by

$$F_\nabla = dA + A \wedge A.$$

Not only can we extend a connection on E to the differential operator on $\Omega^k(M;E)$, but we can also extend a connection on E to a connection on associated vector bundles, such as E^* or $\mathrm{End}\,(E)$, by requiring that the Leibniz rule should hold.

Exercise 8.6 (Dual and endomorphism connections)**.** Let $(E,\nabla) \to M$ be a vector bundle over M. Define a connection ∇^* on E^* by requiring that the Leibniz rule holds

$$d(s^*(s)) = (\nabla^* s^*)(s) + s^*(\nabla s).$$

Similarly, define a connection ∇ on $\mathrm{End}\,(E)$ by requiring that the Leibniz rule holds

$$\nabla(A(s)) = (\nabla A)(s) + A(\nabla s).$$

Theorem 8.9 (Second Bianchi Identity)**.** *Let $(E,\nabla) \to M$ be a vector bundle with connection. Then*

$$\nabla F_\nabla = 0.$$

Proof. To make it easier to follow the computation, it is useful to pick a section $s \in \Gamma(E)$. Then, by definition of the connection induced on endomorphisms, we have

$$(\nabla F_\nabla)s = \nabla(F_\nabla s) - F_\nabla(\nabla s) = \nabla(\nabla^2 s) - \nabla^2(\nabla s) = \nabla^3 s - \nabla^3 s = 0.$$

□

Corollary 8.10. *Given a vector bundle with connection $(E,\nabla) \to M$, $d(\mathrm{Tr}(F_\nabla)) = 0$ and the cohomology class $[\mathrm{Tr}(F_\nabla)]$ is independent of the connection.*

Proof. The first claim follows from the Second Bianchi Identity and the invariance of the trace:

$$d\mathrm{Tr}(\mathrm{F}) = \mathrm{Tr}(\nabla F), \qquad \forall F \in \Omega^\bullet(M;\mathrm{End}\,(E)).$$

The second follows from Exercise 8.5, the previous identity and the fact that $\mathrm{Tr}(A \wedge A) = 0$. [1] □

Definition 8.11. The *curvature class* of a vector bundle E is the cohomology class $\frac{1}{2\pi i}[\mathrm{Tr}(F_\nabla)]$ for any connection ∇ on E.

Notice that for line bundles, $E \otimes E^*$ is trivial and the curvature is just a 2-form, which is closed, by the argument above and represents the curvature class of the line bundle. For higher rank vector bundles we can recover the curvature class from the corresponding determinant vector bundle.

Lemma 8.12. *If $E \to M$ is a rank k vector bundle, then ∇ induces a connection $\tilde\nabla$ on the line bundle $\wedge^k E \to M$ and*

$$\mathrm{Tr}(F_\nabla) = F_{\tilde\nabla}.$$

[1] Any other map

$$\Phi\colon \Omega^\bullet(M;\mathrm{End}\,(E)) \times \cdots \times \Omega^\bullet(M;\mathrm{End}\,(E)) \to \Omega^\bullet(M;\mathbb{C})$$

such that

$$d\Phi(A_1,\dots,A_l) = \Phi(\nabla A_1,\dots,A_l) + \cdots + \Phi(A_1,\dots,\nabla A_l)$$

will produce cohomology classes when applied to the curvature. This way we obtain all higher curvature classes.

Recall that the Čech-to-de Rham isomorphism gives an identification of the sheaf cohomology $H^\bullet(M;\underline{R})$ with the de Rham cohomology.

Theorem 8.13. *For a complex vector bundle $E \to M$, the curvature class of E is the image of the first Chern class of* $\det(E)$ *under the natural inclusion $\mathbb{Z} \to \underline{\mathbb{R}}$:*

$$c_1(E) \in H^2(M;\mathbb{Z}) \to H^2(M;\underline{\mathbb{R}}).$$

Proof. The proof is a matter of chasing the different arrows that arose in the arguments of different proofs we have seen so far. By Exercise **??** the curvature class of E and of $\det(E)$ are the same, hence we can assume without loss of generality that E is a complex line bundle.

We start with the description of a representative of the cohomology class $c_1(E) \in H^2(M;\mathbb{Z})$. Pick an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of M together with trivialisations of E over this open cover to obtain transition functions $\{g^\alpha_\beta\}_{\alpha,\beta\in A}$. Assume further that all the U_α as well as the $U_{\alpha\beta}$ are contractible. The transition functions, $\check{g}$, are a representative of the classifying class of E in $\check{C}^1(M;C^\infty(\mathbb{C}^*),\mathcal{U})$. To obtain a representative of the corresponding class in $\check{C}^2(M;\mathbb{Z})$ we use the connecting homomorphism for the exact sequence of sheaves

$$0 \to \mathbb{Z} \stackrel{2\pi i}{\to} C^\infty(\mathbb{C}) \stackrel{\exp}{\to} C^\infty(\mathbb{C}) \to 0,$$

that is, we chose branches for the log and take $\delta(\frac{1}{2\pi i}\log\check{g})$.

Next the inclusion $\mathbb{Z} \subset \mathbb{R}$ corresponds to considering the cocycle of integers $\delta(\frac{1}{2\pi i}\log\check{g})$ as a cocycle of locally constant real functions and now we follow the Čech-to-de Rham algorithm to produce a 2-form out of this Čech cocycle. The first step consists of considering $\delta(\frac{1}{2\pi i}\log\check{g})$ as a collection of smooth functions and finding a Čech primitive among the smooth function for this cocycle. For this step we are in the luck since $\frac{1}{2\pi i}\log\check{g} \in \check{C}^1(M;\mathcal{U},C^\infty(\mathbb{C}))$ lends itself immediately to this job. Then taking the exterior derivative of this cochain, we obtain the Čech cocycle $\frac{1}{2\pi i}d\log\check{g} \in \check{C}^1(M;\mathcal{U},\Omega^1)$ and the next step is to find a Čech primitive for it, that is, we need to find a collection

$$\check{A} = \{A_\alpha \in \Omega^1(U_\alpha)\colon -(\delta\check{A})_{\alpha\beta} = A_\alpha - A_\beta = d\log g^\alpha_\beta\}.$$

This corresponds to a choice of connection ∇ on E, where $\check{A}$ corresponds to the connection 1-forms.

Then the real 2-form representing the first Chern class is given locally by $\frac{1}{2\pi i}dA_\alpha = \frac{1}{2\pi i}F_\nabla$. $\square$

8.2 Partial connections and the canonical connection

Given complex manifold, M, we can study how connections on a complex vector bundle E over M possibly interact with the complex structure. Similarly to the exterior derivative, we can split a connection into its $(1,0)$ and $(0,1)$-components:

$$\begin{aligned}\nabla &= \nabla^{1,0} + \nabla^{0,1},\\ \nabla^{1,0}&\colon \Gamma(E) \to \Gamma(T^{*1,0}M \otimes E),\\ \nabla^{0,1}&\colon \Gamma(E) \to \Gamma(T^{*0,1}M \otimes E).\end{aligned}$$

$$
\begin{tikzcd}
-\delta\log(\check{g}) & & \\
-\log(\check{g}) \arrow[u, "\delta"] \arrow[r, "d"] & \overbrace{-d\log(\check{g}) - \delta\check{A}}^{0} & \\
& \check{A} \arrow[u, "-\delta"] \arrow[r, "d"] & d\check{A} = F_\nabla
\end{tikzcd}
$$

Figure 8.1: The element $\check{A}-\log(\check{g})$ in the Čech-to-de Rham double complex satisfies $D(\check{A}-\log(\check{g})) = -\delta\log(\check{g}) + F_\nabla$, showing that the real representative of the first Chern class is cohomologous to the curvature class.

In a local trivialisation we can explicitly write both components using the decomposition of the connection 1-form:

$$\nabla = d + A, \qquad \nabla^{1,0} = \partial + A^{1,0}, \qquad \nabla^{1,0} = \overline{\partial} + A^{0,1}.$$

Notice however that since A is complex valued, it is not necessarily the case that $A^{1,0}$ and $A^{0,1}$ are related and hence $\nabla^{0,1}$ is not obtained from $\nabla^{1,0}$ by using complex conjugation.

The Leibniz rule for the connection translates into Leibniz rules for its components:

$$\nabla^{1,0}(fs) = \partial f \otimes s + f\nabla^{1,0}s,$$

$$\nabla^{0,1}(fs) = \overline{\partial} f \otimes s + f\nabla^{0,1}s.$$

Since $\overline{\partial}$ played an important role for holomorphic functions/sections, the operators that behave like $\nabla^{0,1}$ deserve special attention:

Definition 8.14. A *partial connection* on a complex vector bundle over a complex manifold is a linear differential operator

$$\nabla^{0,1}\colon \Gamma(E) \to \Gamma(T^{*0,1}M \otimes E),$$

which satisfies the Leibniz rule

$$\nabla^{0,1}(fs) = \overline{\partial} f \otimes s + f\nabla^{0,1}s.$$

Up to now we only presented how a complex structure allows us to repackage information contained in a connection. Things become more interesting once we start imposing compatibility between these two objects.

Definition 8.15. A (partial) connection ∇ on a holomorphic vector bundle $E \to M$, is compatible with the complex structure if $\nabla^{0,1}s = 0$ for all local holomorphic sections of E.

We see that if $\nabla^{0,1}$ is compatible with the complex structure and we apply it to sections of a local holomorphic frame $\{s_1, \ldots, s_k\}$ of E we have

$$0 = \nabla^{0,1}s_i = A^{0,1}s_i,$$

that is $A^{0,1} = 0$ and conversely if the connection matrix of ∇ is of type $(1, 0)$ in a holomorphic frame, then the connection is compatible with the complex structure.

Putting together a Hermitian structure and the compatibility condition we narrow down the connections on a holomorphic vector bundle to one:

Theorem 8.16. *Let $(E, \langle\cdot,\cdot\rangle) \to M$ be a holomorphic Hermitian vector bundle over M. Then there is a unique connection on E compatible with both the Hermitian structure of E and the complex structure of M.*

Proof. This is proved by showing that locally there is a unique local connection which satisfies the two compatibility conditions.

Let us start with the local uniqueness. Let ∇ be one such connection and let $\{s_1, \ldots, s_k\}$ be a local holomorphic frame for E. Let $h_{ij} = h(s_j, s_i)$[2] and let A_{ij} be the entries of the local connection 1-form in this trivialisation, that is,

$$\nabla(\sum_i f_i s_i) = \sum_i (df_i + \sum_j A_{ij} f_j) s_i.$$

Then compatibility with the metric gives

$$\begin{aligned} dh_{ij} &= \langle \nabla s_j, s_i\rangle + \langle s_j, \nabla s_i\rangle \\ &= \langle \sum_l A_{lj} s_l, s_i\rangle + \langle s_j, \sum_l A_{li} s_l\rangle \\ &= \sum_l (A_{lj}\langle s_l, s_i\rangle + \overline{A_{li}}\langle s_j, s_l\rangle) \\ &= \sum_l (h_{il} A_{lj} + \overline{A_{li}} h_{lj}). \end{aligned}$$

Since the connection matrix is made of $(1, 0)$-forms, taking the $(1, 0)$-component of the equality above we obtain

$$\partial h_{ij} = \sum_l h_{il} A_{lj} = (hA)_{ij},$$

where in the last equality h is the matrix whose entries are h_{ij}, A is the connection matrix and the product is matrix multiplication. Hence

$$A = h^{-1}\partial h \tag{8.2.1}$$

showing that after a choice of holomorphic frame the connection matrix is determined by the Hermitian structure.

To prove existence, we pick an appropriate open cover of M, $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$, and a collection of holomorphic trivialisations $\{\Phi_\alpha\}_{\alpha\in I}$ of E over this open cover, and define the connection matrix, A_α, in each local holomorphic frame by the expression 8.2.1. To prove that this defines a connection, we need to check that it transforms correctly under change of trivialisations, that is, if

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (v, g_\beta^\alpha(x)v),$$

[2]The sneaky use of indices in an unexpected order is to make sure the final expression looks good.

then $A_\alpha - g_\alpha^\beta A_\beta g_\beta^\alpha = g_\alpha^\beta dg_\beta^\alpha$.

First we relate the Hermitian matrices on both frames:

$$\begin{aligned}(h_\alpha)_{ij} = \langle e_j^\alpha, e_i^\alpha \rangle &= \sum_{kl} \langle (g_\beta^\alpha)_{lj} e_l^\beta, (g_\beta^\alpha)_{ki} e_k^\beta \rangle \\ &= \sum_{kl} (g_\beta^\alpha)^*_{ki} (h_\beta)_{kl} (g_\beta^\alpha)_{lj},\end{aligned}$$

where $\bullet^\star$ denotes transposition and conjugation. Therefore the corresponding Hermitian matrices are related by

$$h_\alpha = (g_\beta^\alpha)^* h_\beta g_\beta^\alpha.$$

Hence

$$\begin{aligned}A_\alpha = h_\alpha^{-1}\partial h_\alpha &= (g_\beta^\alpha)^{-1} h_\beta^{-1} (g_\beta^\alpha)^{*-1} \left(((g_\beta^\alpha)^* (\partial h_\beta) g_\beta^\alpha + (g_\beta^\alpha)^* h_\beta (\partial g_\beta^\alpha) \right) \\ &= g_\alpha^\beta (h_\beta^{-1} \partial h_\beta) g_\beta^\alpha + g_\alpha^\beta \partial g_\beta^\alpha \\ &= g_\alpha^\beta A_\beta g_\beta^\alpha + g_\alpha^\beta \partial g_\beta^\alpha,\end{aligned}$$

showing that the proposed local connection 1-forms patch together to define a global connection.

The proof that the connection defined this way is compatible with the metric is essentially the same argument used to show uniqueness. $\square$

Definition 8.17. The unique connection on a holomorphic Hermitian vector bundle compatible with both the complex and the Hermitian structure is the *canonical connection*.

Corollary 8.18. *The curvature of the canonical connection is of type (1,1) and is given by*

$$F_\nabla = \overline{\partial}(h^{-1}\partial h).$$

Proof. Since in a local trivialisation $\nabla = \partial + \overline{\partial} + h^{-1}\partial h$, we see that the $(0,2)$-component of the curvature vanishes and the $(2,0)$-component is given by

$$\begin{aligned}F_\nabla^{2,0} &= \partial(h^{-1}\partial h) + h^{-1}\partial h \wedge h^{-1}\partial h \\ &= -h^{-1}(\partial h) h^{-1} \wedge \partial h + h^{-1}(\partial h) \wedge h^{-1}\partial h = 0,\end{aligned}$$

where we used that $\partial h^{-1} = -h^{-1}(\partial h)h^{-1}$.

Finally, one readily sees that the $(1,1)$-component has the stated form. $\square$

Corollary 8.19. *Let $(L, \langle\cdot,\cdot\rangle) \to M$ be a holomorphic Hermitian line bundle over M and let s be a nontrivial meromorphic section of L. Then the curvature of the canonical connection of L is*

$$F_\nabla = -\partial\overline{\partial} \log \|s\|^2,$$

where the singularities of the 2-form in the right hand side are all removable.

Proof. We compute the right hand side of the expression above in a local trivialisation. If σ is a local nonvanishing section of L, then $s = f\sigma$ for some meromorphic function f. Hence, wherever f is defined and nonzero

$$\begin{aligned}
-\partial\overline{\partial}\log\|s\|^2 &= -\partial\overline{\partial}\log|f|^2\|\sigma\|^2 = -\partial\overline{\partial}\log f\bar{f}\|\sigma\|^2 \\
&= -\partial\overline{\partial}(\log f + \log\bar{f} + \log\|\sigma\|^2) \\
&= -\partial\overline{\partial}\log\|\sigma\|^2 \\
&= \overline{\partial}\left(\frac{\partial\|\sigma\|^2}{\|\sigma\|^2}\right) = F_\nabla.
\end{aligned}$$

□

So far, given a complex vector bundle over a complex manifold, the only way to make it into a holomorphic vector bundle is by choosing wisely local sections so that the transition functions are holomorphic. While this is a clear condition, it is not the easiest to work with since transition functions are tied to a number of choices of trivialisations. We finish this session with an easier description of holomorphic vector bundles.

Theorem 8.20. *Let M be a complex manifold and let $E \to M$ be a complex vector bundle. Let $\nabla^{0,1}$ be a partial connection on E. If $(\nabla^{0,1})^2 = 0$ then E admits the structure of a holomorphic vector bundle for which $\nabla^{0,1}$ is a partial connection compatible with the complex structure.*

8.3 Line bundles and divisors

While isomorphism classes of general vector bundles can be described in terms of transition functions, as in Theorem 5.4, for line bundles a more geometric picture quickly emerges: a line bundle is roughly determined by the vanishing locus of a generic section. The underlying concept is that of a *divisor*.

Definition 8.21. A *divisor* on a manifold M is an ideal sheaf $\mathcal{I}_D \subset C^\infty$ which is locally generated by a single function and whose vanishing locus, D, is nowhere dense on M.

In the life of a differential geometer the ideal defining a divisor may be defined by a real or a complex function. In these notes, we will focus on complex line bundles and hence our ideals will be defined by complex functions.

Notice that the divisor contains more information than its vanishing locus.

Exercise 8.7. Consider the following divisors on $\mathbb{C}$: $\mathcal{I}_1 = \langle z\rangle$ and $\mathcal{I}_2 = \langle z + \overline{z}^2\rangle$. Show that $\mathcal{I}_1$ and $\mathcal{I}_2$ have the same vanishing locus but $\mathcal{I}_1 \neq \mathcal{I}_2$. That is, the vanishing locus does not define the divisor.

Proposition 8.22. *Let $L \to M$ be a line bundle and let $s\colon M \to L$ be a section whose zero locus is nowhere dense, then*

$$\mathcal{I}_s = \{\tau(s)\colon \tau \in \Gamma(L^*)\}$$

is a divisor on M.

Proof. Pick a local frame $\sigma\colon U \to L$, so that $s = f\sigma$ for some function $f\colon U \to \mathbb{C}$. Any section of the dual bundle over U is of the form $\tau = g\sigma^*$, where σ^* the section dual to σ and hence the local functions we obtain by evaluating sections of L^* with s are of the form

$$g\sigma^*(s) = g\sigma^*(f\sigma) = fg \in \mathcal{I}_f,$$

showing that f is a local generator for $\mathcal{I}_s$. □

Therefore, we have a map

$$\begin{aligned}\{(L, s)\colon s \text{ is sufficiently generic}\} &\to \{\text{Divisors}\},\\ (L, s) &\mapsto \mathcal{I}_s.\end{aligned}$$

The converse to Proposition 8.22 also holds.

Proposition 8.23. *Let $\mathcal{I}_D \subset C^\infty(\mathbb{C})$ be a divisor on M, then there is a line bundle $L \to M$ and a section $s\colon M \to L$ such that $\mathcal{I}_D = \mathcal{I}_s$. Further, if (L', s') is another pair of line bundle and section $\mathcal{I}_{s'} = \mathcal{I}_D$, then L' is isomorphic to L via an isomorphism which sends s' to s.*

Proof. Let $\{\mathcal{U}_\alpha\}_{\alpha\in A}$ be a locally finite open cover of M such that for each $\alpha \in A$ there is $f_\alpha \in C^\infty(U_\alpha; \mathbb{C})$ such that $\mathcal{I}_D = \langle f_\alpha\rangle$ on U_α. On the overlap $U_{\alpha\beta}$ we can compare f_α with f_β. Since f_α generates $\mathcal{I}_D$ and $f_\beta \in \mathcal{I}_D$, there is a smooth function such that $f_\beta = g^\alpha_\beta f_\alpha$. Reversing the roles of α and β we conclude that g^α_β is nonvanishing. Further, in the locus where $f_\alpha \neq 0$,

$$g^\gamma_\alpha g^\beta_\gamma g^\alpha_\beta = \frac{f_\alpha}{f_\gamma}\frac{f_\gamma}{f_\beta}\frac{f_\beta}{f_\alpha} = 1.$$

Since this holds in a dense locus, this holds on the whole triple overlap and the collection $\check{g} = \{g^\alpha_\beta\}_{\alpha,\beta\in A}$ determines a line bundle obtained by gluing together $\{U_\alpha \times \mathbb{C}\}_{\alpha\in A}$ using the functions in $\check{g}$. Further, we can define a global section s be declaring that on the trivialization $U_\alpha \times \mathbb{C}$, $s = f_\alpha$. We see immediately that the section s generates the ideal $\mathcal{I}_D$.

Finally, if (L', s') are such that $\mathcal{I}_{s'} = \mathcal{I}_D = \mathcal{I}_s$, then for a cover fine enough, we can find local trivializations in which $s' = f'_\alpha\sigma'_\alpha$ and $s = f_\alpha\sigma_\alpha$. Since both f_α and f'_α are generators of $\mathcal{I}_D$ there is a nonvanishing function h_α such that $f'_\alpha = h_\alpha f_\alpha$, therefore, we can compare the transition functions for these bundles to obtain

$$g'^\alpha_\beta = \frac{f'_\beta}{f'_\alpha} = \frac{h_\beta}{h_\alpha}\frac{f_\beta}{f_\alpha} = \frac{h_\beta}{h_\alpha} g^\alpha_\beta,$$

which shows that L and L' are isomorphic bundles. One can readily check that the isomorphism provided by the collection $\{h_\alpha\}_{\alpha\in A}$ puts s and s' in correspondence. □

Summarising, we have a correspondence

$$\begin{aligned}\frac{\{(L, s)\colon s \text{ is sufficiently generic}\}}{\text{isomorphisms}} &\leftrightarrow \{\text{Divisors}\},\\ [(L, s)] &\leftrightarrow \mathcal{I}_s.\end{aligned}$$

A typical type of section to consider to produce divisors are those transverse to the zero section. This notion belongs to the more general notion of transversality.

Definition 8.24. Let E be a smooth manifold and let $\iota_i \colon M_i \to E$, $i = 0, 1$, be two immersed submanifolds. Let $p_i \in M_i$ be such that $\iota_0(p_0) = \iota_1(p_1) = p$. We say that the immersions are *transverse at* p_0 *and* p_1 if $\iota_{0*}(T_{p_0}M_0) + \iota_{1*}(T_{p_1}M_1) = T_pE$. The immersions ι_0 and ι_1 are *transverse* if they are transverse at all pairs p_0 and p_1 for which $\iota_0(p_0) = \iota_1(p_1)$.

Exercise 8.8. Show that if $\iota_i \colon M_i \to E$ are transverse embeddings, then $\iota_0(M_0) \cap \iota_1(M_1)$ is an embedded submanifold of codimension $\operatorname{codim}(M_0) + \operatorname{codim}(M_1)$.

Definition 8.25. Let $E \to M$ be a vector bundle over M and let $s_i \colon M \to E$, $i = 0, 1$, be two sections. We say that the sections s_0 and s_1 are *transverse* or s_0 *is transverse to* s_1 if they are transverse as embeddings $s_i \colon M \to E$.

A very common condition to consider is that of a section $s \colon M \to E$ transverse to the zero section. When this happens, the zero locus of s, say D, is an embedded submanifold of M whose codimension in M is the rank of E, by Exercise 8.8.

Exercise 8.9. Show that both divisors in Exercise 8.7 are induced by sections transverse to the zero section.

Further, there is a relation between E and the normal bundle of M

Proposition 8.26. *Let $s \colon M \to E$ be a section transverse to the zero section and let $D \subset M$ be the zero locus of s, then*

$$d^\nu s = \pi_E \circ ds \colon TM \to E|_D$$

has kernel TD and induces an isomorphism between $\mathcal{N}_D$, the normal bundle of D, and $E|_D$.

For p a zero of s, the composition of $ds_p \colon T_pM \to T_0E \cong T_pM \oplus E_p$ with the projection onto E_p, $d^\nu s = \pi_E \circ ds$, is known as the *vertical derivative of* s.

Proof. It follows directly from the definition of transversality that s is transverse to the zero section if and only if the vertical derivative of s is surjective at every point $p \in Z$ (since the tangent space of the zero section already takes up all the "horizontal" directions). Further, since s vanishes over Z, we have that for all $X \in TZ$, $d^\nu s(X) = 0$, so the vertical derivative descends to a surjective bundle map, which we still denote by $d^\nu s$

$$d^\nu s \colon \mathcal{N}_Z = TM/TZ \to E|_Z.$$

Since these bundles have the same rank, this map is a bundle isomorphism. □

Returning to divisors, if $s \colon M \to L$ is a section of a line bundle transverse to the zero section and $p \in M$ is in the zero locus of s, write $s = f\sigma$ where σ is a local frame for L defined on a neighbourhood of p. Since, at p, $df \colon T_pM \to \mathbb{C}$ is surjective, f can be extended to a coordinate chart for M in a neighbourhood of p: $(f = x_1 + iy_1, x_2, \ldots, x_{n-1})$. So, in this chart, the ideal defined by s is generated by $x_1 + iy_1$.

It follows from a *general position* argument that every smooth vector bundle admits a section transverse to the zero section and in particular we can describe any complex line bundle via divisors locally generated by a coordinate function $z = x + iy$. When we pass to holomorphic line bundles, this ceases to be the case since, for example, those may have no global sections.

8.4 Holomorphic line bundles and divisors

The key that will allow us to treat holomorphic line bundles using the language of divisors is that we will allow for meromorphic sections.

Definition 8.27. Let M be a complex manifold. A *(holomorphic) divisor* on M is a sheaf $\mathcal{I}_D \subset \mathfrak{M}$ inside the sheaf of meromorphic functions which closed under multiplication by elements of $\mathcal{O}$, the sheaf of holomorphic functions and is locally generated by a single function and whose vanishing locus, D, is nowhere dense on M.

Similarly to the smooth case, all holomorphic divisors arise from meromorphic sections of line bundles, but differently from the smooth case, because of the meromorphic condition, every nonzero section automatically has support M

Proposition 8.28. *There is a correspondence*

$$\frac{\{(L,s)\colon s \text{ meromorphic, } s \neq 0\}}{\text{isomorphisms}} \leftrightarrow \{\text{Holomorphic divisors}\}$$
$$[(L,s)] \leftrightarrow \mathcal{I}_s,$$

where
$$\mathcal{I}_s = \{\tau(s)\colon \tau \in \Gamma(L^*)\}.$$

One advantage of the holomorphic case, is that now the vanishing and singular locus of s together nearly determines the ideal $\mathcal{I}_s$. Part of the game we play now is to add information to (singular) submanifolds so that we can recover the generators of the ideal and hence the divisor. Notice that this is different from the smooth case, in which the vanishing locus of s did not determine the ideal $\mathcal{I}_s$ (c.f. Exercise 8.7)

We start our tour on how to recover $\mathcal{I}_s$ from the zero locus of s considering the case when s is transverse to zero and has simple poles. We make this precise now. Given a meromorphic section s of a line bundle L, there is a meromorphic section s^* of L^* defined by the condition $s^*(s) \equiv 1$. The section s^* has zero and poles that cancel the poles and zeros of s. This allows us to define the relevant notion of transversality in this context.

Definition 8.29. A section s is *transverse to the section at infinity* if the dual section s^* is transverse to the zero section.

If a holomorphic section s is transverse to the zero section and, say, the zero locus of s is $D \subset M$, then, since s is holomorphic, D is an embedded complex submanifold and hence we can find local holomorphic coordinates for which D is the locus $[z_1 = 0]$ and there is a corresponding trivialization of L for which $s(z) = z_1$. In particular, in contrast to Exercise 8.7, there are no local invariants of a divisor associated to a holomorphic section transverse to zero.

Given a meromorphic section $s \in \Gamma(M; L)$ transverse to zero and infinity, let Z_0 denote the zero locus of s and Z_∞ the poles of s. Then, Z_0 and Z_∞ together determine the divisor induced by s which in turn determines L. We denote this divisor by $Z_0 - Z_\infty$. This allows us to describe holomorphic line bundles using (complex) codimension-1 submanifolds of M. Notice that if for $i = 0, 1$, L_i has a section s_i transverse to zero and infinity then $s_0 \otimes s_1 \in \Gamma(M; L_0 \otimes L_1)$ is a meromorphic section whose zero locus/poles is the union of the zero loci/poles of s_0 and s_1

except for cancelations of zeros and poles. If a submanifold $D \subset M$ is in the zero locus for both s_0 and s_1, then D is a zero of order 2 for $s_0 \otimes s_1$, which in this case is not transverse to the zero section. Yet we can keep track of the order of vanishing if we allow integer coefficients in our divisors. So, for example the submanifold $2D_0$, corresponds to the divisor determined by a holomorphic section that vanishes to order 2 along D_0.

Blurring the distinction between submanifolds and divisors, up to now we have a map that associates to a line bundle with meromorphic section (L, σ) a formal sum of submanifolds corresponding to the zeros and poles of s with their respective orders as coefficients:

$$\mathrm{Div}(L, s) := \sum m_i Z_i.$$

Tensor product of sections corresponds to doing a formal sum of divisors:

$$\mathrm{Div}(s_0) = \sum m_i Z_i, \quad \mathrm{Div}(s_1) = \sum n_i W_i \quad \Rightarrow \quad \mathrm{Div}(s_0 \otimes s_1) = \sum m_i Z_i + \sum n_i W_i,$$

where one just adds the coefficients if $Z_i = W_j$.

Conversely, given a collection of submanifolds and integer coefficients, we can associate to it a line bundle (with a meromorphic section):

$$Z \mapsto L_Z.$$

Observe that even the condition that Z is made of embedded submanifolds is not necessary, as long as it is locally defined as the vanishing locus of a single meromorphic function.

Finally one must address the question: When are two divisors associated to the same line bundle? From our previous argument, to each divisor there corresponds a meromorphic section $s \in \Gamma(M; L)$. Given two divisors that come from the same line bundle, we obtain two sections $s_0, s_1 \in \Gamma(M; L)$ which away from their zeros and poles are related by a function $s_0 = f s_1$. By comparing s_0 and s_1 in local coordinates, we see that f is a meromorphic function on f (in particular, a meromorphic section of the trivial bundle) and

$$\mathrm{Div}(s_0) - \mathrm{Div}(s_1) = \mathrm{Div}(f).$$

Definition 8.30. A *principal divisor* is a divisor associated to a meromorphic function.

Definition 8.31. Two divisors are *equivalent* if their difference is a principal divisor.

Finally, even if a line bundle admits a global holomorphic section it may also admit meromorphic sections with poles. Divisors that manifestly express the existence of holomorphic sections do not have negative coefficients. Since that seems to be an important property, these also have a name:

Definition 8.32. An *effective divisor* is one of the form $\sum m_i D_i$ with $m_i \geq 0$.

Exercise 8.10. Let $D = \sum m_i D_i$ be a divisor on M and let $\Sigma \subset M$ be a Riemann surface that only intersects Z in smooth points of D_i and does so transversally. Compute

$$\int_\Sigma c_1([Z]).$$

Exercise 8.11. Let D be an effective divisor. Show that the following is an exacr sequence of sheaves

$$0 \to \Gamma(L \otimes [-D]) \to \Gamma(L) \to \Gamma(L|_D) \to 0,$$

One nagging point about the correspondence given in Proposition 8.28 between line bundles and divisors is the dependence on the section s. This dependence was partially addressed above with the notions of principal divisor and equivalence of divisors. But that discussion still leaves one point missing: the existence of a meromorphic section. If a line bundle has no meromorphic section, there is no associated divisor and the whole theory breaks down at the start.

The quest of finding meromorphic sections leads us to a geometric interpretation of $H^1(\Sigma; L)$ for a Riemann surface Σ and a holomorphic line bundle L. We start with the following question: given a point $p \in \Sigma$ is there a section of L with a simple pole at p (and holomorphic elsewhere)? One can try to answer this question constructively. First we pick a frame σ for L in a neighbourhood U of p and coordinate charts for U centered at p at let $s(z) = z^{-1}\sigma$ be a section of $L|_U$. This section has the desired pole, but may not extend holomorphiclly to Σ. To try and do that, we pick a function ψ which is equal to 1 in a neighbourhood of p and has support in U and form ψs, which now extends to Σ, has the desired pole, but is not holomorphic in the annulus where ψ interpolates between 1 and 0. So to find the global section we are looking for, we need to find a *smooth* section $\tau \in \Gamma(\Sigma; L)$ such that

$$\overline{\partial}(\tau + \psi s) = 0. \tag{8.4.1}$$

This section would be holomorphic by virtue of being in the kernel of $\overline{\partial}$, global and have a simple pole at p, because τ is smooth and ψs has a simple pole at p. Notice that even though ψs has a simple pole at p, it is holomorphic in a neighbouhoor of p (as we mentioned, it fails to be holomorphic in an annulus around p), so $\overline{\partial}\psi s \in \Omega^1(\Sigma; L)$ is a global, smooth, $\overline{\partial}$-closed form, that is, it represents a class in $H^1(\Sigma; L)$. Our quest from Equation (8.4.1) is to determine if $\overline{\partial}\psi s$ is $\overline{\partial}$-exact. If that is the case, we can find a global section with a single pole at p and if not, well... not.

Going a step further, given a collection of points $\{p_1, \ldots, p_n\} \in \Sigma$ we can repeat the local construction from the previous paragraph to produce a collection of local sections $\psi_i s_i$ supported in a neighbourhood of p, with a simple pole at p_i and which fail to be holomorphic in an annulus around p_i. The question of whether it is possible to find a meromorphic section of L with poles only at the points p_i then turns into the question of whether we can find a smooth section τ and a collection of residues λ_i such that

$$\overline{\partial}(\tau + \sum \lambda_i \psi_i s_i) = 0 \Rightarrow \overline{\partial}\tau = -\sum \lambda_i \overline{\partial}(\psi_i s_i).$$

Each summand on the right hand side is a class in $H^1(\Sigma; L)$ and we are looking for a linear relation between those classes. If there is one, we can find a meromorphic section with the prescribed poles.

Therefore we arrive at the following description of $H^1(\Sigma; L)$:

$H^1(\Sigma; L)$ *is the space of obstructions to finding meromorphic sections with prescribed poles.*

Notice that if $H^1(\Sigma; L)$ is finite dimensional, say, $\dim(H^1(\Sigma; L)) = n$, then after allowing for $n+1$ poles the elements $[\overline{\partial}(\psi_i s_i)]$, $i = 1, \ldots, n+1$ will necessarily form a linear dependent set in $H^1(\Sigma; L)$. Therefore, if $H^1(\Sigma; L)$ is finite dimensional, L admits meromorphic sections.

As we will see next chapter, if M is compact, $H^q(M; E)$ is finite dimensional for any holomorphic bundle E. Therefore, for Riemann surfaces we have a correspondence between line bundles and divisors.

In higher dimensions, the construction of the Dolbeault cohomology classes $\overline{\partial}(\psi s)$ fails since we have use partitions of unit to patch local sections over the divisor D. Not only that, but there are examples of complex manifolds and holomorphic line bundles which do not admit meromorphic sections. Those line bundles can not be dealt with the framework of divisors.

Chapter 9

Elliptic operators and Hodge's theorem

9.1 Recap of Sobolev spaces

9.2 Differential operators and their symbol

A differential operator of order k between vector bundles $E, F \to M$ is a map $D\colon \Gamma(E) \to \Gamma(F)$ for which the value of $(Ds)(p)$ only depends on the values of the first k derivatives of the section s at p. Of course the concept "the values" of the first k derivatives of the section s at p requires further explanation because any specific values one can obtain will depend on choices of coordinates and trivializations. Yet, the notion of two sections agreeing to order k (or a section vanishing to order k) is intrinsic and is all we need to make this definition precise:

Definition 9.1. A *differential operator of order* k between vector bundles $E, F \to M$ is a map $D\colon \Gamma(E) \to \Gamma(F)$ such that if s_1 and s_2 agree to order k at p, then $(Ds_1)(p) = (Ds_2)(p)$.

A *differential operator* between vector bundles $E, F \to M$ is a map $D\colon \Gamma(E) \to \Gamma(F)$ which is locally a differential operator of order k for some k.

If we pick coordinates for M and trivializations of the bundles involved, sections can be identified with functions and two functions agree to order k only if their first k-derivatives agree, hence a differential operator acquires the form

$$(Ds)(p) = \tilde{D}\left(s_\alpha(p), \left.\frac{\partial}{\partial x_i} s_\alpha\right|_p, \dots, \left.\frac{\partial^I}{\partial x_I} s_\alpha\right|_p\right), \quad |I| \leq k \tag{9.2.1}$$

for some function $\tilde{D}$ where s_α are the functions corresponding to the local expression for s in the given trivialization. Often we need to concretely handle differential operators and the expression (9.2.1) allows us to do precisely that.

Notice that in Definition 9.1, we allowed for the order of a differential operator to vary from point to point, which opens the door for "pathological" cases in which an operator does not have a global order.

Example 9.2. Let $\{\psi_i\colon \mathbb{R} \to \mathbb{R}\}_{i\in\mathbb{N}}$ be collection of smooth functions such that the support of ψ_i is in the interval $(i, i+1)$ and let

$$D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \qquad D(f) = \sum_{i>0} \psi_i \frac{\partial^i}{\partial x^i}.$$

Then D is a differential operator between two copies of the trivial bundle over $\mathbb{R}$. In each interval $[i, i+1]$, D has order i, but D is not an operator for order k for any $k \in \mathbb{N}$. ■

Among all differential operators, there is a class that is easier to deal with, appears with frequency in real situations and often gives us a good guidance on what to expect from general differential operators:

Definition 9.3. A *linear differential operator* is a differential operator $D\colon \Gamma(E) \to \Gamma(F)$ which is also a linear map.

Exercise 9.1. A linear differential operator of order zero $D\colon \Gamma(E) \to \Gamma(F)$ is a tensor.

Most of the differential operators we encountered so far were either outright tensors, such as the curvature of a connection, or first order linear differential operators, such as d, ∂, $\overline{\partial}$, ∇. Yet, one can construct higher order differential operators by composing lower order ones.

Exercise 9.2. Let $D_1\colon \Gamma(E_1) \to \Gamma(E_2)$ and $D_2\colon \Gamma(E_2) \to \Gamma(E_3)$ be differential operators of order k_1 and k_2 respectively, then $D_2 \circ D_1$ is a differential operator of order $k_1 + k_2$. Observe that this also holds if we declare that the zero operator has order $-\infty$.

For example, in a complex manifold $\partial\overline{\partial}$ is an operator of order two. Other operators that are candidates to being order two operators include d^2, $\overline{\partial}^2$ and ∇^2, but in all these cases, the actual order of the operator is lower than expected.

What determines the order of a differential operator are the terms with the highest derivatives and it is a good idea to get a conceptual grip into that part. Bear in mind that to get an expression such as (9.2.1), we need to pick coordinates around the point in question and trivializations of the bundles, so the term with the highest derivatives in that expression is hardly an invariant of the differential operator, but there are couple of considerations that lead us to the invariant object.

Firstly, we notice that if we change coordinates the derivatives in the expression for D will transform accordingly and each derivative can either act on the section itself or on the functions that pop up because of the change of coordinates. This way, an order i derivative present in $\tilde{D}$ will yield an order i derivative in the expression for D in the new coordinates which changes as an element in $\mathrm{Sym}^i TM$ plus a bunch of lower order terms. A concrete example may make this point clearer.

Example 9.4. Consider the Laplacian in $\mathbb{R}^2$, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and let us try to write it in polar coordinates.

We have

$$x = r\cos\theta \quad \Rightarrow \quad dx = \cos\theta dr - \sin\theta r d\theta,$$

$$y = r\sin\theta \quad \Rightarrow \quad dy = \sin\theta dr + \cos\theta r d\theta,$$

hence

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Therefore, as tensors we have

$$\begin{aligned}\frac{\partial}{\partial x}\odot\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\odot\frac{\partial}{\partial y} &= (\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})^2 + (\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})^2 \\ &= \cos^2\theta\frac{\partial}{\partial r}\odot\frac{\partial}{\partial r} - 2\cos\theta\frac{\sin\theta}{r}\frac{\partial}{\partial r}\odot\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta} + \\ &+ \sin^2\theta\frac{\partial}{\partial r}\odot\frac{\partial}{\partial r} + 2\sin\theta\frac{\cos\theta}{r}\frac{\partial}{\partial r}\odot\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta} \\ &= \frac{\partial}{\partial r}\odot\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\odot\frac{\partial}{\partial\theta}.\end{aligned}$$

While the change of coordinate for the Laplacian is

$$\begin{aligned}\triangle &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \\ &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\circ\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\circ\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) \\ &= \cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + 2\frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} \\ &+ \sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - 2\frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}.\end{aligned}$$

Here we see concretely that the degree two component corresponds to the change of $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ thought of as an element in $\mathrm{Sym}^2T\mathbb{R}^2$ but lower degree operators appear because of derivative terms acting on the change the functions for change of coordinates. ■

Secondly, D maps $\Gamma(E)$ to $\Gamma(F)$ so if anything, the domain and codomain of the highest degree part should be related to E_p and F_p. Once again, the highest degree derivative must act of the section itself so the functions for change of trivialization just relate the different identifications of E_p and F_p with the reference vector space.

With all of this we hope to have argued that the term that takes the k^{th} derivatives in an operator of order k is an element

$$\sigma \in \mathrm{Sym}^kTM\otimes\mathrm{End}\,(E;F)$$

Definition 9.5 (Principal symbol – moral definition)**.** Given a linear differential operator of order k, $D\colon\Gamma(E)\to\Gamma(F)$, the *principal symbol* of D is the associated tensor

$$\sigma(D)\in\mathrm{Sym}^kTM\otimes\mathrm{End}\,(E;F)$$

We can concretely get our hands on $\sigma(D)$. First, we observe that TM is the dual to T^*M and hence we can regard $\sigma(D)$ as a symmetric map

$$\sigma(D)\colon\mathrm{Sym}^kT^*M\to\mathrm{End}\,(E;F).$$

Further, we observe that a symmetric map is fully determined by the values it takes in the small diagonal, that is, on elements of the form $(\xi, \ldots, \xi)$, hence we can regard $\sigma(D)$ as

$$\sigma(D)\colon T^*M \to \operatorname{End}(E;F), \quad \sigma(D)(\lambda\xi) = \lambda^k \sigma(D)(\xi).$$

And finally we can concretely compute the expression above as follows. Given $\xi \in T_p^*M$, let f be a function such that $f(p) = 0$ and $df|_p = \xi$. For $v \in E_p$, let $s \in \Gamma(E)$ be a section such that $s(p) = v$, then $f^k s$ vanishes to order $k-1$ at p, hence when computing $D(f^k s)$, only the k^{th} order derivatives of D will contribute to the result (at the point p) and this contribution will only come about if all the derivatives are applied to the coefficient f^k, one derivative per copy of f, that is, only the value of $s(p)$ and $df|_p$ contribute to the result, and the outcome is precisely $\sigma(D)(\xi)(v)$.

All of this should be the justification for the following definition

Definition 9.6 (Principal symbol – practical definition)**.** Given a linear differential operator of order k, $D\colon \Gamma(E) \to \Gamma(F)$, the *principal symbol* of D is the map

$$\sigma(D)\colon T^*M \times E \to F,$$

defined by

$$\sigma(D)|_p(\xi; v) = D(f^k s)|_p,$$

where f is any function satisfying $f(p) = 0$ and $df|_p = \xi$ and s is any section of E such that $s(p) = v$.

The principal symbol satisfies several properties

Exercise 9.3. Let $D_1\colon E_1 \to E_2$ be a linear differential operator of order k_1 and $D_2\colon E_2 \to E_3$ be a linear differential operator of order k_2. Then

- $\sigma(D_1)(\lambda\xi; v) = \lambda^{k_1}\sigma(D_1)(\xi; v)$,
- $\sigma(D_1)(\xi; \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1\sigma(D_1)(\xi; v_1) + \lambda_2\sigma(D_1)(\xi; v_2)$,
- $\sigma(D_2 \circ D_1)(\xi; v) = \sigma(D_2)(\xi, \sigma(D_1)(\xi, v))$,
- If $\sigma(D_1) = 0$, then D_1 is an operator of order $k_1 - 1$.

Now it is time to get our hands dirty and compute the symbols of a few operators we have encountered so far.

Example 9.7. Consider $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$. This an order 1 operator, so to following the recipe above, to compute its symbol we pick $\xi \in T_p^*M$, $\rho \in \Omega^k(M)$ and f such that $f(p) = 0$ and $df|_p = \xi$. Then

$$\sigma(d)(\xi; \rho(p)) = (d(f\rho))|_p = (df \wedge \rho + f d\rho)| + p = \xi \wedge \rho(p),$$

That is, $\sigma(d)(\xi, \bullet) = \xi \wedge \bullet$. ■

Example 9.8. Let $E \to M$ be a vector bundle and ∇ be any connection on E. As in the previous example, ∇ is an order 1 operator, so to compute its symbol we pick $\xi \in T_p^*M$, $s \in \Gamma(E)$ and f such that $f(p) = 0$ and $df|_p = \xi$ and compute

$$\sigma(\nabla)(\xi; s(p)) = (\nabla(fs))|_p = (dfs + f\nabla s)| + p = \xi s(p),$$

That is, $\sigma(d)(\xi, \bullet) = \xi \wedge \bullet$.

This should not be a surprise since locally $\nabla = d + A$, or, said another way, locally $\nabla - d = A$ is an operator of order zero, hence the symbols of ∇ and d agree. ■

Exercise 9.4. Compute the symbols of ∂, $\overline{\partial}$ and $\partial\overline{\partial}$.

The class of operators we will focus on are called elliptic.

Definition 9.9. A linear differential operator of order k, $D\colon \Gamma(E) \to \Gamma(F)$ is *elliptic* if for all p and all $\xi \in T_p^*M \backslash \{0\}$

$$\sigma(D)(\xi, \bullet)\colon E_p \to F_p$$

is an isomorphism.

9.3 Formal adjoints

Given a vector bundle over a compact orientable manifold, $E \to M$, if we fix a volume form, σ, on M and a fiberwise inner product $\langle \cdot, \cdot \rangle$ on E we obtain an inner product on $\Gamma(E)$ by declaring

$$\langle\langle s_1, s_2 \rangle\rangle_E = \int_M \langle s_1, s_2 \rangle \sigma.$$

Given a linear differential operator $D\colon \Gamma(E) \to \Gamma(F)$, if we fix a volume form, σ on M and fiberwise inner products on E and F we can try to find a *formal adjoint* of D, that is a map $D^*\colon \Gamma(F) \to \Gamma(E)$ such that

$$\langle\langle Ds, t \rangle\rangle_F = \langle\langle s, D^* t \rangle\rangle_E, \qquad \forall s \in \Gamma(E), t \in \Gamma(F).$$

Such a map D^* always exists and is again a differential operator.

Proposition 9.10. *The formal adjoint of a differential operator of order k exists, is unique and is a differential operator of order k.*

Proof. Uniqueness follows from non degeneracy of the inner product. If D^* and $\tilde{D}^*$ are formal adjoints of D, then for all $s \in \Gamma(E)$ and $t \in \Gamma(F)$,

$$\langle\langle s, (D^* - \tilde{D}^*)t \rangle\rangle_E = \langle\langle (D - D)s, t \rangle\rangle_F = 0.$$

Since this is true for all s, $(D^* - \tilde{D}^*)t = 0$ for all t and hence $D^* = \tilde{D}^*$.

The strategy to prove that D^* exists and is a differential operator of the same order as D is quite simple: use a partition of unity to write explicit expressions for the differential operator, integrate by parts and then use nondegeneracy of the fiberwise metric on E to re-write the result as an inner product. As it is, there is no conceptual difference between the proofs for a first order operator and a higher order operator, so we will deal only with the former case.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of M over which we can find trivializations of E and F and charts $U_\alpha \to \mathbb{R}^n$. Let $\{\psi_\alpha\}_{\alpha\in A}$ be a partition of unit subordinate to the cover $\mathcal{U}$ and let g_α^{ij} and h_α^{ij} be the functions that make up the inner products of E and F in the trivializations over U_α. Denote by D_α the expression of D in the coordinate chart defined on U_α and let $s_\alpha = \sum s_i^\alpha e_\alpha^i$, $t_\alpha = \sum t_i^\alpha f_\alpha^i$ be sections of E and F respectively expressed in terms of the local frame over U_α. Then

$$D_\alpha s_\alpha = \sum_{i,j,l} (a_j^{0,i} s_i^\alpha + a_j^{1,l,i} \frac{\partial s_i^\alpha}{\partial x_l}) f_\alpha^j.$$

And we can compute

$$\begin{aligned}
\langle\langle Ds, t\rangle\rangle_F &= \langle\langle Ds, \sum_\alpha \psi_\alpha t\rangle\rangle_F = \sum_\alpha \langle\langle Ds, \psi_\alpha t\rangle\rangle_F = \sum_\alpha \langle\langle D_\alpha s_\alpha, \psi_\alpha t_\alpha\rangle\rangle_F \\
&= \sum_{i,j,k,\iota,\alpha} \langle\langle (a_j^{0,i} s_i^\alpha + a_j^{1,l,i} \frac{\partial s_i^\alpha}{\partial x_l}) f_\alpha^j, \psi_\alpha t_\iota^\alpha f_\alpha^\iota\rangle\rangle_F \\
&= \sum_{i,j,k,\iota\alpha} h^{\iota j} (a_j^{0,i} s_i^\alpha + a_j^{1,l,i} \frac{\partial s_i^\alpha}{\partial x_l}) \psi_\alpha t_\iota^\alpha \\
&= \sum_{i,j,k,\iota\alpha} s_i^\alpha (a_j^{0,i} - a_j^{1,l,i} \frac{\partial}{\partial x_l}) h^{\iota j} \psi_\alpha t_\iota^\alpha \\
&= \sum_{i,j,k,\iota,\kappa,\gamma,\alpha} g^{i\kappa} g_{\kappa\gamma} s_i^\alpha (a_j^{0,i} - a_j^{1,l,i} \frac{\partial}{\partial x_l}) h^{\iota j} \psi_\alpha t_\iota^\alpha \\
&= \sum_{i,j,k,\iota,\kappa,\gamma,\alpha} \langle\langle s_i^\alpha e_\alpha^i, g_{\kappa\gamma} (a_j^{0,i} - a_j^{1,l,i} \frac{\partial}{\partial x_l}) h^{\iota j} \psi_\alpha t_\iota^\alpha e_\alpha^\iota\rangle\rangle_E,
\end{aligned}$$

where in the second line we used the local expression for D, in the third line we wrote the inner product on $\Gamma(F)$ explicitly, in the fourth line we integrated by parts and in the fifth line we induced the appearance of the inner product on $\Gamma(E)$.

This shows that D^* exists and has the unpleasant shape

$$D^* t = \sum_{i,j,k,\iota,\kappa,\gamma,\alpha} g_{\kappa\gamma} (a_j^{0,i} - a_j^{1,l,i} \frac{\partial}{\partial x_l}) h^{\iota j} \psi_\alpha t_\iota^\alpha e_\alpha^\iota$$

which expresses it as a sum of local linear differential operators of order 1. □

For specific bundles, operators and inner products we can find expressions for the adjoint that are easier on the eye. For example, given a metric and orientation on a manifold, the space of forms obtains a fiberwise inner product as determined by the Hodge star operator (see Definition 3.16):

$$\langle \phi, \psi\rangle = \star(\phi \wedge \star\psi).$$

Also M inherits a natural volume form, $\star 1$, hence we have an inner product on forms:

$$\langle\langle \phi, \psi\rangle\rangle = \int_M \star(\phi\wedge\star\psi)(\star 1) = \int_M \phi \wedge \star\psi,$$

where in the last equality we used that for forms of complimentary degree, $\star\alpha \wedge \star\beta = \alpha\wedge\beta$.

With this inner product we can readily determine an expression for the adjoint of the exterior derivative without passing through local coordinates:

Lemma 9.11. *Let M be an n-dimensional, compact, oriented manifold. With the inner product on forms described, when acting on k-forms $d^* = (-1)^k \star^{-1} d\star$*

Proof. As in Proposition 9.10, the proof is just integration by parts. For $\phi \in \Omega^{k-1}(M)$ and $\psi \in \Omega^k(M)$ we have

$$\langle\langle d\phi, \psi\rangle\rangle = \int_M (d\phi)\wedge\star\psi = \int_M (-1)^k \phi\wedge d(\star\psi) = \int_M (-1)^k \phi\wedge\star\star^{-1} d(\star\psi) = \langle\langle \phi, (-1)^k \star^{-1} d(\star\psi)\rangle\rangle$$

□

Exercise 9.5. Following the steps below, compute the formal adjoint of $\overline{\partial}$.

- Show that for complex valued forms, if we extend the Hodge star operator to be complex linear, the following is a fiberwise Hermitian inner product

  $$\langle \phi, \psi\rangle = \star\left(\phi\wedge\star\overline{\psi}\right).$$

  For a shorthand we introduce the operator $\bar{\star}$ which acts on complex forms by $\bar{\star}\psi = \star\bar{\psi}$.

- Using integration by parts, show that $\overline{\partial}^* = (-1)^{k+1} \star^{-1} \overline{\partial}\star$

Exercise 9.6. Let $E \to M$ be a vector bundle with inner product and oriented fibers over a compact oriented manifold and let $\nabla\colon \Omega^\bullet(M;E) \to \Omega^{\bullet+1}(M;E)$ be a connection on E. Find an expression for ∇^*.

Exercise 9.7. Let $E \to M$ be a holomorphic vector bundle with Hermitian inner product over a compact complex manifold. Find an expression for $\overline{\partial}^*\colon \Omega^q(M;E) \to \Omega^{q-1}(M;E)$.

Exercise 9.8. Let $D\colon \Gamma(E) \to \Gamma(E)E$ be a self-adjoint elliptic operator. Show that $\ker(D)$ is orthogonal to $\mathrm{Im}(D)$.

9.4 Elliptic complexes and their Laplacians

In definition 9.9, we introduced the class of elliptic operators and claimed that they would be our object of study. Yet, strangely enough, according to Examples 9.7 to Exercise 9.4, none of the operators we have been considering up to now is elliptic. In this section we see how to produce elliptic operators from the familiar operators we have considered so far.

Definition 9.12. A sequence linear differential operators, $D_i\colon \Gamma(E_i) \to \Gamma(E_{i+1})$ of order k, for $i \in \mathbb{Z}$ is *elliptic at E_i* if for all p and all $\xi \in T_p^*M\backslash\{0\}$ the symbol sequence

$$E_{i-1}|_p \overset{\sigma(D_{i-1})(\xi,\bullet)}{\longrightarrow} E_i|_p) \overset{\sigma(D_i)(\xi,\bullet)}{\longrightarrow} E_{i+1}|_p$$

is exact at the middle space.

An *elliptic complex* is a complex of linear differential operators

$$\cdots \longrightarrow \Gamma(E_{i-1}) \overset{D_{i-1}}{\longrightarrow} \Gamma(E_i) \overset{D_i}{\longrightarrow} \Gamma(E_{i+1}) \longrightarrow \dots,$$

that is $D_i \circ D_{i-1} = 0$, which is elliptic at all E_i.

It follow from Examples 9.7 to Exercise 9.4 that d, ∂, $\overline{\partial}$ give rise to elliptic complexes while a connection ∇ gives rise to an elliptic sequence.

The relation between elliptic sequences and elliptic operators comes from two simple computations.

Lemma 9.13. *Let $A_i\colon V_i \to V_{i+1}$, $i \in \mathbb{Z}$ be a collection of linear maps between finite dimensional vector spaces with inner product and let A_i^* be their duals. Then if*

$$V_{i-1} \stackrel{A_{i-1}}{\to} V_i \stackrel{A_i}{\to} V_{i+1}$$

is exact at V_i then $A_i^*A_i + A_{i-1}A_{i-1}^*\colon V_i \to V_i$ is an isomorphism.

Proof. Assume that $v \in \ker(A_i^*A_i + A_{i-1}A_{i-1}^*)$, then

$$0 = \langle (A_i^*A_i + A_{i-1}A_{i-1}^*)v, v\rangle = \langle A_i v, A_i v\rangle + \langle A_{i-1}^*v, A_{i-1}^*v\rangle = \|A_i v\|^2 + \|A_{i-1}^*v\|^2,$$

hence $A_i v = 0$ and $v \in \ker A_i = \mathrm{Im}(A_{i-1})$. But since $\|A_{i-1}^*v\|^2 = 0$, we conclude that $v \in \ker A_{i-1}^* = \mathrm{Im}(A_{i-1})^\perp$, showing that $v = 0$. $\square$

Lemma 9.14. *Let $D\colon \Gamma(E) \to \Gamma(F)$ be a linear differential operator between vector bundles with inner product. Then $\sigma_{D^*}(\xi, \bullet) = (\sigma_D(\xi, \bullet))^*$.*

Proof. Unfortunately I do not see a better way to prove this than chasing down the symbol from the operator obtained in Proposition 9.10. $\square$

Putting these two together we have the following relation between elliptic sequences and elliptic operators:

Proposition 9.15. *Let $D_i\colon \Gamma(E_i) \to \Gamma(E_{i+1})$, $i \in \mathbb{Z}$, be a sequence linear differential operators between vector bundles with a fiberwise inner product. If the sequence is elliptic at E_i then*

$$\triangle_D = D_i^*D_i + D_{i-1}D_{i-1}^*\colon \Gamma(E_i) \to \Gamma(E_i)$$

is a self adjoint elliptic operator.

Furthermore if the sequence is finite and is elliptic at every E_i, then

$$\not{D}_{ev} = \sum_i D_{2i} + D_{2i-1}^*\colon E_{ev} \to E_{od}$$

$$\not{D}_{od} = \sum_i D_{2i-1} + D_{2i}^*\colon E_{od} \to E_{ev}$$

are also elliptic.

Proof. The symbol of $\triangle_D$ can be computed by composing the symbols of the operators that make it up:

$$\sigma_{\triangle_D}(\xi) = \sigma_{D_i^*}(\xi) \circ \sigma_{D_i}(\xi) + \sigma_{D_{i-1}}(\xi) \circ \sigma_{D_{i-1}^*}(\xi) = \sigma_{D_i}^*(\xi) \circ \sigma_{D_i}(\xi) + \sigma_{D_{i-1}}(\xi) \circ \sigma_{D_{i-1}}^*(\xi),$$

which is invertible by Lemma 9.13.

Next we consider $\not{D}_{ev}$ and $\not{D}_{od}$. The operator $\not{D}_{od} \circ \not{D}_{ev}$ applied to an element in E_{2i} has three components: Two move the index by ± 2, but have zero symbol by the elliptic condition and one preserves the index and is equal to the Laplacian, as indicated below

$$\begin{tikzcd}
{} \arrow[d, "\not{D}_{ev}"] & & & E_{2i} \arrow[dl, "D^*_{2i-1}"'] \arrow[dr, "D_{2i}"] \arrow[dd, "\triangle_D"] & & & \\
{} \arrow[d, "\not{D}_{od}"] & & E_{2i-1} \arrow[dl, "D^*_{2i-2}"'] \arrow[dr, "D_{2i-1}"] & & E_{2i+1} \arrow[dl, "D^*_{2i}"'] \arrow[dr, "D_{2i+1}"] & & \\
{} & E_{2i-2} & & E_{2i} & & E_{2i+2} &
\end{tikzcd}$$

Similarly for $\not{D}_{ev} \circ \not{D}_{od}$, therefore, $\sigma_{\not{D}_{ev}}(\xi) \circ \sigma_{\not{D}_{od}}(\xi)$ and $\sigma_{\not{D}_{od}}(\xi) \circ \sigma_{\not{D}_{ev}}(\xi)$ agree with the symbol of $\triangle_D$ at the appropriate spaces, and since $\sigma_{\triangle_D}(\xi)$ is invertible for $\xi \neq 0$, so are the other two symbols. $\square$

For an elliptic complex, one often denotes all the differential operators D_i by the same symbol, say D, with the understanding that the argument determines which D_i one should use. This is completely analogous to how we denote all exterior derivatives by d, instead of $d_k \colon \Omega^k(M) \to \Omega^{k+1}(M)$.

Proposition 9.16. *Let*

$$\dots \to \Gamma(E_{i-1}) \xrightarrow{D} \Gamma(E_i) \xrightarrow{D} \Gamma(E_{i+1}) \to \dots$$

be an elliptic complex between Hermitian vector bundles and let D^* be the formal adjoint. Denote $\mathcal{H} = \ker(\triangle_D)$, then

1. *A section $s \in \mathcal{H}$ if and only if $Ds = 0$ and $D^* s = 0$*
2. *The spaces $\mathcal{H}$, $\mathrm{Im}(D)$ and $\mathrm{Im}(D^*)$ are pairwise orthogonal,*
3. $\ker(D)$ *is orthogonal to* $\mathrm{Im}(D^*)$,
4. $\ker(D^*)$ *is orthogonal to* $\mathrm{Im}(D)$,

Proof. *1.* If $Ds = 0$ and $D^* s = 0$, then $\triangle_D s = (DD^* + D^* D)s = 0$. Conversely, if $\triangle_D s = 0$, then

$$0 = \langle\langle \triangle_D s, s \rangle\rangle = \langle\langle (DD^* + D^* D)s, s \rangle\rangle = \langle\langle D^* s, D^* s \rangle\rangle + \langle\langle Ds, Ds \rangle\rangle = \|D^* s\|^2 + \|Ds\|^2,$$

showing that both $Ds = 0$ and $D^* s = 0$.

2. All the orthogonality relations are proved similarly, for example, to show that $\mathcal{H}$, $\mathrm{Im}(D)$ are orthogonal, we simply compute the inner product of two elements in those spaces, say, pick $s \in \ker(\triangle)$ and $Dt \in \mathrm{Im}(D)$, then

$$\langle\langle s, Dt \rangle\rangle = \langle\langle D^* s, t \rangle\rangle = 0,$$

where we used *1.* to conclude that $D^* s = 0$.

3 & 4. Follow from the definition of adjoint. $\square$

The main result about elliptic operators is the following

Theorem 9.17. *Let $E \to M$ be a vector bundle with fiberwise inner product over M and let $\triangle\colon \Gamma(E) \to \Gamma(E)$ be a self adjoint elliptic operator on E, then*

1. *we have an orthogonal decomposition $\Gamma(E) = \mathrm{Im}(\triangle) \oplus \mathcal{H}$,*
2. *there is a self adjoint pseudo-differential operator $G\colon \Gamma(E) \to \Gamma(E)$ such that $\ker G = \ker \triangle$, $G \circ \triangle = \triangle \circ G$ and*
$$\triangle \circ G + \Pi_{\mathcal{H}},$$
where $\Pi_{\mathcal{H}}$ denotes the orthogonal projection onto $\mathcal{H}$, the kernel of $\triangle$.
3. *$\mathcal{H}$ is finite dimensional.*

This result has many implications for elliptic complexes as we see next.

Corollary 9.18. *Let*
$$\dots \xrightarrow{D} \Gamma(E_{i-1}) \xrightarrow{D} \Gamma(E_i) \xrightarrow{D} \Gamma(E_{i+1}) \xrightarrow{D} \dots$$
be an elliptic complex, then

1. *D and D^* commute with G,*
2. *We have an orthogonal decomposition*
$$\Gamma(E_i) = \mathrm{Im}(D) \oplus \mathrm{Im}(D^*) \oplus \mathcal{H}. \tag{9.4.1}$$
3. *$\ker(D) = \mathrm{Im}(D) \oplus \ker \triangle_D$ and the natural projection $\ker(D) \to H_D$ gives rise to an isomorphism*
$$\mathcal{H}^k \cong H^k_D = \frac{\ker(D\colon \Gamma(E_k) \to \Gamma(E_{k+1}))}{\mathrm{Im}(D\colon \Gamma(E_{k-1}) \to \Gamma(E_k))}.$$
In particular H^k_D is finite dimensional.

Proof. 1. For a section s we have
$$GDs = GD(\triangle_D G + \Pi_{\mathcal{H}})s = GD\triangle Gs,$$
where we have used that $D \circ \Pi_{\mathcal{H}} = 0$. Similarly
$$DGs = (G\triangle_D + \Pi_{\mathcal{H}})DGs = G\triangle_D DGs = GD\triangle_D Gs,$$
where we have used that $\Pi_{\mathcal{H}} \circ D = 0$ and $D\triangle_D = \triangle_D = DD^*D$. A similar proof hold for D^*.

2. We already established that the three spaces in (9.4.1) are mutually orthogonal. We only need to show that together they generate all sections, but for $s \in \Gamma(E_i)$ we have
$$s = (\triangle G + \Pi_{\mathcal{H}})s = DD^*Gs + D^*DGs + \Pi_{\mathcal{H}}s \in \mathrm{Im}(D) \oplus \mathrm{Im}(D^*) \oplus \mathcal{H}.$$

3. We already have that $\ker D$ is orthogonal to $\mathrm{Im}(D^*)$ and hence lies in the sum $\mathrm{Im}(D) \oplus \mathcal{H}$, which is the orthogonal complement of $\mathrm{Im}(D^*)$, according to *2.* Conversely any element in $\mathrm{Im}(D) \oplus \mathcal{H}$ is in the kernel of D because $D^2 = 0$ and $\mathcal{H}$ is made of elements that are closed and co-closed. Hence $\ker D = \mathrm{Im}(D) \oplus \mathcal{H}$. From this, it follows directly that the degree k D-cohomology is isomorphic to the space of degree k harmonic sections, $\mathcal{H}^k$. □

Exercise 9.9. Let (M, g) be a compact oriented Riemannian manifold, let $(E_i, D)_{i\in\mathcal{N}}$ be an elliptic complex of Hermitian vector bundles and let $s \in H^k_D$ be a cohomology class. Show that $\Pi_{\mathcal{H}} s$ can be characterised by the following two properties:

1. $[\Pi_{\mathcal{H}} s] = [s]$,
2. $\|\Pi_{\mathcal{H}} s\|^2 \leq \|\tilde{s}\|^2$ for all $\tilde{s}$ cohomologous to s.

That is, the harmonic representative is the section or least norm within the cohomology class.

9.5 Dualities

The isomorphism $\mathcal{H}^k \cong H^k_D$ has several consequences which derive from the observation that any linear map $A\colon \mathcal{H} \to \mathcal{H}$ gives rise to a linear map on H_D. To describe linear maps with this property we need a more concrete grip on the operator D

Lemma 9.19. *Let M be a compact, orientend manifold and let g be a Riemannian metric on M. Then $\alpha \in \Omega^k(M)$ is closed/exact if and only if $\star\alpha$ is co-closed/co-exact.*

Proof. The claim follows directly from the identity

$$d^*\alpha = (-1)^k \star^{-1} d \star \alpha.$$

□

Theorem 9.20 (Poincaré duality)**.** *Let M^n be a compact oriented manifold, then there is a nondegenerate pairing*

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}, \qquad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

In particular $b_k(M) = b_{n-k}(M)$.

Proof. Pick a Riemannian metric on M, then $\star\colon \mathcal{H} \to \mathcal{H}$ since it maps closed/co-closed forms to co-closed/closed forms. For $a \in H^k(M)$ a nonzero class, let $\alpha \in \mathcal{H}^k$ be the unique harmonic representative of a, then $\star\alpha \in \mathcal{H}^{n-k}$ is also harmonic and

$$\int_M \alpha \wedge \star\alpha = \|\alpha\|^2 \neq 0,$$

showing that the pairing is nondegenerate. □

A completely analogous result holds for $\overline{\partial}$:

Theorem 9.21 (Serre duality I)**.** *Let M^{2n} be a compact complex manifold, then there is a nondegenerate pairing*

$$H^{p,q}_{\overline{\partial}}(M) \times H^{n-p,n-q}_{\overline{\partial}}(M) \to \mathbb{C}, \qquad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

In particular $h^{p,q}(M) = h^{n-p,n-q}(M)$.

Proof. The proof is identical to the proof of Poincaré duality with two changes. Firstly, the real inner product on forms has to be extended to a Hermitian inner product and for that we introduce the operator $\bar{\star}$:

$$\bar{\star}\alpha = \star\bar{\alpha}.$$

This way the Hermitian inner product on forms is

$$\langle\langle \alpha, \beta \rangle\rangle = \int_M \alpha \wedge \bar{\star}\beta.$$

Then it follows that acting of k forms, $\overline{\partial}^* = (-1)^k \bar{\star}^{-1} \overline{\partial} \bar{\star}^{-1}$.

Also, it is a direct computation that $\bar{\star}\colon \wedge^{p,q} T^*M \to \wedge^{n-p,n-q} T^*M$. Therefore $\bar{\star}\colon \mathcal{H}^{p,q}_{\overline{\partial}} \to \mathcal{H}^{n-p,n-q}_{\overline{\partial}}$ gives rise to the pairing alluded to in the statement of the theorem. □

Exercise 9.10 (Serre Duality II)**.** Let $E \to M$ be a holomorphic vector bundle over a compact complex manifold. Then there is a nondegenerate pairing

$$H^q(M; \wedge^{p,0} T^*M \otimes E) \times H^{n-q}(M; \wedge^{n-p,0} T^*M \otimes E^*) \to \mathbb{C}, \qquad ([\alpha], [\beta]) \mapsto \int_M \langle \alpha, \beta \rangle,$$

where $\langle \bullet, \bullet \rangle$ denotes the natural pairing between E and E^* together with the exterior product of forms.

In particular, we have a nondegenerate pairing

$$H^q(M; E) \times H^{n-q}(M; \wedge^{n,0} T^*M \otimes E^*) \to \mathbb{C}$$

9.6 Pseudo-differential operators

9.7 A parametrix for elliptic differential operators

9.8 The index, Gauss–Bonnet

Chapter 10

Hodge theory

10.1 Kähler manifolds

Among complex manifolds, there is a special class for which magic things happen

Definition 10.1. A *Kähler manifold* is a Hermitian manifold (M, I, g) for which the associated 2-form $\omega = g(\bullet, I, \bullet)$ is closed.

Since the symplectic structure in turn also determines the metric, one can alternatively define a *Kähler manifold to be a manifold with a pair of compatible complex and symplectic structures.*

Example 10.2. Euclidean space, $\mathbb{C}^n$ with the standard symplectic form

$$\omega = \frac{1}{2i}\sum dz_i \wedge d\bar{z}_i = \frac{1}{2i}\sum \partial\overline{\partial}(z_i\bar{z}_i) = \frac{1}{2i}\sum \partial\overline{\partial} r^2.$$

is a Kähler manifold, where r^2 is the square distance to the origin. ■

Example 10.3 (Riemann surfaces)**.** Let Σ be any oriented surface and let g be a Riemannian metric on Σ. The orientation together with the metric give rise to a complex structure on Σ given by counterclockwise rotation by $\pi/2$ (see Theorem 3.15). Notice that by construction the complex structure is compatible with the metric and the associated symplectic form is a volume form and hence is closed, therefore an arbitrary choice of metric and orientation yields a Kähler structure on Σ.

The same holds if we approach Riemann surfaces from the complex point of view. Let (Σ, I) be a Riemann surface and let g be an arbitrary Riemannian metric on Σ. Then $\tilde{g}$ defined by

$$\tilde{g}(X, Y) = \frac{1}{2}(g(X, Y) + g(IX, IY))$$

is compatible with I and the corresponding symplectic form, being a volume form on Σ is closed, hence $(\Sigma, I, \tilde{g})$ is a Kähler manifold. That is, not only are all orientable surfaces Kähler, they are Kähler for any choice of complex structure on Σ. ■

Example 10.4 ($\mathbb{C}P^n$)**.** The projective space is a Kähler manifold. We start with $\mathbb{C}^{n+1}\backslash\{0\}$. In the affine coordinate patch corresponding to the j^{th} coordinate, let $r^2 = \sum_{i\neq j} z_i\bar{z}_i$ be the square distance to the origin and let

$$\omega_j = \frac{1}{2i}\sum \partial\overline{\partial}\log(1 + r^2).$$

The change from the affine coordinates corresponding to the j^{th} patch to the k^{th} patch are given by $w_k = \frac{1}{z_k}$ and $w_i = \frac{z_i}{z_k}$, hence

$$\begin{aligned}
\omega_j &= \frac{1}{2i} \sum \partial\overline{\partial} \log(1 + \sum_{i \neq j} z_i \bar{z}_i) \\
&= \frac{1}{2i} \sum \partial\overline{\partial} \log(\frac{1}{z_k \bar{z}_k}(1 + \sum_{i \neq j} z_i \bar{z}_i)) \\
&= \frac{1}{2i} \sum \partial\overline{\partial} \log(\frac{1}{z_k \bar{z}_k} + \sum_{i \neq j,k} \frac{z_i}{z_k}\frac{\bar{z}_i}{\bar{z}_k} + 1) \\
&= \frac{1}{2i} \sum \partial\overline{\partial} \log(1 + \sum w_i \bar{w}_i) \\
&= \omega_k,
\end{aligned}$$

so these forms patch together to give rise to a global 2-form on $\mathbb{C}P^n$.

Next we show that this form together with the complex structure gives rise to a metric... [compute, compute, compute]. ■

With this example at hand, we can produce many more examples of Kähler manifolds by fiding complex submanifolds

Proposition 10.5. *Let (M, I, g) be a Kähler manifold and let $\iota\colon N \to M$ be a complex submanifold. Then the restriction of the metric of M to N makes N into a Kähler manifold.*

Proof. The complex structure on N is just the restriction of the complex structure on M to TN, which is invariant under I. Since I and g are compatible on T_pM for every $p \in M$, their restriction to T_pN is also compatible. Further, $\omega_N = (\iota^* g)(\bullet, I\bullet) = \iota^*\omega_M$ is therefore closed. □

Notice that a Kähler manifold is in particular symplectic, hence known obstructions to the existence of symplectic structures are also obstructions to the existence of Kähler structures. We give one example.

Proposition 10.6. *Let M^{2n} be a compact manifold. If M admits a symplectic structure, there is a cohomology class $a \in H^2(M)$ such that $a^n \neq 0 \in H^{2n}(M)$. In particular $H^{2i}(M) \neq 0$ for $0 \leq i \leq n$.*

Proof. Indeed, by nondegeneracy ω^n is pointwise a volume form, hence it is a volume form on M and

$$\int_M \omega^n \neq 0.$$

Therefore the class $a = [\omega]$ has the stated properties. □

Example 10.7. The Hopf manifold from Example 3.9 is diffeomorphic to $S^1 \times S^{2n-1}$ has no degree 2 cohomology for $n > 1$ (by the Kunneth formula) and hence is not Kähler.

Similarly, compact connected Lie groups have the same cohomology ring as a product of odd dimensional spheres, with the torus corresponding to the product of circles [4]. That means that for any compact Lie group different from a torus, there is no class $a^n \in H^2(G)$ whose top power is nonzero and the only compact connected Lie groups that admit symplectic structures are tori (which are in fact Kähler) ■

10.2 A bit of symplectic geometry

Prove that symplectic reduction of Kähler manifolds at central elements gives Kähler manifolds. Revisit $\mathbb{C}P^n$.

10.3 Statement of the Hodge theorem and simple consequences

As we have seen a complex manifold comes equipped with a few cohomology operators, namely d, ∂ and $\overline{\partial}$. While the de Rham cohomology is reasonably easy to compute because it is topological (it is the same as singular cohomology with real coefficients), we have been struggling to compute $\overline{\partial}$ cohomology since its introduction in Chapter 3. The closest things we have to a tool are the holomorphic Poincaré lemma and spectral sequence arguments. For Kähler manifolds, however a miracle happens and we can relate Dolbeault cohomology more readily with de Rham cohomology.

This is the content of Hodge"s theorem:

Theorem 10.8 (Hodge)**.** *Let (M, I, g) be a Kähler manifold. Then*

$$\triangle_d = 2\triangle_\partial = 2\triangle_{\overline{\partial}}.$$

Due to the theory of elliptic operators developed last chapter this has deep consequences.

Corollary 10.9. *Let $\phi \in \Omega^k(M;\mathbb{C})$ be a harmonic form in a Kähler manifold and let $\phi^{p,q} \in \Omega^{p,q}(M)$ be its (p,q)-components. Then each $\phi^{p,q}$ is harmonic. In particular*

$$H^k(M;\mathbb{C}) = \mathcal{H}^k_d = \oplus_{p+q=k}\mathcal{H}^{p,q}_{\overline{\partial}} = \oplus_{p+q=k}H^{p,q}_{\overline{\partial}}.$$

Proof. Since $\triangle_{\overline{\partial}}$ preserves the (p,q)-decomposition of forms, we have that

$$0 = \triangle_d\phi = 2\sum_{p+q=k}\triangle_{\overline{\partial}}\phi^{p,q}$$

where in the last sum each term is in a different space so they all must vanish independently. □

Corollary 10.10. *In a compact Kähler manifold $h^{p,q} = h^{q,p}$.*

Proof. Indeed, since complex conjugation swaps ∂ and $\overline{\partial}$ as well as ∂^* and $\overline{\partial}^*$ it maps $\overline{\partial}$-harmonics to ∂-harmonics, but since the Laplacians agree these two spaces are the same and we have

$$h^{p,q} = \dim H^{p,q}_{\overline{\partial}} = \dim \mathcal{H}^{p,q}_{\overline{\partial}} = \dim \mathcal{H}^{q,p}_{\partial} = \dim \mathcal{H}^{q,p}_{\overline{\partial}} = h^{q,p}.$$

□

Corollary 10.11. *Let (Σ, I) be a compact Riemann surface of genus g. Then*

$$h^{0,0}(\Sigma) = h^{1,1}(\Sigma) = \mathbb{C}, \qquad h^{1,0}(\Sigma) = h^{0,1}(\Sigma) = \mathbb{C}^g.$$

Proof. We have that $h^{0,0}(\Sigma) = \mathbb{C}$ since $h^{0,0}(\Sigma)$ is the space of global holomorphic functions and Σ is compact. By Serre dualtiy we have $h^{1,1}(\Sigma) = h^{0,0}(\Sigma) = \mathbb{C}$. Finally, since Σ is Kähler, $h^{1,0} = h^{0,1}$ and $2g = b_2 = h^{1,0} + h^{0,1} = 2h^{1,0}$. □

Corollary 10.12 (Riemann–Roch Theorem)**.** *Let $L \to \Sigma$ be a holomorphic line bundle of degree d over a Riemann surface of genus g, then*

$$h^0(\Sigma; L) - h^1(\Sigma; L) = d - g + 1.$$

Proof. Since $\overline{\partial}$ is an elliptic operator, $H^1(\Sigma; L) = \mathcal{H}^{0,1}(\Sigma; L)$ is finite dimensional. By the discussion in page 106, L admits meromorphic sections and therefore is determined by a divisor. Then due to Corollary 7.46 we have

$$h^0(\Sigma; L) - h^1(\Sigma; L) = d + h^{0,0}(\Sigma) - h^{0,1}(\Sigma) = d + 1 - g.$$

□

Corollary 10.13. *In a compact Kähler manifold the odd indexed Betti numbers are even.*

Proof. Indeed, we have

$$b_{2k+1} = \sum_{p+q=k} h^{p,q} = \sum_{p+q=k, p<q} h^{p,q} + \sum_{p+q=k, p>q} h^{p,q} = \sum_{p+q=k, p<q} h^{p,q} + \sum_{p+q=k, p<q} h^{q,p} = 2\left(\sum_{p+q=k, p<q} h^{p,q}\right).$$

□

Corollary 10.14. *In a Kähler manifold*

$$\overline{\partial}^* \partial + \partial \overline{\partial}^* = \partial^* \overline{\partial} + \overline{\partial} \partial^* = 0$$

Proof. We compute the difference of the Laplacians

$$\begin{aligned} 0 &= \triangle_d - \triangle_\partial - \triangle_{\overline{\partial}} \\ &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) - \partial\partial^* - \partial^*\partial - \overline{\partial}\overline{\partial}^* - \overline{\partial}^*\overline{\partial} \\ &= \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial}. \end{aligned}$$

Since $\partial\overline{\partial}^* + \overline{\partial}^*\partial$ maps $\Omega^{p,q}(M)$ to $\Omega^{p+1,q-1}$ while $\overline{\partial}\partial^* + \partial^*\overline{\partial}$ maps $\Omega^{p,q}(M)$ to $\Omega^{p+1,q-1}(M)$ these two terms must vanish independently. □

Corollary 10.15 ($\partial\overline{\partial}$-Lemma)**.** *In a compact Kähler manifold, the following holds*

$$\ker(\partial) \cap \operatorname{Im}(\overline{\partial}) = \ker(\overline{\partial}) \cap \operatorname{Im}(\partial) = \operatorname{Im}(\partial\overline{\partial}).$$

Proof. We will only prove the inclusion $\ker(\partial) \cap \operatorname{Im}(\overline{\partial}) \subset \operatorname{Im}(\partial\overline{\partial})$ as the converse is automatic and the other equality follows from this by taking complex conjugates. Assume that a form β is $\overline{\partial}$-exact and ∂-closed, that is $\beta = \overline{\partial}\alpha$ for some α and $\partial\beta = \partial\overline{\partial}\alpha = 0$. We need to show that $\beta = \partial\overline{\partial}\gamma$ for some γ. Decomponsing α into $\overline{\partial}$-exact, $\overline{\partial}^*$-exact and harmonic components, we have

$$\beta = \overline{\partial}(\overline{\partial}\tilde{\alpha} + \overline{\partial}^*\tilde{\alpha} + \alpha_{\mathcal{H}}) = \overline{\partial}(\overline{\partial}^*\tilde{\alpha} + \alpha_{\mathcal{H}}),$$

so we may assume that $\overline{\partial}^*\alpha = 0$. The we compute

$$\overline{\partial}\alpha = \overline{\partial}((\triangle_\partial G + \Pi_{\mathcal{H}})\alpha) = \overline{\partial}((\partial\partial^* G + G\partial^*\partial)\alpha) = \overline{\partial}\partial\partial^* G\alpha + G\partial^*\partial\overline{\partial}\alpha = -\partial\overline{\partial}(\partial^* G\alpha).$$

□

Exercise 10.1. Let M be a Kähler manifold and let $d^c = i(\partial - \overline{\partial})$. Show that $\triangle_{d^c} = \triangle_d$.

Exercise 10.2. Let (M, I, g) be a compact Kähler manifold.

1. Consider $(\Omega_c^\bullet(M), d) = \ker d^c \subset (\Omega^\bullet(M;\mathbb{C}), d)$. Show that $(\Omega_c^\bullet(M), d)$ is a subcomplex closed under wedge product of forms.

2. Let $(H_c^\bullet(M), d)$ be the d^c-cohomology. Show that the exterior derivative, d, induces a linear operator on $H_{d^c}^\bullet(M)$ whose square is zero and for which the graded Leibniz rule holds.

3. Show that the natural inclusion $\iota\colon (\Omega_c^\bullet(M), d) \to (\Omega^\bullet(M;\mathbb{C}), d)$ preserves products and induces an isomorphism in cohomology.

4. Show that the projection $(\Omega_c^\bullet(M), d) \to (H_{d^c}^\bullet(M;\mathbb{C}), d)$ induces an isomorphism in cohomology.

5. Show that the differential in $(H_c^\bullet(M), d)$ vanishes.

6. Conclude that the sequence of maps

$$((H_d(M), 0) \xleftarrow{I^*} H_{d^c}^\bullet(M;\mathbb{C}), 0) \xleftarrow{\pi} (\Omega_c^\bullet(M), d) \xrightarrow{\iota^*} (\Omega^\bullet(M;\mathbb{C}), d)$$

   give a quasi-isomorphism between the de Rham cohomology of M and the space of differential forms on M.

10.4 Weird linear algebra

10.5 Hodge identities

10.6 Topological consequences

10.7 Hodge theorem for the ball and Newlander–Niremberg (sketch)

Chapter 11

Kodaira's embedding theorem

11.1 Hodge manifolds

11.2 Kodaira's vanishing theorem

11.3 Quadratic transformations

11.4 Kodaira's embedding theorem

Bibliography

[1] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[2] D. Hilbert, *Ueber die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), no. 4, 473–534.

[3] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326.

[4] H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. (2) **42** (1941), 22–52.

[5] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404.

[6] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1915), no. 1, 89–92.