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Mumford Curves with Maximal Automorphism Group II: Lamé Type Groups in Genus 5–8

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Abstract. A Mumford curve of genus $g = 5, 6, 7$ or 8 over a non-Archimedean field of characteristic p (such that if $p = 0$, the residue field characteristic exceeds 5) has at most $12(g-1)$ automorphisms. In this paper, all curves that attain this bound and their automorphism groups (called of Lamé type) are explicitly determined.

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Introduction

Let $(k, |\cdot|)$ be a non-Archimedean valued field of characteristic $p \geqslant 0$, and X a Mumford curve of genus g over k (i.e., such that its stable reduction is a union of rational curves intersecting in rational points over the residue class field $\bar{k}$ of k). It is known (cf. [6, 2]) that for $g \in \{5, 6, 7, 8\}$, X has at most $12(g-1)$ automorphisms (where we assume for $p = 0$ that $\mathrm{char}(\bar{k}) > 5$). The aim of this work is to determine explicitly all (families of) Mumford curves that attain this bound. The main result is the following:

THEOREM. *Let X be a Mumford curve of genus $g \in \{5, 6, 7, 8\}$ with $12(g-1)$ automorphisms over a non-Archimedean field $(k, |\cdot|)$, such that $\mathrm{char}(\bar{k}) > 5$ if $p = 0$. Then actually $p > 3$ and either*

(a) $g = 5$ *and* $\mathrm{Aut}(X) \cong S_4 \times \mathbf{Z}_2$; *or*
(b) $g = 6$, $\mathrm{Aut}(X) \cong A_5$ *and* $p \neq 5$.

Furthermore, the normalizers of the corresponding Schottky groups are isomorphic to the following amalgams of groups:

in case (a) *to either* (a1) $S_4 *_{\mathbf{Z}_4} D_4$ *or* (a2) $D_3 *_{\mathbf{Z}_2} D_2$;
in case (b) *to either* (b1) $A_5 *_{\mathbf{Z}_5} D_5$, (b2) $A_4 *_{\mathbf{Z}_3} D_3$ *or* (b3) $D_3 *_{\mathbf{Z}_2} D_2$.

We then turn to determining the strata in the moduli space of curves of genus g with automorphism group as above, and then determine the loci of Mumford curves in these strata.

For case (a), we have the following result:

PROPOSITION A. *Let X be a Mumford curve of genus 5 with automorphism group $S_4 \times \mathbf{Z}_2$. Then X occurs in the family of curves C_α constructed as follows; Let $C_\alpha^\sim \to \mathbf{P}_\alpha^1$ be the following one-parameter family of genus 3 curves*

$$C_\alpha^\sim : x^4 + y^4 + z^4 + \alpha(x^2y^2 + y^2z^2 + x^2z^2) = 0;$$

then $\mathrm{Aut}(C_\alpha^\sim)$ *contains S_4, and the 4 lines $x \pm y \pm z = 0$ are bitangent to $C_\alpha^\sim$ at 8 points, which form two disjoint S_4-orbits, say $\{p_i\}_{i=1}^4$ and $\{p_i'\}_{i=1}^4$. The divisor $\sum_{i=1}^4 (p_i - p_i')$ corresponds to an S_4-invariant two-torsion point of* $\mathrm{Jac}(C_\alpha^\sim)$*, so defines an étale $\mathbf{Z}_2$-cover C_α of $C_\alpha^\sim$.*

More precisely, C_α is a Mumford curve exactly for

$$(\mathrm{a1})\ |\alpha| > 1 \quad \textit{or} \quad (\mathrm{a2})\ |\alpha + 2| < 1,$$

where the numeration is compatible with that of the normalizers in the theorem.

The curves in (b) belong to the one-parameter family of genus six curves on the Del Pezzo quintic that was studied by Edge in [4, 5], and its Mumford loci were considered in [8]. For the sake of completeness, we state the result.

PROPOSITION B. *Let X be a Mumford curve of genus $g = 6$ with automorphism group A_5. Then X occurs in the family E_α of genus 6 curves in the Del Pezzo quintic surface which is the strict transform of the family of sextics $E_\alpha^\sim$ in $\mathbf{P}^2$ given by $E_\alpha^\sim : T + \alpha S = 0$ where*

$$T = x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2$$
$$S = (y^2 - z^2)(z^2 - x^2)(x^2 - y^2)$$

More precisely, E_α is a Mumford curve precisely when α is in one of the loci

$$(\mathrm{b1})\ 0 < |\alpha \pm 5\sqrt{5}| < 1; \quad (\mathrm{b2})\ 0 < |\alpha \pm \sqrt{-3}| < 1 \quad \textit{or} \quad (\mathrm{b3})\ |\alpha| > 1$$

where the numeration is compatible with that of the normalizersin the theorem (*and the two different signs in* (b1) *and* (b2) *actually correspond to the same locus in moduli space*).

Notice that for $g \notin \{0, 1, 5, 6, 7, 8\}$ and $p > 0$, the maximal number of automorphisms of a Mumford curve of genus $g \geqslant 2$ is $2\sqrt{g}(\sqrt{g} + 1)^2$ (cf. [2]), and in [3], the corresponding families of curves that attain this bound were explicitly found. Thus, the above theorem and proposition complete the determination of all positive characteristic Mumford curves with maximal automorphism group begun in [3], i.e., the non-Archimedean analogue of the determination of all Hurwitz groups (cf. Conder, [1]).

Here is a short outline of the proofs. We first show that $X \mapsto \mathrm{Aut}(X)\backslash X$ is a cover of $\mathbf{P}^1$ ramified tamely above four points with indices (2, 2, 2, 3) (this follows from manipulating the Hurwitz formula for ordinary curves and uses the main result from [2]). Once the ramification type has been fixed, we use the techniques of [2] to make a finite list of possible (abstract types of) normalizers of the corresponding Schottky groups (there turn out to be only four). Namely, such N form a tree product, which we can combinatorially rewrite as a product over a simpler tree with only four ends (corresponding to the four ramification points). We then look at what finite groups of order $12(g-1)$ can be quotients of these N. Actually, most can be excluded using elementary, but long-winded group theory (of which we defer some to an appendix). To explicitly determine the corresponding $S_4 \times \mathbf{Z}_2$-family, we decompose it into an étale $\mathbf{Z}_2$-part and a genus three S_4-cover of $\mathbf{P}^1$. We then use classical algebraic geometry to make the cover explicit as in the proposition, and then determine the Mumford loci in the moduli by computing a semistable model for the degenerate fibers by the methods from [8].

Notice that a Riemann surface has at most $84(g-1)$ automorphisms, and if this occurs, then it corresponds to a (2, 3, 7)-cover of $\mathbf{P}^1$ — possible automorphism groups are then called *Hurwitz groups*. Here, we are dealing with (2, 2, 2, 3)-covers of $\mathbf{P}^1$ by 'non-Archimedean Riemann surfaces', and as these ramification indices remind us of the Lamé equation, we would like to baptize the corresponding possible automorphism groups of *Lamé type*.

1. Classification of Normalizers of Schottky Groups

1.1. MUMFORD CURVES ([10])

Let $(k, |\cdot|)$ be a non-Archimedean valued field of characteristic $p \geqslant 0$, and X a Mumford curve of genus g over k. This means that there exists a semi-stable model over the valuation ring of k such that the stable reduction of X over the residue field $\bar{k}$ of k is a union of rational curves intersecting in $\bar{k}$-rational points. Equivalently, as a rigid analytic space over k, the analytification X^{an} of X is isomorphic to an analytic space of the form $\Gamma \backslash (\mathbf{P}_k^{1,\mathrm{an}} - \mathcal{L})$, where Γ (the so-called Schottky group of X) is a discrete free subgroup of $\mathrm{PGL}(2, k)$ of rank g (acting in the obvious way on $\mathbf{P}_k^{1,\mathrm{an}}$) with $\mathcal{L}$ as its set of limit points.

PROPOSITION 1.2. *Let X be a Mumford curve of genus $g \in \{5, 6, 7, 8\}$ over a non-Archimedean field $(k, |\cdot|)$ (with $\mathrm{char}(\bar{k}) > 5$ if $p = 0$), such that the order of its automorphism group $\mathrm{Aut}(X)$ is $12(g-1)$. Then $p = 0$ or $p > 3$, the quotient curve $\mathrm{Aut}(X)\backslash X$ is isomorphic to $\mathbf{P}^1$ and the ramification in the corresponding quotient map is tame of type $(2, 2, 2, 3)$.*

Proof. It follows from the proof of Satz 3 in [12] that if $|\mathrm{Aut}(X)| = 12(g-1)$, then $Y := \mathrm{Aut}(X)\backslash X$ is of genus zero and $X \to Y$ ramifies above at most four points. However, if it branches above strictly less than four points, then it is known for $p > 0$ that $|\mathrm{Aut}(X)| \leqslant 2\sqrt{2}(\sqrt{g}+1)^2$ (cf. [2], Section 6), which is strictly less than $12(g-1)$ for g in the prescribed range. On the other hand, if $p = 0$ and less than four points ramify, then the main theorem of [9] implies that $\mathrm{char}(\bar{k}) \leqslant 5$ (it actually classifies all such Mumford curves).

Since Mumford curves are ordinary ([2], (1.2)), the ramification groups at those four points are of the form $\mathbf{Z}_p^{t_i} \rtimes \mathbf{Z}_{n_i}$ for some integers t_i, n_i with $n_i \,|\, p^{t_i} - 1$ and $i = 1, \dots, 4$ (this is because the second ramification group of an ordinary curve is trivial, cf. [11]). Then the Riemann–Hurwitz formula (including higher ramification groups, cf. [11]) in this case implies

$$\sum_{i=1}^{4} \frac{1}{n_i}\left(1 - \frac{2}{p^{t_i}}\right) = -\frac{11}{6}.$$

Suppose that w of the four points are wildly ramified (so $t_i \neq 0$ for them); then the corresponding term on the left-hand side is $> 1/2$, whereas for the $4 - w$ tame points, it is $> -1/2$. Hence, the left-hand side exceeds $w - 2$. As it should equal $-11/6$, we get that $w = 0$ and all ramification is tame. It is then easy to see that $\{n_i\} = \{2, 2, 2, 3\}$ is the unique solution to the above equation. As the ramification is tame, $p = 0$ or $p > 3$. □

1.3. TREES

The group $\mathrm{Aut}(X)$ is (abstractly) isomorphic to the quotient N/Γ, where $\Gamma \subseteq \mathrm{PGL}(2, k)$ is the Schottky group corresponding to X and N is its normalizer in $\mathrm{PGL}(2, k)$. The group N is a finitely generated discrete subgroup of $\mathrm{PGL}(2, k)$, and it can be written as the amalgamated product along a certain tree of finite groups (cf. [2], Section 2). More precisely, let $\mathcal{T}$ be the Bruhat–Tits tree of $\mathrm{PGL}(2, k)$, let $\mathcal{T}_N^*$ be its subtree generated by the limit points and the fixed points of torsion elements of N, seen as ends of $\mathcal{T}$; and let T_N^* be the quotient of $\mathcal{T}_N^*$ by N. Then N is the tree product over T_N^*, if we make it into a tree of groups by labelling an edge or vertex by the stabilizer of any of its lifts to $\mathcal{T}_N^*$, seen as acted upon by N. The ends of T_N^* (which are usually contracted so N becomes a finite tree product) are in bijection with the branch points $X \to \mathrm{Aut}(X)\backslash X$, and their stabilizers are the corresponding ramification groups.

LEMMA 1.4. *Let X be as in* 1.2*, and N the normalizer of the corresponding Schottky group. Then any finite subgroup of N has order coprime to p.*

Proof. It suffices to prove that if we write $N = *_{N_i} N_j$, no group N_i has order divisible by p, as any finite subgroup of N is conjugated to a subgroup of some N_i. But if this were so, then [2] (4.4) (ii) implies that there is wild ramification in $X \to \mathrm{Aut}(X)\backslash X$, which contradicts 1.2. □

CONVENTION 1.5. We let D_n denote the dihedral group of order $2n$.

PROPOSITION 1.6. *Let X be as in* 1.2, *and N the normalizer of the corresponding Schottky group. Then N is* (*abstractly*) *isomorphic to one of the following:*

(i) $D_2 *_{\mathbf{Z}_2} D_3$; (ii) $D_3 *_{\mathbf{Z}_3} A_4$; (iii) $D_4 *_{\mathbf{Z}_4} S_4$; (iv) $D_5 *_{\mathbf{Z}_5} A_5$,

where (iv) *only occurs if* $p > 5$.

Proof. We know $p = 0$ of $p > 3$ by 1.2. To proceed, we use the technique of [2]. Proposition 1 of [2] says that if T^* is a subtree of T_N^* having the same ends as T_N^*, then N is still the tree product over this T^*. Since there are four ends, we can assume T^* to be of the form 'two lines (= four ends) connected by a segment'. We know the stabilizers of these ends have to be cyclic groups of order (2, 2, 2, 3) respectively (say, (2, 2) occurs on the left line and (2, 3) on the right line).

We now use the fact that if a finite subgroup G of N stabilizes a vertex, then the edges going out of the vertex have to be stabilized by cyclic groups of order the branching indices of $\mathbf{P}^1 \to G \setminus \mathbf{P}^1$ (at least in the tame case, cf. [2] (2.10)).

A vertex group, say, G, is a subgroup of PGL(2, k) of order prime-to-p, so by the classification of Dickson ([2], (2.9)), it occurs in the following list:

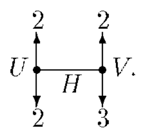
$$G \in \{\mathbf{Z}_n, D_n, A_4, S_4, A_5\}$$

with branching indices of the corresponding $\mathbf{P}^1$-quotient $\mathbf{P}^1 \to G \backslash \mathbf{P}^1$ given by

$$(n, n), (2, 2, n), (2, 3, 4), (2, 4, 4), (2, 3, 5)$$

respectively. We also know that $p > 5$ in the very last case.

If the segment has an inner vertex, only two edges can emanate from it, hence by inspection of the above list, its stabilizer should be a cyclic group. Then actually, any stabilizer of a neighbouring vertex should be the same cyclic group (by the fact that cyclic stabilizers should be maximally cyclic subgroups of corresponding vertex groups, cf. [6], Lemma 1 and [2] (4.1)). Now the next edge has to be stabilized by the same $\mathbf{Z}_n$ (corresponding to the second ramification point in the $\mathbf{Z}_n$-quotient of $\mathbf{P}^1$). Proceeding in the same way, all stabilizers of edges and vertices on the inner segment are the same cyclic group, so we do not change the abstract structure of the corresponding tree product by replacing the segment by just one edge with this cyclic group. Thus, T_N^* looks as in the following picture (where arrows indicate ends):



The possibility that U is cyclic can be excluded, because maximality of cyclic subgroups then implies that H is trivial, but then there is no choice for V that leads to the desired stabilizers of ends (2, 3).

Further inspecting the above list, we see that only $U = D_n$ is possible (so the left line is stabilized by elements of order two). Then the middle segment has to be stabilized by

$H = \mathbf{Z}_n$. If the right vertex also has a dihedral group, then it has to be $V = D_3$ (because we need a (2, 3)-ramification) and $n = 2$, so we in case (i). Similarly, if the right vertex is A_4, S_4 or A_5, then $n = 3, 4, 5$ and we are in case (ii), (iii) and (iv), respectively. □

2. Classification of Groups of Lamé Type

2.1. REMINDER

Let X be a Mumford curve of genus $g \in \{5, 6, 7, 8\}$ with automorphism group $G := \mathrm{Aut}(X)$ of order $12(g-1)$ (which is 48, 60, 72 and 84, respectively). We now know that G is a quotient of some group N as in proposition 1.6 by a free group Γ of rank g. Our aim in this paragraph is to systematically determine all possible such G from these facts. We start with an obvious lemma:

LEMMA 2.2. *Let N be a finitely generated discrete subgroup of* PGL $(2, k)$ *that is abstractly isomorphic to an amalgam $U *_Z V$ of finite groups, and assume that we are given a surjective group homomorphism $\phi\colon N \to G$ to some finite group G. Then if the kernel $\Gamma := \ker(\phi)$ is free of finite type, it maps U and V isomorphically into G, and their images generate G.*

Proof. The morphism ϕ restricted to U is injective, since an element of U in the kernel would give an element of finite order in Γ, which contradicts the fact that Γ is a free group. Since any element of N is already a word in letters from U and V and ϕ is surjective, any element of G is a word in the images of letters from U and V. □

In the next few lemmas, we exclude most possibilities for G using the elementary theory of groups, applying Lemma 2.2. For the reader to appreciate the drastic result that only two groups remain, let us remark that there are 52, 13, 50 and 15 non-isomorphic groups of order 48, 60, 72 and 84, respectively.

Most of the arguments involve counting Sylow groups and a discussion of split extensions. As a compromise to readability, we have deferred the two most technical such arguments to an appendix, and present here only the shorter ones.

Some of the claims below can also be checked by computer, and the authors have used the system GAP ([14]) for double checking a few of the results.

LEMMA 2.3. *In the situation of the main theorem, if $N \cong D_4 *_{\mathbf{Z}_5} A_5$, then $g = 6$, $p \neq 5$ and $G \cong A_5$.*

Proof. By the previous lemma, G has a subgroup isomorphic to A_5, hence the order of G has to be 60, and $g = 6$. Also, $p \neq 5$ by Dickson's classification. □

LEMMA 2.4. *In the situation of the main theorem, if $N \cong D_4 *_{\mathbf{Z}_4} S_4$, then $g = 5$, and $G \cong S_4 \times \mathbf{Z}_2$.*

Proof. Let U and V denote the respective isomorphic images of D_4 and S_4 in N. As G contains V, its order has to be 48 or 72 and, correspondingly, $g = 5$ or $g = 7$.

2.4.1. *The case* $g = 5$. V is of index two in G, hence normal, and as $U \not\subset V$ (since otherwise, $G = V$) and $U \cap V$ contains all elements of order four in U, there is an element of exact order two outside V, so the sequence $1 \to V \to G \to \mathbf{Z}_2 \to 1$ splits. Since $\mathrm{Aut}(S_4) = S_4$ and S_4 has two conjugacy classes of elements of order two, there are at most two such nonisomorphic nontrivial split extensions, and if we write $G = S_4 \rtimes \mathbf{Z}_2, \mathbf{Z}_2 = \langle\gamma\rangle$, they are given by $\gamma \mapsto (12)$ and $\gamma \mapsto (12)(34)$. However, both of these are isomorphic to the trivial one by multiplying on the left with (12) and (12)(34), respectively. Hence, $G = S_4 \times \mathbf{Z}_2$, and then there does exist a surjective homomorphism $N \to G$ (the image of D_4 is $\langle(12)(34)\gamma, (1234)\rangle$, and Z maps to $\langle(1234)\rangle$).

2.4.2. *The case* $g = 7$. The Sylow theorem implies that there are either one or four 3-Sylow groups. Suppose there is only one, say, P (of order 9), then G contains at most four subgroups of order 3 (as all are contained in P), but already V contains four such groups, hence $P \subseteq V$, so $9|24$, a contradiction.

Hence G has precisely four 3-Sylows, say, $\{P_i\}_{i=1}^4$. G acts on these by conjugation, so gives a homomorphism $\phi: G \to S_4$, and $\ker(\phi) = \bigcap_{i=1}^4 N(P_i)$, where $N(-)$ denotes normalizer. This kernel is nontrivial, since $|G|/|S_4| = 3$. For all i, $N(P_i)$ contains 18 elements. Since the $N(P_i)$ cannot have any P_i (of order nine) in common, $\ker(\phi)$ contains three or six elements. The restriction of ϕ to V gives a homomorphism $V \cong S_4 \to A_4$ whose kernel has order dividing 6. But $V \cong S_4$ does not have normal subgroups of order 2, 3, or 6, so $V \cap \ker(\phi) = \{1\}$ and hence $|\ker(\phi)| = 3$.

Therefore, the exact sequence $1 \to \mathbf{Z}_3 \to G \to S_4 \to 1$ splits. The semi-direct product is nontrivial, since otherwise, there would be no copies of D_4 outside S_4 in G. There is a unique such nontrivial semi-direct product given by the signature homomorphism $S_4 \to \mathrm{Aut}(\mathbf{Z}_3) = \mathbf{Z}_2$. If $\gamma \in \mathbf{Z}_3$ and $\alpha \in S_4$, we have $\gamma\alpha = \alpha\gamma^{\mathrm{sgn}(\alpha)}$ in $G = \mathbf{Z}_3 \rtimes S_4$.

The number of 2-Sylows of G is 1, 3 or 9, but there are already three in V and one more in U, so there must be 9. For every two-Sylow Q of $V \cong S_4$ (of which there are three), we get three two-Sylows of G (namely, the conjugates by $\langle\gamma\rangle$) and these are all two-Sylows of G. Now all of these nine 2-Sylows of G are isomorphic to D_4, and hence these are all the subgroups of G isomorphic to D_4. One computes immediately that the 2-Sylows of S_4 intersect in a four group, and then the same holds for all D_4 in G. Hence no D_4 in G can share a cyclic $\mathbf{Z}_4$ with V, and so there is no map $N \to G$. □

LEMMA 2.5. *In the situation of the main theorem, if* $N \cong D_3 *_{\mathbf{Z}_3} A_4$, *then* $g = 6$ *and* $G = A_5$.

Proof. This time, all g can occur. Let U and V denote the respective isomorphic images of A_4 and D_3 in N.

2.5.1. *The case* $g = 5$. This case is excluded by the Theorem A.1 in the appendix.

2.5.2. *The case* $g = 6$. If there were a unique 5-Sylow P in G, then $G/P \cong A_4$ and as $P \cap D_3 = \{1\}$, we would get $D_3 \subseteq A_4$ in the image, a contradiction. Therefore, $G \cong A_5$, as is shown in the appendix A.8.

2.5.3. *The case* $g = 7$. See appendix A.1.

2.5.4. *The case* $g = 8$. Then G has a unique normal 7-Sylow group P, and $U \cong A_4$ and $V \cong D_3$ should still be subgroups of G/P, hence $G/P \cong A_4$, and so $D_3 \subseteq A_4$, a contradiction. □

LEMMA 2.6. *In the situation of the main theorem, if* $N \cong D_2 *_{\mathbf{Z}_2} D_3$, *then* $g = 5$ *and* $G = S_4 \times \mathbf{Z}_2$ *or* $g = 6$ *and* $G = A_5$.

Proof. Again, all g can occur. Let U and V denote the respective isomorphic images of D_2 and D_3 in N.

2.6.1. *The case* $g = 5$. See appendix A.1.

2.6.2. *The case* $g = 6$. If there is a unique 5-Sylow P, then the quotient is a group of order 12 in which D_3 is normal (since of index two); hence it has a unique 3-Sylow, and pulling it back to G we get a normal subgroup of order 15, hence a $\mathbf{Z}_{15}$, hence a unique 3-Sylow in G, which intersects D_3 in $\mathbf{Z}_3$. The quotient of G by this 3-group should have order 20 and be generated by D_2, a contradiction. Hence, the number of 5-Sylows of G is 6. The appendix A.8 shows that $G \cong A_5$.

2.6.3. *The case* $g = 7$. See appendix A.1.

2.6.4. *The case* $g = 8$. There is a unique normal 7-Sylow group P; let $\pi: G \to G/P$ be the corresponding quotient map.

The number of 3-Sylows is 1, 3, or 7. If there are 7, their normalizers are of order 12, so map by π injectively into G/P (which is of order 12). Hence, π gives a splitting $G \cong \mathbf{Z}_7 \rtimes M$ for some such normalizer M. The product is not direct as G cannot have a quotient $\mathbf{Z}_7$ (it is generated by D_2 and D_3). The extension corresponds to a map $M \to \mathrm{Aut}(\mathbf{Z}_7) = \mathbf{Z}_6$. If $M = D_6$, there is only one such nontrivial homomorphism with image $\mathbf{Z}_2$, and then G contains a normal $\mathbf{Z}_7 \times \mathbf{Z}_6$. So there is a quotient $G/\mathbf{Z}_6$ of order 14, which cannot be generated by D_2 and D_3. The second possibility is $M \cong \mathbf{Z}_3 \rtimes \mathbf{Z}_4$, and this leads similarly to a normal $\mathbf{Z}_7 \times \mathbf{Z}_6$.

If there are 4 3-Sylows, then the kernel of the action $G \to S_4$ has 7 or 21 elements, but if there were 21, G would have one or seven 3-Sylows. Hence, the kernel has 7 elements, so equals P, and the image of this map is A_4, which cannot contain D_3.

Hence there is a unique 3-Sylow, which intersects D_3 in a $\mathbf{Z}_3$, so the quotient of G by this 3-Sylow is a group of order 28 generated by D_2, a contradiction. □

These lemmas finish the proof of the main theorem. □

3. Realization as Automorphism Groups

3.1. INTRODUCTION

In this section, we show that the possible pairs (N, G) that are left by the group theoretical arguments of the previous paragraph can indeed be realized by Mumford curves. We note that it suffices to prove that such N can be realized as discrete subgroups of $\mathrm{PGL}(2, k)$, and for this, one could use known methods (e.g., Herrlich, [7]). However, we will take a different approach as follows. For a given g and G, we first explicitly construct the family of algebraic curves of genus g whose automorphism group contains G. This is not too difficult as g is not too large. We then calculate stable models for the bad fibers of these families, and this allows us to read off the Mumford loci and the corresponding normalizers of Schottky groups. Recall that we only have to do this for the remaining case of proposition A.

3.2. PROOF OF PROPOSITION A

Let X be an algebraic curve with $g = 5$ and $G = \mathrm{Aut}(X) = S_4 \times \mathbf{Z}_2$. Let Y be the quotient of X by $\mathbf{Z}_2$, which is an S_4-Galois cover of $\mathbf{P}^1$, whose ramification is tame above at most four points with indices taken from (2, 2, 2, 3). One checks immediately using the Riemann–Hurwitz formula for $Y \mapsto \mathbf{P}^1$ that the only possibility is that it ramifies over four points with indices (2, 2, 2, 3), and then, Y is of genus three. This implies that X is an unramified cover of Y.

Now all curves of genus three with octahedral automorphism group (or larger) have been determined by Wiman in [13], using the fact that the canonical embedding maps such curves to the plane with a number of natural sets of fixed points (inflection, double tangent and sextactic). He finds exactly the one-parameter family $C_\alpha^\sim$ as in proposition A, and observes that the bitangets $x \pm y \pm z = 0$ touch $C_\alpha^\sim$ in 8 points. One calculates immediately that these fall naturally into two S_4-orbits, one of which is given by the four points

$$\{p_i\}_{i=1}^4 = \{(1:\omega:\omega^2), (1:\omega:-\omega^2), (1:-\omega:-\omega^2), (1:-\omega:\omega^2)\},$$

where ω is a fixed third root of unity. The other S_4-orbit is given by $\{p_i'\} = \{p_i^\sigma\}$ for $\langle\sigma\rangle = \mathrm{Gal}(\mathbf{Q}(\omega)/\mathbf{Q})$.

Since X is an unramified 2-cover of $Y = C_\alpha^\sim$, it should correspond to an S_4-invariant two-torsion point in $\mathrm{Jac}(Y)$, i.e., a degree zero S_4-invariant divisor on Y. Now the two lines $L_1, L_2 = 0$ through (p_1, p_2) and (p_3, p_4) together with the two lines $L_3, L_4 = 0$ through (p_1, p_3) and (p_2, p_4) generate a pencil of conics $P_\lambda : L_1 \cdot L_2 + \lambda L_3 \cdot L_4 = 0$ through all four $\{p_i\}$, and it is tangent to one (hence, all) of the bitangents above precisely if $\lambda = \omega^2$. Taking Galois conjugates gives a similar pencil P_λ', and so finally $P_\lambda'' : \ P_\omega + \lambda P_{\omega^2}' = 0$ gives a pencil of conics that are tangent to the four lines $x \pm y \pm z = 0$.

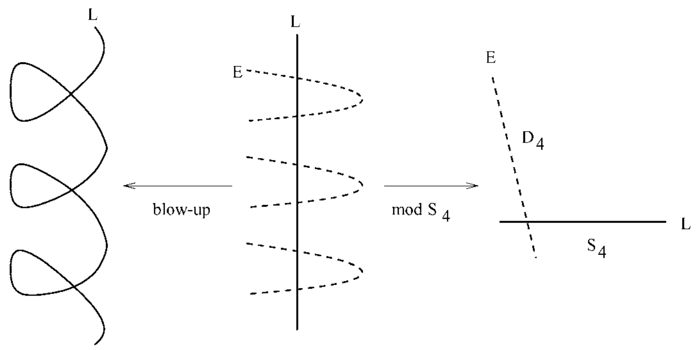
Now suppose $\alpha \neq -1$. Then $C_\alpha^\sim$ does not intersect P_λ'' at its four base points $(1:\pm 1:\pm 1)$, so there is a well defined function $f\colon C_\alpha^\sim \to \mathbf{P}^1\colon P \mapsto \lambda_P$, where λ_P is the value of λ such that P_λ'' goes through P. This function f has divisor $\mathrm{div}(f) = 2(\sum_{i=1}^4 (p_i - p_i'))$, as $\lambda = 0$ (respectively, $\lambda = \infty$) give precisely the conics tangent at $\{p_i\}$ (respectively, $\{p_i'\}$). Finally, $D = \sum_{i=1}^4 (p_i - p_i')$ is a two-torsion element in $\mathrm{Jac}(C_\alpha^\sim)$, which is clearly S_4-invariant and nontrivial (since the four points p_i are not collinear). Hence we have found our desired family C_α.

Note that if $\alpha = -1$, we a priori have such a map f defined outside the base points of the pencil P_λ'', but then it can be extended to all of $C_{-1}^\sim$.

We still have to prove that we have found all such curves, i.e., that there is no further S_4-invariant 2-torsion point in $\mathrm{Jac}(C_\alpha^\sim)$. However, this will be shown at the end of the proof using one of the degenerate fibers which we now study first.

Those degenerate fibers of the family $C_\alpha^\sim$ occur precisely at $\alpha \in \{\pm 2, \infty\}$ (easily seen calculating singularities)—as a matter of fact, the singular fibers of C_α occur at precisely the same values of α, since the cover is étale. We calculate as in [8], Section 3. It is known that $C_\alpha^\sim$ is a Mumford curve precisely when α coincides with a degeneration point over the residue field $\bar{k}$, such that the fiber over that degeneration point has a stable model which is the union of rational curves intersecting in rational points. We will prove that this is the case for $\alpha = \infty$ and $\alpha = -2$, whereas $\alpha = 2$ has (potentially) good reduction.

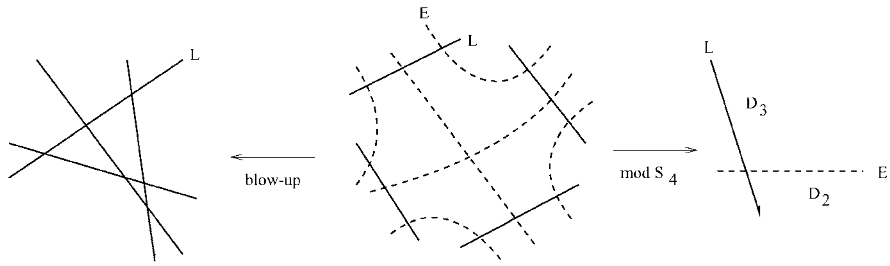
3.2.1. If $\alpha = \infty$, then the special fiber is a rational curve with three nodes, which, after blowing up the double points, becomes a line (solid in the picture) intersecting each of three lines (the special fibers; dashed in the picture) in exactly two points.



Now S_4 acts without inversion on the intersection dual graph of this configuration. The original line is stabilized by S_4, and one sees that the exceptional divisors are stabilized by D_4 (the only group of index three in S_4 up to isomorphism), and these two groups intersect in a $\mathbf{Z}_4$. Hence, if we quotient by S_4, we get two lines (the image of the original line and of the exceptional divisor) which are stabilized by S_4 and D_4, respectively, and so that the intersection point is stabilized by $\mathbf{Z}_4$. So the quotient of a stable model of $C_\alpha^\sim$ over the valuation ring of k for $|\alpha| > 1$ is a rational curve with intersection graph of the special fiber equal to the tree product $S_4 *_{\mathbf{Z}_4} D_4$, and so we are in case (a1).

Now we pass to C_α. The support of the divisor D restricts to smooth points of the rational curve L. Therefore, a semi-stable model of C_∞ is given by two rational curves connected by 6 nonintersecting exceptional divisors (which are rational lines)—the intersection dual of this consists of two points connected by six edges. Thus, one calculates in the same way that case (a1) occurs.

3.2.2. If $\alpha = -2$, the degenerate fiber consists of 4 lines intersecting pairwise, each of which is stabilized by a D_3 (index 4 in S_4), and on which S_4 acts transitively. If we blow up all 6 double points, S_4 acts without inversion on the intersection graph, and the quotient consists of two lines with one double point (the image of a line and the image of an exceptional divisor).



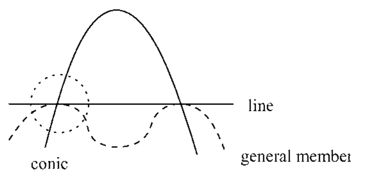
The image of a line is stabilized by D_3 and the exceptional divisor by D_2 (the stabilizer of a double point before the blow-up – note that it is a group of index 6 in S_4 all of whose elements have to have order two, since they should interchange the two lines through the double point), and these intersect in a $\mathbf{Z}_2$. Hence, the quotient of a stable model of $C_\alpha^\sim$ over the valuation ring of k for $|\alpha + 2| < 1$ is a rational curve with intersection graph of the special fiber equal to the tree product $D_3 *_{\mathbf{Z}_2} D_2$, and so we are in case (a2).

Again, the support of the divisor D restricts to smooth points, and so passing from $C_{-2}^\sim$ to C_{-2} just causes the solid lines in the above picture to be doubled. Again, we see that we are in case (a2).

3.2.3. We will now show that curves around $\alpha = 2$ have good reduction. In our model of the pencil $C_\alpha^\sim$, we see only a double conic at $\alpha = 2$, so we change the pencil by picking two new generators to

$$C_t^* : (x^2 + y^2 + z^2)^2 + t(x + y + z)(x - y + z)(x + y - z)(x - y - z) = 0$$

We first look at what happens étale locally around a bitangent point (in the dashed circle below), where the conic $x^2 + y^2 + z^2 = 0$ and the line $x + y - z = 0$ can be replaced by $X = 0$ and $Y = 0$, and the pencil looks like $X^2 - tY = 0$.



To split the multiple fiber, set $t = \tau^2$, then the corresponding surface in (X, Y, τ)-space $X^2 - \tau^2 Y = 0$ has a one-dimensional singularity along $X = \tau = 0$, which blows up to an exceptional divisor $\tau^2 = 0$ and a conic $\left(\frac{X}{\tau}\right)^2 - Y = 0$.

If we are not around a bitangent point, setting $t = \tau^2$ just leads to two transversally intersecting components.

All of this shows that, in the pencil C_t^*, blowing up along $x^2+y^2+z^2$ gives a curve which is a double cover of this conic ramified over eight (bitangent) points, so is a hyperelliptic curve of genus three, so we have good reduction.

Finally, we turn to the proof that the divisor D is the unique S_4-invariant two torsion point in $J_\alpha := \mathrm{Jac}(C_\alpha^\sim)$. For this, notice that $J_\alpha[2]$ is a locally *constant* group scheme $(\underline{\mathbf{Z}/2\mathbf{Z}})^6$, hence to show that there is a unique S_4-invariant vector in this space, we can check it at any particular fiber. We choose $\alpha = 2$, then by the calculation in the previous paragraph the curve is a hyperelliptic cover of $\mathbf{P}^1$ ramified above precisely eight points, corresponding to the eight bitangent points $\{p_i, p_i'\}_{i=1}^4$, and this prescribes the action of S_4 on them. Now the 2-torsion in the Jacobian of a hyperelliptic curve like $C_2^\sim$ is well known to be spanned by divisors supported at those ramification points (= Weierstrass points). Suppose D' is an S_4-invariant element in $J_2[2]$. We know $\mathrm{supp}(D') \subseteq \{p_i, p_i'\}_{i=1}^4$. If a p_i occurs, say, p_1, then its whole S_4-orbit $\{p_i\}$ should occur with the same multiplicity. As the degree of D' should be zero, an element outside this orbit should also occur, hence some p_i', hence its whole S_4-orbit, so $D' = n\sum_{i=1}^4 (p_i - p_i')$ for some odd n, which is the same element of $J_2[2]$ as D. This finishes the proof of the proposition. □

Appendix A. More Group Theory

THEOREM A.1. *Let G be a finite group containing two subgroups U and V which generate G. Consider the following cases:*

(a) $U \cong D_3$, $V \cong D_2$, *and* $U \cap V \cong \mathbf{Z}/2\mathbf{Z}$,
(b) $U \cong D_3$, $V \cong \wedge_4$, *and* $U \cap V \cong \mathbf{Z}/3\mathbf{Z}$.

Then

(1) *If* $|G| = 48$ *and* (a) *holds, G is isomorphic to* $S_4 \times \mathbf{Z}/2\mathbf{Z}$, *whereas* (b) *cannot hold,*
(2) *there is no group G with* $|G| = 72$ *such that either* (a) *or* (b) *holds.*

Proof. We will use the following presentation of the groups: set $G = \langle x, y, z\rangle$ and in case (a): $U = \langle x, y \mid x^3 = y^2 = 1, yxyx = 1\rangle$, $V = \langle y, z \mid y^2 = z^2 = 1, yz = zy\rangle$, $U \cap V = \langle y\rangle$; in case (b): $U = \langle x, y \mid x^3 = y^2 = 1, yxyx = 1\rangle$, $V = \langle x, z \mid x^3 = z^2 = (zx)^3 = 1$, $U \cap V = \langle x\rangle$.

LEMMA A.2. *The 2-Sylow subgroup of G are not normal.*

Proof. Let P_2 be a 2-Sylow subgroup of G. The elements y and xy are of order 2, while the product $xy \cdot y = x$ is of order 3. If P_2 were normal, hence unique, both y and xy would belong to P_2, and, hence, $x \in P_2$, which is absurd. □

LEMMA A.3. *The 3-Sylow subgroups of G are not normal.*

Proof. Suppose a 3-Sylow subgroup P_3 is normal.

Case (a). Since $x \in P_3$, we have $G/P_3 = \langle yP_3, zP_3 \rangle \cong D_2$ and, hence, $|G/P_3| = 4$. It follows that the order of G must be of the form $2^2 \cdot 3^k$, which is not the case when $|G| = 48$ or 72.

Case (b). In A_4 there are two elements of order 3 of which the product is of order 2; e.g., $(123)(124) = (13)(24)$. This implies P_3 contains an element of order 2, which is a contradiction. □

LEMMA A.4. *There are precisely four* 3-*Sylow subgroups in* G.

Proof. Case $|G| = 72$: Immediate from Lemma A.3. and Sylow's theorem.

Case $|G| = 48$. The number of 3-Sylow subgroups is either 1, 4, 16. We know already it is not 1. Suppose it is 16. Then there are $2 \cdot 16 = 32$ elements of order 3 and, hence, all the rest have to form the unique 2-Sylow subgroup, violating Lemma A.2. □

To proceed, we fix the transitive representation $\rho\colon G \to S_4$, induced from the action of G on the set of all 3-Sylow subgroups. We denote its kernel by K.

LEMMA A.5. *The subgroup* K *does not contain a* 3-*Sylow subgroup. Moreover, we have* $|\rho(G)| = 12$ *or* 24.

Proof. Suppose a 3-Sylow subgroup P_3 is contained in K. Then elements of P_3 normalize another 3-Sylow subgroup P', $P_3 \neq P'$. This means that G contains a subgroup isomorphic to a semi-direct product of P_3 and P', which is a 3-group of order strictly larger than that of P_3, whence a contradiction. Hence, the first assertion is proved. It then follows that $|\rho(G)|$ is divisible by 3 since, otherwise, ρ would map each 3-Sylow subgroup to the unit element. Hence, $\rho(G)$ contains an element of order 3, which implies in particular that $\rho(G)$ is doubly transitive. Now $|\rho(G)|$ is $4 \cdot 3$ times the order of the stabilizer of two points, whence the result. □

LEMMA A.6. $x \notin K$.

Proof. If $|G| = 48$, then $|K| = 2$ or 4, hence the assertion. Suppose $|G| = 72$ and $x \in K$. Since in this case $|K| = 3$ or 6, $\langle x \rangle$ is the unique 3-Sylow subgroup of K, hence is normal in G. In Case (b), this contradicts the fact that in V, 3-Sylow subgroups are not normal. In Case (a), we would have $G/K = \langle yK, zK \rangle \cong D_2$, which contradicts Lemma A.5. □

LEMMA A.7. *The order of* $\rho(G)$ *is actually* 24. *In particular,* $|K| = 2$ (*if* $|G| = 48$) *or* 3 (*if* $|G| = 72$).

Proof. Suppose $|\rho(G)| = 12$, $\rho(G) \cong A_4$. Since $x \notin K$ (Lemma A.6), we may assume $\rho(x) = (123)$. It holds that $yxy = x^{-1}$, but there is no element σ of order 2 in A_4 such that $\sigma(123)\sigma = 132$. This is a contradiction. □

Proof of Theorem A.1. Since $G = \langle x, y, z \rangle$, we know $\rho(G) = \langle \rho(x), \rho(y), \rho(z) \rangle = S_4$. As in the proof of Lemma A.7, we may set $\rho(x) = (123)$, and, since $\rho(y)$ has to normalize (123), $\rho(y) = 12$. Since $\rho(G) = S_4$, $\rho(z)$ has to be of order 2, which commutes with (12). Such an element in S_4 is either (12), (34), or (12)(34), where the first one can be discarded since $\rho(G) = S_4$. We can without loss of generality suppose $\rho(z) = (12)(34)$, since if $\rho(z) = (34)$, we can replace z by yz. Set $W = \langle x, z\rangle$. Note that $\rho(W) = \langle(123), (12)(34)\rangle = A_4$, hence in particular, $y \notin W$ (otherwise, the image of ρ would be A_4). We set $K = \langle w\rangle$, where w is of order 2 or 3 for $|G| = 48$ or $|G| = 72$ respectively.

Case (a) *with* $|G| = 48$ ($|K| = 2$): Since K is normal in G and of order 2, we have $K \subseteq Z(G)$. We claim that K is contained in W. Indeed, since y normalizes W (recall: $yxy = x^{-1}$ and $yzy = z$), G is a semi-direct product of W and $\langle y\rangle$, and hence $|W| = 24$. Since $|\rho(W)| = 12$, we have $K \subset W$.

Now set $u = x^{-1}zx$ and $v = x^{-1}ux$. We have $\rho(u) = (13)(24)$ and $\rho(v) = (14)(23)$. In particular, $\rho(zuv) = 1$, whence $zuv \in K = \langle w\rangle$. If $zuv = 1$, then $\{1, z, u, v\}$ forms a subgroup normalized by x, which would imply that W is a semi-direct product of $\langle x\rangle$ and $\{1, z, u, v\}$, hence of order 12, which is a contradiction. Hence, $zuv = w$. Set $r = zw$, $sx^{-1}rx$, $t = x^{-1}sx$. Then we have $rst = 1$. Hence, we obtain a subgroup $\{1, r, s, t\}$ of order four normalized by x. Hence, $\langle x, r\rangle \cong A_4$.

Now since $w \in K \subseteq Z(G)$, one sees easily that y normalizes $\langle x, r\rangle$. Hence, $\langle x, y, r\rangle$ is of order 24. Since $\rho(r) = \rho(z)$, and since $\langle\rho(x), \rho(y), \rho(z)\rangle = S_4$, ρ gives the isomorphism $\langle x, y, r\rangle \cong S_4$.

Since $w \notin \langle x, y, r\rangle$, and $w \in Z(G)$, we have $G \cong S_4 \times \langle w\rangle \cong S_4 \times \mathbf{Z}_2$.

Case (a) *with* $|G| = 72$ ($|K| = 3$): An easy calculation shows $\rho((xz)^3) = 1$, which implies $(xz)^3 = 1, w$, or w^{-1}. If $(xz)^3 = 1$, then we see $W \cong A_4$, and, since y normalizes it, we would have $|G| = |\langle x, y, z\rangle| = 24$, which is absurd. Replacing w by its inverse if necessary, we may suppose $(xz)^3 = w$. Since $\mathrm{Aut}(\mathbf{Z}_3) \cong \mathbf{Z}_2$, we have $xwx^{-1} = w$. Since K is normal, $zwz = w$, or w^{-1}. We claim $zwz = w$. Indeed, otherwise, $w = (xz)^{-3}$ $w(xz)^3 = w^{-1}$, contradiction. Hence, K is contained in the center of $H = \langle x, z, w\rangle$. Note that $\rho(H) = \rho(W) = A_4$.

Let N be the unique normal subgroup of order 4 (and, hence, index 3) in A_4, and set $L = \rho^{-1}(N)$. Then L is a subgroup of H of index 3, hence of order 12. Let S be a 2-Sylow subgroup of L. Then S is isomorphic to $N \cong (\mathbf{Z}_2)^2$. Since K in included in the center of H, it follows that $L \cong K \times S$. Since L is a normal subgroup of H, and S is the unique 2-Sylow subgroup of L, we deduce that S is normal in H, and hence, S is the unique 2-Sylow subgroup of H. In particular, $z \in S$.

Next consider $\langle x, S\rangle \subset H$. In this, S is a normal subgroup, and hence $\langle x, S\rangle$ is a semi-direct product of $\langle x\rangle$ and S, so it is of order 12. Now, since $z \in S$, we see $W \subseteq \langle x, S\rangle$, while $\rho(W) = A_4$ is also of order 12. Hence, $W = \langle x, S\rangle$, so $|W| = 12$, which would imply that, since y normalizes W, $|\langle x, y, z\rangle| = 24$, which is a contradiction.

Case (b) *with* $|G| = 48$ ($|K| = 2$): Since $\rho(yzyz) = 1$, $(yzyz) = 1$ or w. If $(yzyz) = 1$, then the group G with U and $V' = \langle y, z\rangle$ fits in with Case (a), hence is isomorphic to $S_4 \times \mathbf{Z}_2$. But then, there is no subgroup V of G as in the assumption. Hence, we have $yzyz = w$. Since $K \subseteq Z(G)$, we have $yzy = zw$. Then, $1 = y(xz)^3y = (yxyyzy)^3 =$

$(x^{-1}zw)^3 = (x^{-1}z)^3 w$. But, since $(xz)^3 = 1$ and z is of order 2, we have $(x^{-1}z)^3 = 1$, which implies $w = 1$. This is absurd.

Case (b) with $|G| = 72$ ($|K| = 3$): Since K is normal and $\mathrm{Aut}(\mathbf{Z}_3) \cong \mathbf{Z}_2$, we have $xwx^{-1} = w$, i.e, $xw = wx$. We have $zwz = w$, or w^{-1}. If $zwz = w^{-1}$, then $w = (xz)^{-3}w(xz)^3 = zwz = w^{-1}$ (recall: $(xz)^3 = 1$), which is absurd. Hence, $zwz = w$. Therefore, K belongs to the center of $\langle x, z, w \rangle$. Since $\rho(yzyz) = 1$, we have $yzyz = 1, w$, or w^{-1}. The case $yzyz = 1$ is discarded by the same reasoning as before. If $yzyz = w$, then $w^2 = z^2w^2 = (zw)^2 = (yzy)(yzy) = 1$, contradiction. The other case $yzyz = w^{-1}$ can be excluded similarly. □

THEOREM A.8. *If G is a group of order* 60 *with more than one* 5-*Sylow group, then* $G \cong A_5$.

Proof. The number of 5-Sylow groups of G is then 6. The number of 3-Sylow groups of G is either 1, 4 or 10.

If there is a unique 3-Sylow, then the quotient by it is a group of order 20, hence has a unique 5-Sylow that pulls back to a normal subgroup of order 15 in G. This group has to be $\mathbf{Z}_{15}$, so its 5-Sylow is unique. Hence, also G has a unique 5-Sylow (here, we use that Sylows are conjugate, so if one of them belongs to a normal subgroup (here, $\mathbf{Z}_{15}$) of G, then all do). If there are 4 3-Sylows, then we get a transitive action $\phi: G \to S_4$, of which the kernel is the intersection of the normalizers of these Sylows. The normalizers are of order 15, so this kernel is of order 5, 10 or 15. All groups of these orders have a unique 5-Sylow, which is then also normal in G (by the same argument as before).

Hence, there are 10 3-Sylows. The number of 2-Sylows in G is 1, 3, 5 or 15. If there is only one, the corresponding quotient is $\mathbf{Z}_{15}$, which has only one 5-Sylow, and the same argument shows this is impossible. If there are 3, then there is an action $G \to S_3$ with kernel of order 10 or 20, but both of these have a unique 5-Sylow, so the same applies. Now 15 2-Sylows don't fit into G together with the other Sylows, so G has 5 2-Sylows.

Hence, there is an action $G \to S_5$; let K be its kernel. The normalizer of a 2-Sylow has 12 elements, so K has 1, 2, 3, 4 or 6 elements, but if it has four, all normalizers have a common 2-Sylow (of order four). If it is 3 or 6, then K contains a unique 3-Sylow, and the previous argument excludes this possibility. If K has order two, the image has order 30. This image has 1 or 6 5-Sylows, and 1 or 10 3-Sylows, so counting orders, there is at least a normal 5-Sylow or a normal 3-Sylow. Pulling back to G gives a normal subgroup of order 6 or 10, hence a unique 3- or 5-Sylow in G, a contradiction. Hence, K is trivial, and G is a subgroup of S_5 of order 60, so $G \cong A_5$. □

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