Reconstructing global fields using noncommutative geometry

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(joint work with Matilde Marcolli)

The **general philosophy** can be described by the motto *pluralizing zeta.* Zeta functions ζ_X are counting devices associated to invariants of objects X. For example, counting ideals of a given norm in a number field, counting the spectrum of the Laplace-Beltrami operator on a Riemannian manifold, counting points of varieties over finite fields, lengths of geodesics, periodic orbits, and what not. However, such a single zeta function does not always characterize the object up to isomorphism: for number fields, this is the phenomenon of arithmetic equivalence; for Riemannian manifolds, that of isospectrality; for curves over finite fields, Tate's theory of isogenies of Jacobians, etc. We put this single zeta function into a family of zeta functions, indexed by some algebra, and the problem disappears. For number fields, the algebra is the group ring of the maximal abelian extension of the field, for Riemannian manifolds, the smooth functions on it, and so on. In this talk, I focussed on the case of number fields. For other examples, see [4], [5], [7].

I will list various **objects that do/don't determine a number field K.** The Dedekind zeta function of K is $\zeta_K(s) := \sum_{0\neq\mathfrak{a}} N_K(\mathfrak{a})^{-s}$, where the sum runs over all non-zero ideals $\mathfrak{a}$ of the ring of integers of K, and N_K is the norm from K to $\mathbf{Q}$. Knowing ζ_K is the same as knowing the inertia degree $f(\mathfrak{p}|p)$ for all prime ideals $\mathfrak{p}$. A theorem of Mihály Bauer (1903 [1]) says that if K, L are two number fields *that are Galois over* $\mathbf{Q}$, then $K \cong L$ is equivalent to $\zeta_K = \zeta_L$. However, a result of Gaßmann from 1926 [8] says that in general, there do exist non-isomorphic number fields K, L with $\zeta_K = \zeta_L$. Actually, he proves that $\zeta_K = \zeta_L$ is equivalent to the following statement: fix a common extension N of K and L that is Galois over $\mathbf{Q}$ with Galois group G, and let H_K and H_L denote the Galois groups of N/K and N/L, respectively. Then $\zeta_K = \zeta_L$ if and only if each G-conjugacy class intersects H_K and H_L in the same number of elements. A result from Perlis from 1977 [15] says that the smallest degree of a field $K/\mathbf{Q}$ with $\zeta_K = \zeta_L$ but $K \ncong L$ is 7, and an example is given by $K = \mathbf{Q}(\alpha), L = \mathbf{Q}(\beta)$ with $\alpha^7 - 7\alpha + 3 = 0$ and $\beta^7 + 14\beta^4 - 42\beta^2 - 21\beta + 9 = 0$. Here are some further attempts at finding objects that determine isomorphism of number fields K and L: an *isomorphism of adele rings* $\mathbf{A}_K \cong \mathbf{A}_L$ is strictly stronger than equality of zeta functions, but still does not imply field isomorphism (Komatsu, 1976 [10]); an example is $K = \mathbf{Q}(\sqrt[8]{18})$and $L = \mathbf{Q}(\sqrt[8]{288})$. An isomorphism of abelian Galois groups $G_K^{\mathrm{ab}} \cong G_L^{\mathrm{ab}}$ is not enough either: Kubota [11] determined the isomorphism type of G_K^{ab} (its *Ulm invariants*) in terms of K, and Onabe (1976 [14]) gave explicit examples, such as $G^{\mathrm{ab}}_{\mathbf{Q}(\sqrt{-2})} \cong G^{\mathrm{ab}}_{\mathbf{Q}(\sqrt{-3})}$. At the other side of the spectrum, an isomorphism of absolute Galois groups $G_K \cong G_L$ does imply that $K \cong L$! This is due to Neukirch (1969 [13]) when K, L are Galois over $\mathbf{Q}$ and Uchida (1976 [17]) in general. This last theorem is the first manifestation of what Grothendieck called **an**abelian theorems. We conclude that the objects listed above, that are *internal* to a number field K (i.e., can be described in terms of ideals of K), such as $\zeta_K, \mathbf{A}_K$ or G_K^{ab} (which is internal by class field theory), lead to *failure*, whereas a mysterious object G_K, that is *external* to K (described in terms of extensions of K, or via the Langlands program in terms of automorphic forms), leads to *success* …Can we do better, and have "internal success"? A first example is the result of Connes and Consani [3] that the two adele class spaces $\mathbf{A}_K/K^* \cong \mathbf{A}_L/L^*$ are isomorphic as hyperrings over the Krasner hyperfield if and only if K and L are isomorphic.

We go on and look for a good topological space (rather than ring) that does it. And this space will turn out to be *noncommutative*. The method is to consider **class field theory as (noncommutative) dynamical system,** as follows. Let J_K denote the group of fractional ideals of K, J_K^+ the semigroup of integral ideals of K, $\vartheta_K \colon \mathbf{A}_K^* \to G_K^{\mathrm{ab}}$ the Artin reciprocity map and $\hat{\mathcal{O}}_K$ the integral finite adeles of K. Choose a section s of the natural map $\mathbf{A}_{K,f}^* \to J_K \colon (x_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \prod \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$. These objects were used by Ha and Paugam in 2005 [9] [12] to construct a dynamical system associated to K (for $K = \mathbf{Q}$, this is the famous Bost-Connes system [2]), as follows: we make a *topological space* $X_K = G_K^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_K^*} \hat{\mathcal{O}}_K$, consisting of classes $[(\gamma, \rho)]$ for $\gamma \in G_K^{\mathrm{ab}}$ and $\rho \in \hat{\mathcal{O}}_K$, defined by the equivalence $(\gamma, \rho) \sim (\vartheta_K(u^{-1}) \cdot \gamma, u\rho)$ for all $u \in \hat{\mathcal{O}}_K^*$. Then we consider the *action* of $\mathfrak{n} \in J_K^+$ on X_K given by $\mathfrak{n} * [(\gamma, \rho)] := [(\vartheta_K(s(\mathfrak{n}))^{-1}\gamma, s(\mathfrak{n})\rho)]$. In this way, we get a dynamical system (X_K, J_K^+).

Theorem. *For two number fields K and L, an isomorphism $K \cong L$ is equivalent to a norm-preserving isomorphism of dynamical systems $(X_K, J_K^+) \cong (X_L, J_L^+)$.*

By *isomorphism of dynamical systems*, we mean a homeomorphism $\Phi \colon X_K \xrightarrow{\sim} X_L$ and a group homomorphism $\varphi \colon J_K^+ \xrightarrow{\sim} J_L^+$ such that $\Phi(\mathfrak{n} * x) = \varphi(\mathfrak{n}) * \Phi(x)$ for all $x \in X_K$ and $\mathfrak{n} \in J_K^+$; and *norm-preserving* means that $N_L(\varphi(\mathfrak{n})) = N_K(\mathfrak{n})$ for all $\mathfrak{n} \in J_K^+$. The proof is really to "hit the dynamical system with a hammer until enough isomorphic objects jump out".

The result has a reformulation using **quantum statistical mechanics**, by encoding the dynamics in Banach algebra language. We set $A_K := C(X_K) \rtimes J_K^+$ to be the semigroup crossed product C^*-algebra corresponding to the dynamical system. Physically, it corresponds to the *algebra of observables*. If we let $\mu_{\mathfrak{n}}$ and $\mu_{\mathfrak{n}}^*$ denote the partial isometries of the algebra corresponding to $\mathfrak{n} \in J_K^+$, then we also need the non-involutive subalgebra $A_K^\dagger$ of A_K generated by $C(X)$ and $\langle \mu_{\mathfrak{n}} \rangle_{\mathfrak{n} \in J_K^+}$ (but not the $\mu_{\mathfrak{n}}^*$). We also consider a one-parameter subgroup of automorphisms of A_K, denoted $\sigma_K \colon \mathbf{R} \hookrightarrow \mathrm{Aut}(A_K)$, defined by $\sigma_K(t)(f) = f$ and $\sigma_K(t)(\mu_{\mathfrak{n}}) = N_K(\mathfrak{n})^{it}\mu_{\mathfrak{n}}$. The algebra with this so-called *time evolution* is an abstract *quantum statistical mechanical system*. A slightly stronger statement than the main theorem is the following (the proof is similar to that of Davidson and Katsoulidis in [6], combined with ergodicity):

Theorem. *Two number fields K and L are isomorphic if and only if there is an isomorphism of quantum statistical mechanical systems $(A_K, \sigma_K) \xrightarrow{\sim} (A_L, \sigma_L)$ that maps $A_K^\dagger$ to $A_L^\dagger$.*

In a sense, these theorems show that a *suitable combination of failure* (ζ_K, which will be the partition function of the system, G_K^{ab} and $\mathbf{A}_K$, which occur in the system) *may lead to success.* It gives an "internal" description of the isomorphism type of a number field by a noncommutative topological space. One may replace "anabelian" by "noncommutative" …

From the main theorem, we deduce our answer to the problems outlined before:

Theorem. *An isomorphism of number fields $K \cong L$ is equivalent to the existence of an isomorphism $\psi\colon G_K^{\mathrm{ab}} \xrightarrow{\sim} G_L^{\mathrm{ab}}$, such that* ***all*** *abelian L-series match: $L_K(\chi) = L_L((\psi^{-1})^*\chi)$ for all $\chi \in \mathrm{Hom}(G_K^{\mathrm{ab}}, S^1)$.*

The L-series of the trivial character is the zeta function, so this theorem does solve the number theoretical riddle we outlined before. What is more, we discovered this theorem because L-series occur as evaluations of low temperature equilibrium states of the system at particular test functions related to the character. Our proof of this theorem is to deduce from L-series equality an isomorphism of dynamical systems, which basically boils down to a bit of character theory, and then using the main theorem.

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