Collatz – problems

An algebraic approach to the $3n+1$-problem

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Soli Deo Gloria

The fear of the Lord is the beginning of knowlegde;
but fools despise wisdom and instruction.
PROVERBS 1:7

Preface

The first time that I saw the 3n+1-problem was in a book of professor Beukers [Beukers, 1999]. The problem intrigued me, but at that time I was more interested in prime factorization. But after a while, as I learned more about mathematics (I just started college), I realized that prime factorization was to difficult for me to solve. About then the 3n+1-problem became my main interest again.

I started to verify the conjecture, mostly by hand to gain intuition about the behavior of the trajectories. I've looked at the problem in it's binary form trying to find any system. So far I'm not much closer to solving the problem. But I did gain insight in the problem and it's generalizations. Like Paul Erdös said: "Mathematics is not yet ready for such problems.", I agree with him on that.

But since the conjecture first surfaced in mathematics a lot of people tried to solve the problem in many different ways. But one thing amazes me more and more that only so little algebra has been used in trying to solve the problem. In this article we will give natural generalizations of the $3n + 1$-function, in more than one variable as well as over a arbitrary principal ideal domain. We will furthermore give an abstract algebraic definition of a generalized Collatz problem. We will see that these functions form a generalization of ordinary polynomials. I do not attempt to solve the problem but I will try to make mathematics more ready for this kind of problems.

I like to thank Gunther Cornelissen who suffered much while proof reading this paper, without his help this paper wouldn't be what it is today. His suggestions made this article much more readable and formal. I'd also like to thank my fellow students for always listening and always showing interest to me when I was telling them about new ideas.

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Contents

Chapter 1

Introduction

We will start this paper at the place where it all started for me, namely at the $3n+1$-problem. In the first section we will define the actual problem, in the middle section we will shortly discuss the origin of the problem and in the last section we will refer to other articles on the $3n+1$-problem and give a short overview of the rest of the article. But as promised first the definition.

1.1 Definition

The $3n+1$-problem is a very simple one to explain, and can be used at a party to show that mathematics can be fun, but is a very hard problem to prove. The best way to explain the problem to people is to actually let them do some calculations themselves. Pick a number, if it is odd multiply it by 3 and add 1, else divide by 2, and repeat this procedure. This is quite a algorithmic way of describing the $3n+1$-problem, but most people, after doing some calculations, notice that when multiplying by 3 and adding 1 a odd number always becomes even. So we can optimize our “algorithm” by taking this into account. If we now look at the algorithm again we see that it becomes a function mapping from $\mathbb{N}$ to $\mathbb{N}$ of the form:

$$f(n) = \begin{cases} \dfrac{3n+1}{2} & n \equiv 1 \pmod 2 \\ \dfrac{n}{2} & n \equiv 0 \pmod 2 \end{cases} \tag{1.1}$$

Now of course this function is not all this impressive, so why do we wish to study it. Well the real magic of this formula only surfaces if one looks at the iterating it, in other words if one looks at: n, $f(n)$, $f(f(n)) = f^{(2)}$, $\ldots$. If one looks at the iterations of f on elements of $\mathbb{N}$ then one gets to the following conjecture:

Conjecture 1.1 *Let* $n \in \mathbb{N}$ *then there is a* $m \in \mathbb{N}$ *such that* $f^{(m)}(n) = 1$.

Proof: I'd wish. ☹
□

There is a natural extension of this conjecture, which is less magical but generalizes the problem a bit. This conjecture arises if one extends the problem to $\mathbb{Z}$, then one gets the following conjecture:

Conjecture 1.2 *Let* $z \in \mathbb{Z}$ *then there is a* $n \in \mathbb{N}$ *such that* $f^{(n)}(z) \in \{-17, -5, -1, 0, 1\}$.

Or the somewhat weaker form:

Conjecture 1.3 *There exists a finite subset* C *of* $\mathbb{Z}$ *such that for all* $z \in \mathbb{Z}$ *there is an* $m \in \mathbb{N}$ *such that* $f^{(m)}(z) \in C$.

And in fact this conjecture is actually composed of two even weaker conjectures, namely that there are only finitely many cycles and the conjecture that all iterations will eventually end in a cycle. Neither of these two conjectures however, have been proven so far. So why should we believe in the conjecture? Well looking at the problem we see that a random number N has a "chance" of 50% to be multiplied by roughly 1.5 or by 0.5, so basic statistical methods tell us that N should become smaller after sufficient iterations. Of course heuristic argument is hardly a proof and it also fails terribly if we look at the iterations of f on $-\mathbb{N}$ since -17 and -5 (and so also their iterates) never become -1 while the heuristics doesn't change. So we will have to use a much more subtle approach to the problem. But that will not be our (main) goal, but of course a paper on the $3n+1$-problemcan't ever entirely escape from an attempt to solve it. Our main goal will be to define a much more general concept of Collatz-problems.

A rather nice generalization is given by functions of the form:

$$T(x) = \frac{m_i x - r_i}{d} \text{if } x \equiv i \pmod{d},\ 0 \leq i < d \tag{1.2}$$

With $r_i \equiv i m_i \pmod{d}$. We will keep these functions in the back of our mind while generalizing the $3n+1$-problemin the next chapter. There is also a rather nice conjecture [Lagarias, 1985] [Matthews, 2002] for generalizations of this form:

Conjecture 1.4 *If gcd$(d, m_i) = 1$ for all i then if $|m_0 \dots m_{d-1}| < d^d$ then all sequences will eventually cycle. If $|m_0 \dots m_{d-1}| > d^d$ then almost all sequences are divergent. The number of cycles is in both cases finite.*

In the next section we will take a look at the origin of the $3n + 1$-problem.

1.2 Origin

In this section I will tell something about the origin of the $3n + 1$-problem, as well as what we will do in the rest of this paper. This section is based on the first section of an article of Jeffrey Lagarias [Lagarias, 1985], he did an excellent job documenting the origin and many properties of the $3n+1$-function and it's generalizations [Lagarias, 1985] and the research that has been done [Lagarias, 2004]. Some other interesting articles on the $3n + 1$-problem can be found on the homepage of Ken Monks (http://math.scranton.edu/monks/).

The origin of the $3n + 1$-problem is not really known. Therefore there are many names related to the problem. The problem is traditionally is credited to Lothar Collatz. Though we have no actual prove that he was the one who studied the problem first, we do know that he studied a similar problem in 1932. But he never published any of his results concerning this problem. He did however circulated them at the International Congress of Mathematics in 1950 in Cambridge, Massachusetts.

In 1952 the $3n+1$-problem was "discovered" by B. Thwaites. Since than it was given many names like: "Hasse's algorithm", "Syracuse problem", "Kakutani's problem", "Ulam's problem", and the numbers in a sequence of the $3n + 1$-function are sometimes referred to as "hailstone numbers".

Even though the problem hasn't been solved a lot of work has been done in a large number of fields to get a grip on this elegant but slippery problem. More then a hundred articles have been published about the problem and it's generalizations. There has been a conference about the $3n+1$-problem in 1999 in Eichstätt, Germany.

In the next chapter we will take a closer look at the $3n + 1$-function and try to find natural generalizations. Once we found these generalizations will define a ring structure on these function, we will see that such rings always have a ring of polynomials as subring. We will spend a section on studying a few properties of the ideals of these rings. We will also define the generalizations of polynomial maps, that is based on Collatz-functions instead of polynomials. We will see that these maps induce homomorphisms, and sometimes even endomorphisms. In the fourth chapter we will exploit this fact to generalize the $3n + 1$-problem. We will be able to formulate these problems with concepts coming from basic algebra, of course the Collatz rings play an important role to find this generalization but we do not restrict ourselves to these rings.

But as promised in the next chapter we will start by taking a closer look at the $3n + 1$-function and we will define the general notion of a Collatz-function over a principal ideal domain R.

Chapter 2

Collatz-functions

In this chapter we will take a closer look at the $3n+1$-problem, and in particular to the $3n + 1$-function, to see what is actually going on. We then use this view in the next chapters to define new generalizations of the $3n + 1$-problem.

2.1 The 3n+1 function a closer look

To formulate the $3n + 1$-problem in it's most general case we must of course first study the $3n + 1$-problem itself. The usual way of explaining the behavior of the $3n + 1$-function is, if the number is odd then multiply by 3 and add 1, if the number is even then divide by 2, and of course iterate. But then again we all know that an odd number is always mapped to an even number, so we could extend the map to the form in which it is usually presented:

$$f(n) = \begin{cases} \dfrac{3n+1}{2} & n \equiv 1 \pmod 2 \\ \dfrac{n}{2} & n \equiv 0 \pmod 2 \end{cases} \tag{2.1}$$

If we take a closer look at the function we see that, like the algorithm describes, there are two stages. First we take a element of $\mathbb{Z}$ and determine the equivalence class modulo (2), take the function that belongs to that equivalence class to map the element to the ideal (2). After that we apply a map on the ideal (2) to send the image of our element back to the ring $\mathbb{Z}$. So we get the following commutative diagram which shows us what is actually going on.

$$\begin{array}{ccccc} \mathbb{Z} & & \xrightarrow{\quad f \quad} & & \mathbb{Z} \\ \Big\downarrow{\scriptstyle id\times q} & & & & \Big\uparrow{\scriptstyle d} \\ \mathbb{Z}\times\mathbb{Z}/(2) & \xrightarrow{id\times s} & \mathbb{Z}\times f_{(2)} & \xrightarrow{c} & (2) \end{array} \tag{2.2}$$

In which $f_{(2)}$ is the set $\{n \mapsto n, n \mapsto 3n+1\}$, in other words the functions that are applied in the first stage. The functions in this diagram are defined as follows:

$$\begin{array}{cc} f & \text{The } 3n+1\text{—function} \\ id \times q & x \mapsto (x, x \bmod 2) \\ id \times s & (x,i) \mapsto (x, f_i) \\ c & (x, f_i) \mapsto f_i(x) \\ d & y \mapsto \frac{y}{2} \end{array} \tag{2.3}$$

We note that the functions $id \times q$, $id \times s$ and c are the functions used in stage one, where we select the proper function to map the element of $\mathbb{Z}$ to (2). The function d represents the second stage in which we map the ideal (element) back to the ring.

Now that we have identified all steps in the process we can look at the requirements of the steps, so that we can define them in the more general context of any ideal of a principal ideal domain R. Let us first look at the first function that we encounter, namely $id \times q$ we don't have any choice here, because there's only one identity and only one canonical map from $\mathbb{Z}$ to $\mathbb{Z}/(2)$. So let's look at the second function $id \times s$. Again when it comes to this function we don't have much choice, except at this point we can choose $f_{(2)}$, the next function requires that for any f_i $f_i(i) \in (2)$. If we further more require that the functions in $f_{(2)}$ are polynomials then that is sufficient to comply with the requirements. The last function of the first stage, c, leaves us again no choice. The canonical choice here is the composition of f_i with x. Note that the property of c that it maps to (2) gave limitations on our choice of the set $f_{(2)}$. When we look at the properties of d we see that first of all it is surjective, and in fact bijective. But further more if we look at (2) and $\mathbb{Z}$ as $\mathbb{Z}$-modules then we see that d is in fact a $\mathbb{Z}$-module homomorphism (in fact an isomorphism).

In the next section we will generalize this diagram in order to get the notion of a Collatz-function. Which, as we shall prove, form a generalization of ordinary polynomials.

2.2 pre-Collatz sets

In this section we will start to work towards the general definition of a Collatz-function over a domain R. We will take the first step by defining a pre-Collatz set over an ideal $I \neq (0)$. In this section R will be an arbitrary domain, and I and J will denote ideals of R which are not equal to (0), and furthermore n is assumed arbitrary but fixed. The use of the definition of a (pre-)Collatz set will probably not be immediately clear. So I ask you to be a bit patient, but I will give the commutative diagram which we will construct.

$$
\begin{array}{ccccc}
R^n & & \xrightarrow{\quad f_{I,d} \quad} & & R \\
\downarrow {\scriptstyle id\times\varphi} & \searrow^{f_I} & & & \uparrow {\scriptstyle d} \\
R^n \times (R/I)^n & \xrightarrow{id\times f_{I,I}} & R^n \times R[x_1,\dots,x_n] & \xrightarrow{\ \circ\ } & I
\end{array} \tag{2.4}
$$

Notice the great similarities between this diagram and diagram 2.2. But we will start by defining a general case of the dotted arrow.

Definition 2.1 *A pre-Collatz set* f *over an ideal* I *of* R *is a map (of sets) from* R^n *to* $R[x_1,\dots,x_n]$ *such that if* $p \mapsto f_p$ *then* f_p *evaluated in the point* p *is an element of* I, $f_p(p) \in I$.

Now strictly speaking a pre-Collatz set is not a set, but a map. On the other hand it does define a set, namely $\mathrm{Img}(f)$. And further more it also assigns to an element of R^n an element of $\mathrm{Img}(f)$. So in fact it doesn't only define the set $\mathrm{Img}(f)$ but also a structure on $\mathrm{Img}(f)$.

Now it is time for an example of a pre-Collatz set, to show that these objects actually exist. If we take $I = R$ then we see that a pre-Collatz set is nothing more than an arbitrary map (of sets) from R^n to $R[x_1,\dots,x_n]$.

In fact if we take a good look at the set of pre-Collatz sets over R then we will see that they actually form a ring. But first let us introduce a notation for the set of pre-Collatz sets over an ideal I.

Definition 2.2 *We denote the set of pre-Collatz sets over* I *by* R^n_I.

Now let us restate our claim in terms of a lemma and prove it.

Lemma 2.3 *We can define a commutative ring structure on* R^n_R*: if* $f, g \in R^n_R$ *then* $f + g : p \mapsto f_p + g_p$ *and* $f \cdot g : p \mapsto f_p \cdot g_p$. *The pre-Collatz set* $0 : p \mapsto 0$ *is the additive identity and* $1 : p \mapsto 1$ *is the multiplicative identity of the ring* $(R^n_R, +, \cdot)$.

Proof: Let us start with proving that $(R^n_R, +)$ is an abelian group. Lets first prove associativity. Let $f, g, h \in R^n_R$ then $(f+g)+h : p \mapsto (f_p+g_p)+h_p$ and $f+(g+h) : p \mapsto f_p+(g_p+h_p)$ but since $f_p+(g_p+h_p), (f_p+g_p)+h_p \in R[x_1, \dots, x_n]$ and because $(R[x_1, \dots, x_n], +)$ is a group we see that $(f + g) + h = f + (g + h)$. Now we want to prove that $0 : p \mapsto 0$ is the additive identity of $(R^n_R, +)$. This is relatively simple $f + 0 : p \mapsto f_p + 0$ and $0 + f : p \mapsto 0 + f_p$ but since $f_p + 0, 0 + f_p \in R[x_1, \dots, x_n]$ and since 0 is the identity of $(R[x_1, \dots, x_n], +)$ we have $f + 0 = f = 0 + f$. Now we have to prove that we have a additive inverse. I claim that $-f : p \mapsto -f_p$ is the additive inverse of $f : p \mapsto f_p$. We see that $f - f : p \mapsto f_p - f_p$ and now we use that $-f_p$ is the additive inverse of f_p in $(R[x_1, \dots, x_n], +)$. And finally we need to prove that $f+g = g+f$ this is another triviality, $f+g : p \mapsto f_p+g_p$ and $g+f : p \mapsto g_p+f_p$ but since $(R[x_1, \dots, x_n], +)$ is commutative we have $f_p + g_p = g_p + f_p$.

Now that we proved that $(R^n_R, +)$ is an abelian group we need to prove that $(R^n_R, +, \cdot)$ is a ring. Let us start with associativity, $(f \cdot g) \cdot h : p \mapsto (f_p g_p)h_p$ and $f \cdot (g \cdot h) : p \mapsto f_p(g_p h_p)$. Now since $(f_p g_p)h_p, f_p(g_p h_p) \in R[x_1, \dots, x_n]$ they are equal because $R[x_1, \dots, x_n]$ is a ring. Next we will prove that $1 : p \mapsto 1$ is the multiplicative identity. $f \cdot 1 : p \mapsto f_p 1$ and $1 \cdot f : p \mapsto 1 \cdot f_p$ but $1f_p = f_p = f_p 1$ in $R[x_1, \dots, x_n]$. Now we will prove the distributive laws. $f \cdot (g + h) : p \mapsto f_p(g_p + h_p)$ but $f_p(g_p + h_p) = f_p g_p + f_p h_p$ but $f \cdot g + f \cdot h : p \mapsto f_p g_p + f_p h_p$ so we have $f \cdot (g + h) = f \cdot g + f \cdot h$. $(f + g) \cdot h : p \mapsto (f_p + g_p)h_p$ but we have $(f_p + g_p)h_p = f_p h_p + g_p h_p$ but $f \cdot h + g \cdot h : p \mapsto f_p h_p + g_p h_p$ so we have that $(f + g) \cdot h = f \cdot h + g \cdot h$. Now all we have left to prove is that $f \cdot g = g \cdot f$, $f \cdot g : p \mapsto f_p g_p$ but $f_p g_p = g_p f_p$ but $g \cdot f : p \mapsto g_p f_p$ so we see $f \cdot g = g \cdot f$.
□

Now of course we wonder what about R^n_I? We start the study of these sets with a very simple but important lemma.

Lemma 2.4 *If* $I \subseteq J$ *then* $R^n_I \subseteq R^n_J$.

Proof: The proof is rather simple. Let $f \in R^n_I$ then $\forall p \; f_p(p) \in I$ but since $I \in J$ we see $\forall p \; f_p(p) \in J$ so $f \in R^n_J$.
□

The next corollary is an trivial but important result of the previous lemma.

Corollary 2.5 *If* $f \in R_I^n$ *then* $f \in R_R^n$.

Proof: This is an immediate consequence of lemma 2.4, since $I \subseteq R$.
□

Now we are able to state the most important property of pre-Collatz sets.

Corollary 2.6 *The pre-Collatz sets in* n *variables form a ring. We will denote it by* R_R^n.

Proof: This is an immediate consequence of lemma 2.3 and corollary 2.5. Since if $f \in R_I^n$ and $g \in R_J^n$ then $f, g \in R_R^n$ and R_R^n is a ring.
□

We are now able to understand the properties of R_I^n fully.

Lemma 2.7 *Let* I *be an ideal of* R *then* R_I^n *is an ideal of* R_R^n.

Proof: Let us first prove that R_I^n is closed under addition. Suppose $f, g \in R_I^n$ then $f : p \mapsto f_p$ and $g : p \mapsto g_p$. Furthermore $f_p(p), g_p(p) \in I$ but since I is closed under addition we have $f_p(p)+g_p(p) \in I$. We also have $f+g : p \mapsto f_p+g_p$ so $f+g \in R_I^n$. Now all we have to prove is that R_I^n is closed under multiplication by elements of R_R^n. Suppose $f \in R_I^n$ and $r \in R_R^n$ we have that $r \cdot f : p \mapsto r_p f_p$. Furthermore $f_p(p) \in I$ and $r_p(p) \in R$ and since I is closed under multiplication by elements of R we have $r_p(p) f_p(p) \in I$. Therefore $r \cdot f \in R_I^n$.
□

It is obvious (take f to be $p \mapsto f_p$ and $\forall p' \neq p \; p' \mapsto 0$ and look at the ideal generated by f) of course that the ideals R_I^n are not the only ideals of R_R^n, but these are the most interesting for us. So we will study them in some more detail.

Lemma 2.8 *Let* I *and* J *be to ideals of* R *then we have the following properties:*

1. $I \subseteq J \Leftrightarrow R_I^n \subseteq R_J^n$
2. $R_I^n \cap R_J^n = R_{I \cap J}^n$
3. $R_I^n R_J^n \subseteq R_{IJ}^n$

Proof: In lemma 2.4 we proved already that $I \subseteq J \Rightarrow R_I^n \subseteq R_J^n$ so we only have to prove the other implication. Suppose that $\forall f \in R_I^n$ we have $f \in R_J^n$ of course we have $f_p(p) \in I$ but by assumption we also have $f_p(p) \in J$ but since $f_p(p)$ can be any arbitrary element of I we have $I \subseteq J$.

Now let us prove property 2. Suppose $f \in R_I^n$ and $f \in R_J^n$ then $\forall p f_p(p) \in I$ and $\forall p\ f_p(p) \in J$ so $\forall p\ f_p(p) \in I \cap J$. This gives us $R_I^n \cap R_J^n \subseteq R_{I\cap J}^n$. The other inclusion follows from property 1, since $I \cap J \subseteq I$ and $I \cap J \subseteq J$ so $R_{I\cap J}^n \in R_I^n$ and $R_{I\cap J}^n \in R_J^n$ and therefore $R_{I\cap J}^n \subset R_I^n \cap R_J^n$.

Now all we need to prove is property 3. Suppose $f \in R_I^n$ and $g \in R_J^n$ then $\forall p\ f_p(p)g_p(p) \in IJ$ since $f_p(p) \in I$ and $g_p(p) \in J$. So $f \cdot g \in R_{IJ}^n$, now since R_{IJ}^n is an ideal it is closed under addition so all sums of products of element in R_I^n and R_J^n are in R_{IJ}^n. So we have $R_I^n R_J^n \subseteq R_{IJ}^n$.
□

We notice that equality need not hold at property 3, because if $f \in R_{IJ}^n$ and $\forall p f_p$ is a irreducible polynomial then f_p can't be written as a product let alone f as a product of two pre-Collatz sets g, h such that $g \in R_I^n$ and $h \in R_J^n$. We will formulate a trivial but important result of the previous lemma.

Corollary 2.9 *If* $f \in R_I^n$ *and* $g \in R_J^n$ *then* $fg \in R_{IJ}^n$.

Proof: This has already been proven in the proof of lemma 2.8.
□

Next we will give a definition that will make in easier to talk about pre-Collatz sets.

Definition 2.10 *Let* f *be a pre-Collatz set then we call* f_p *the branch of* f *at* p.

If we look more closely at the definition of the ring structure on R_R^n we see that it is defined branch wise. That is addition is defined by addition of branches, and multiplication is defined by multiplication of branches. Now let us fix p and study the possible choices of branches above p. We will state the possibilities in the following lemma.

Lemma 2.11 *Let* p *and* I *be fixed then the possible choices for a branch at* p *of a pre-Collatz set over* I *form an ideal of* $R[x_1, \ldots, x_n]$ *which we will denote by* I_p.

Proof: Let $f, g \in I_p$ then $f(p), g(p) \in I$ and since I is an ideal we have $f(p) + g(p) \in I$ so $f + g \in I_p$. Let $r \in R[x_1, \ldots, x_n]$ then $r(p) \in R$ and since I is an ideal we have $r(p)f(p) \in I$ so $rf \in I_p$.
□

We see some similarities between a vector field and a pre-Collatz set, if R^n plays the role of differentiable manifold and I_p plays the role of the tangent space at p. Of course a pre-Collatz set is not a vector field, at least not in the cases we will be interested in.

In the next section we will look at pre-Collatz sets with a bit more structure or even better symmetry, which gives rise to the definition of a Collatz set.

2.3 Collatz sets

In this section we will construct pre-Collatz sets such that they fit along the dotted arrow in diagram 2.4, such that it factors like given by the commutative triangle in the diagram. We will start first by giving the definition of a Collatz function and then continue by looking which structures of pre-Collatz sets also hold for Collatz sets.

Definition 2.12 *A Collatz set* $f_{I_1,\ldots,I_n,I}$ *over ideals* $I_1, \ldots, I_n$ *is a map from* $\overset{n}{\underset{i=1}{\times}} R/I_i$ *to* $R[x_1, \ldots, x_n]$ *such that* $\bigcap_{i=1}^{n} I_i = I$ *and there is a pre-Collatz set* f_I *over* I *such that we have the following commutative diagram:*

$$\begin{array}{ccc} R^n & \xrightarrow{f_I} & \\ \downarrow \varphi & & R[x_1, \ldots, x_n] \\ \overset{n}{\underset{i=1}{\times}} R/I_i & \xrightarrow{f_{I_1,\ldots,I_n,I}} & \end{array} \tag{2.5}$$

Since φ is the product of the canonical maps $\varphi_i : R \to R/I_i$ we can always reconstruct f_I from $f_{I_1,\ldots,I_n,I}$ and vice versa. In other words we can see a Collatz-set $f_{I_1,\dots,I_n}$ as a pre-Collatz set f_I with the following property: if $\forall i\ p_i \equiv p'_i \pmod{I_i}$ then $f_{Ip} = f_{Ip'}$. So f_I is symmetric with respect to $I_1, \dots, I_n$.

Definition 2.13 *Let f_I and $f_{I_1,\dots,I_n,I}$ be as in definition 2.12, then we call f_I the pre-Collatz set associated to the Collatz set $f_{I_1,\dots,I_n,I}$.*

Now of course it is very well possible that two different Collatz sets have the same associated pre-Collatz set. Therefore we give the following definition.

Definition 2.14 *We call two Collatz sets equivalent iff their associated pre-Collatz sets are equal.*

We generalize the notion of a branch to Collatz sets.

Definition 2.15 *Let $f_{I_1,\dots,I_n,I}$ be a Collatz set then we call f_i with $i \in \mathop{\times}\limits_{i=1}^{n} R/I_i$ the branch of $f_{I_1,\dots,I_n,I}$ at i.*

Notice that this definition makes sense because if $\forall j\ p_j \equiv p'_j \pmod{I_j}$ then $f_p = f_{p'}$ and therefore f_i is uniquely defined.

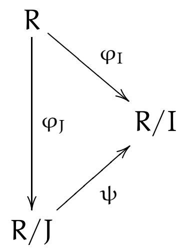
We give a definition of a special case of Collatz sets, although we will prove later on that this is not such a special case at all.

Definition 2.16 *We call a Collatz set $f_{I_1,\dots,I_n,I}$ monideal (over I) if $I_1 = \dots = I_n = I$.*

The word monideal comes from mono ideal. Now we will state exactly what we mean by "not such a special case at all".

Lemma 2.17 *Every Collatz set over $I_1, \dots, I_n$ is equivalent to a monideal Collatz set over $\bigcap\limits_{i=1}^{n} I_i$.*

Proof: If we have $J \subseteq I$ then from elementary algebra we know that we have the following commutative diagram:



(2.6)

In which φ_I, φ_J and ψ are canonical maps. From this we see that if a pre-Collatz set factors over $I_1, \dots I_n$ and $J_1 \subseteq I_1, \dots, J_n \subseteq I_n$. So when it comes to symmetry there are no problems, so far. But we know that $f_p(p) \in \bigcap_{i=1}^{n} I_i$ so if we want our Collatz set to be a Collatz set over $J_1, \dots, J_n$ we'd better ensure that $\bigcap_{i=1}^{n} J_i = \bigcap_{i=1}^{n} I_i$. But if these two conditions are satisfied then our original Collatz set is a Collatz set over $J_1, \dots, J_n$. Since we can take $f_j = f_i$ if $\psi(j) = i$. Now we the proof is simple take $J_i = \bigcap_{c=1}^{n} I_c$, since $\forall c \bigcap_{c=1}^{n} I_c \subseteq I_c$ and the intersection of n equivalent ideals is such an ideal which is $\bigcap_{c=1}^{n} I_c$ so both conditions are satisfied.
□

Notice the uniqueness of the monideal representation, as in we can't find $J_1, \dots, J_n$ such that $\forall i J_i \subseteq \bigcap_{c=1}^{n} I_c$ and $\bigcap_{i=1}^{n} J_i = \bigcap_{i=1}^{n} I_i$. So since this is unique let give a notation for it.

Definition 2.18 *Let $f_{I_1,\dots,I_n,I}$ be a Collatz set over $I_1, \dots, I_n$ then we denote it's equivalent monideal Collatz set by $f_{I,I}$. A branch of a monideal Collatz set $f_{I,I}$ is denoted by: $f_{i_1,\dots,i_n,I}$ or by abuse of notation simply by $f_{i,I}$ or even f_i.*

Notice that for the previous lemma to be true it is essential that R is a domain. Now it may not be immediately clear to the reader, but the difference in definition of a Collatz set with respect to the definition of a pre-Collatz set is rather big. In other words, most pre-Collatz sets do not give rise to a Collatz set. That is because if $J \subseteq I$ then if a pre-Collatz set f factors over I it most certainly factors over J but for f to be a Collatz set over J we also need that $f_p(p) \in J$ but for an arbitrary Collatz-set over I we can only assume that $f_p(p) \in I$. So in other words if you have a lot of choice for the the image of f (for instance $I = R$) then we have a lot of symmetry (in case of $I = R$ there is in fact only one branch). On the other have little symmetry we have also less options for our image $f_p(p)$.

We will now state a lemma which explains the importance of a monideal Collatz set.

Lemma 2.19 *A map from* $(R/I)^n$ *to* $R[x_1, \dots, x_n]$ $f_{I,I}$ *is a Collatz set iff* $\forall i_1, \dots, i_n$ $f_{i_1,\dots,i_n,I}(i_1, \dots, i_n) \in I$ *with* $i_j \in R/I$. *In other words it is sufficient to check* $f_p(p) \in I$ *only for one representative of* $p \in (R/I)^n$ *of each residue class.*

Proof: This is a direct consequence of the binomium of Newton, in which one should interpret the binomial coefficients as $\mathbb{Z}$-actions on the additive structure of R. One sees that one gets two types of terms when one looks at $f_{i_1,\dots,i_m,I}(i_1 + p_1, \dots, i_n + p_n)$ with $p_c \in I$, namely those with a p_c (for an arbitrary c) and those without. It is obvious that the terms with a p_c in it are elements of I (since I is closed under scalar multiplication by R). Furthermore it is obvious that their sum is in I (since I is closed under addition). So the only terms we have to worry about are those without a p_c, but those terms are exactly the terms of $f_{i_1,\dots,i_n,I}(i_1, \dots, i_n)$ and we know that their sum is in I so we have $f_{i_1,\dots,i_n,I}(i_1 + p_1, \dots, i_n + p_n) \in I$.
□

Now that we have defined the notion of a Collatz set lets see which structures that we have defined on pre-Collatz sets can also be defined on Collatz sets. But first of all we introduce another notation:

Definition 2.20 *We denote the set of all monideal Collatz sets in* n *variables over* I *by* $R_I[x_1, \dots, x_n]$.

Now we state the following lemma.

Lemma 2.21 $R_R[x_1, \dots, x_n]$ *forms a ring and* $R_R[x_1, \dots, x_n] \cong R[x_1, \dots, x_n]$.

Proof: The proof of this lemma is rather straightforward. Since $\forall p \in R^n p \equiv p'$ (mod R) we have that if $f \in R_R[x_1, \dots, x_n]$ then f has only one branch, f_0. Furthermore for all $gR[x_1, \dots, x_n]$ we have $\forall p \in R^n g(p) \in R$. So we see that f_0 can be any polynomial in n variables (over R). For the addition we can use branch wise addition, we can do the same with multiplication. The proof that

this forms a ring structure follows from the fact that we have a ring structure on R_R^n and that for any Collatz set we can reconstruct the associated pre-Collatz set. Now we only have to notice that the sum/product still factors over $(R/R)^n$. Another way of proving this lemma is by noticing that f has only one branch, defining addition and multiplication branch wise, and noticing that f_0 can be any polynomial in $R[x_1, \dots, x_n]$. This also proves our second statement since $R_R[x_1, \dots, x_n]$ inherits its ring structure from $R[x_1, \dots, x_n]$.
□

Now we of course wonder about the structure on the set $R_I[x_1, \dots, x_n]$ for arbitrary I. For those we can state the following lemma.

Lemma 2.22 $R_I[x_1, \dots, x_n]$ *forms an abelian group (under addition) and in fact it forms a* $R_R[x_1, \dots, x_n]$*-module.*

Proof: The proof of this is again straightforward, and again we define addition branch wise. So $(f+g)_i = f_i + g_i$. The fact that this group is abelian follows from the fact that $(R[x_1, \dots, x_n], +)$ is. We see that $0 := \forall i 0_i \equiv 0$ is the additive identity of this group, since $(f+0)_i = f_i + 0_i = f_i + 0 = f_i = 0 + f_i = 0_i + f_i = (0+f)_i$. Furthermore let $r \in R_R[x_1, \dots, x_n]$ and $f \in R_I[x_1, \dots, x_n]$ then we define the $R_R[x_1, \dots, x_n]$ branch wise by $f_i \mapsto r_0 f_i$. Since $r_0(p) \in R$ and $f_i(p) \in I$ we have $r_0(p) f_i(p) = r_0 f_i(p) \in I$. So we see that $rf \in R_I[x_1, \dots, x_n]$. Now we have to show that this action of $R_R[x_1, \dots, x_n]$ respects the group structure of $R_I[x_1, \dots, x_n]$. In other words we have to prove that $r(f+g) = rf + rg$. The proof of this is again straightforward $r(f+g)_i = r_0(f_i + g_i) = r_0 f_i + r_0 g_i = rf_i + rg_i$. And of course we have to check that $(rs)f = r(sf)$. So lets do this: $((rs)f)_i = (rs_0)f_i = r_0 s_0 f_i = r_0(s_0 f_i) = (r(sf))_i$, so $((rs)f)_i = (r(sf))_i$ for all i and therefore $(rs)f = r(sf)$. Next we have to check that $(r+s)f = rf + sf$. Again this is done branch wise, $((r+s)f)_i = (r+s)_0 f_i = (r_0 + s_0) f_i = r_0 f_i + s_0 f_i = rf_i + sf_i$ again this holds for all i. And last but not least we have to check that $1f = f$, of course this is just too easy: $(1f)_i = 1_0 f_i = 1 f_i = f_i$ for all i so $1f = f$.
□

The fact that $R_I[x_1, \dots, x_n]$ forms a $R_R[x_1, \dots, x_n]$ module is a direct result of the following more general statement.

Lemma 2.23 *Let* $f \in R_I[x_1, \dots, x_n]$ *and* $g \in R_J[x_1, \dots, x_n]$ *then we can define* $f \cdot g$ *and* $f \cdot g \in R_{IJ}[x_1, \dots, x_n]$. *Furthermore let* $f \in R_I[x_1, \dots, x_n]$ *then* f *defines a module-homomorphism from* $R_J[x_1, \dots, x_n]$ *to* $R_{IJ}[x_1, \dots, x_n]$.

Proof: Let us first define the multiplication of f and g (as given by the lemma). Surprisingly we will do this branch wise this time. First we note that since $f_i(x) \in I$ if $x \equiv i \pmod{I}$ and $g_j(x) \in J$ if $x \equiv j \pmod{J}$, so we have $f_i(x)g_j(x)IJ$ if $x \equiv i \pmod{I}$ and $x \equiv j \pmod{J}$. We recall from algebra that if $I \subseteq J$ then there is a canonical map $\psi : R/I \to R/J$. Since we have that $IJ \subseteq I, J$ we have canonical maps from R/IJ to R/I and R/J, so to each $k \in R/IJ$ we can associate a unique $i \in R/I$ and $j \in R/J$. So now we can define fg branch wise $fg_k := f_i g_j$ if $x \equiv k \pmod{IJ}$, $x \equiv i \pmod{I}$ and $x \equiv j \pmod{J}$. We already noticed that i and j only depend on k so this is well defined. In fact to multiply f and g we could also just multiply their corresponding pre-Collatz sets and then notice that it is a pre-Collatz set over IJ which is also symmetric with respect to IJ.

Now we have to prove the second part of the lemma namely that this multiplication induces a module homomorphism. So let us first prove that we have a group homomorphism. Lets start with $f0 = 0$, this proof is predictably simple, $(f0)_k = f_i 0_j = f_i 0 = 0$ for all k so $f0 = 0$. Next we have to prove that $f(g+h) = fg+fh$, again straightforward, $(f(g+h))_k = f_i(g+h)_j = f_i(g_j+h_j) = f_i g_j + f_i h_j = (fg)_k + (fh)_k$ for all k so $f(g+h) = fg+fh$. So we have established that it is a group homomorphism, now all we have to show is that it respects the $R_R[x_1, \ldots, x_n]$ action. So now all we have to prove is that $rfg = frg$, so lets do that. $(rfg)_k = r_0(fg)_k = r_0 f_i g_j = f_i r_0 g_j = f_i(rg)_j = (frg)_k$, this holds for all k so $rfg = frg$.
□

Of course these are not the only module homomorphisms from $R_J[x_1, \ldots, x_n]$ to $R_I J[x_1, \ldots, x_n]$ in fact any pre-Collatz set over I which is symmetric with respect to IJ would have done the trick.

So we see that we are able to multiply (two) arbitrary Collatz sets, but we are not able to add (two) arbitrary Collatz sets. This is of course an immediate consequence of the definition of a Collatz set, since of course we have $(f_{I,I} + g_{J,J})_{ij}(p) \in I + J$ but we can't guarantee that $(f_{I,I} + g_{J,J})$ factors over $I + J$. But we don't need a ring structure on the Collatz sets of R. In fact the structures that we have defined above are sufficient (on the side of Collatz sets) to define the notion of a Collatz-function.

In the next section we will specify when a domain R admits Collatz functions and furthermore we will prove that there are domains which admit Collatz functions. And of course we will define the notion of a Collatz function.

2.4 Collatz-functions

So far we have we have only required that R is a domain, now we will state some more restrictions on R which we need in order to define Collatz functions and a ring structure on them.

We will start with a definition.

Definition 2.24 *A basis for Collatz-functions over I is a surjective R-module homomorphism $d : I \to R$.*

In order to be able to define Collatz functions for a ring R we need at least one d for each ideal I. But we need more than that so we will define when a ring admits Collatz-functions.

Definition 2.25 *We say that a domain R admits Collatz-functions if it satisfies the following conditions:*

1. *For every ideal $I \neq (0)$ there is a basis for Collatz-functions.*
2. *Let d and d' both be a basis for Collatz-functions over I. Then there is a module isomorphism $\varphi : I \to I$ such that $d = d'\varphi$.*
3. *Let d_I be a basis for Collatz-functions over I, and d_J a basis for Collatz-functions over J and let $I \subseteq J$. Then there is a module homomorphism $\varphi : J \to I$ such that $d_J = d_I\varphi$.*
4. *Let d_I be a basis for Collatz-functions over I, and d_J a basis for Collatz-functions over J. Then $d_I d_J = d_J d_I$ and $d_I d_J$ is a basis for Collatz-functions over IJ.*

We will first prove that there is a ring that actually admits Collatz-functions.Then we will give the definition of a Collatz-function.

Lemma 2.26 *Let R be a principal ideal domain. Then R admits Collatz-functions.*

Proof: Let $I \neq (0)$ be an ideal, then $I = (r)$ for some $d \in R$. Now we can define the needed surjective module homomorphism d by defining it on the generator: $d : r \mapsto 1$. Note that this map is not only a surjective module homomorphism but in fact a bijective module homomorphism. Furthermore the generator r is well defined up to a unit, so let u we see that $d' : ur \mapsto 1$ is also a surjective module homomorphism. But we see that the module isomorphism $\varphi : r \mapsto ur$

has the required property: $d = d'\varphi$. Now suppose $(s) \subseteq (r)$ then the map $\varphi : r \mapsto s$ is a module homomorphism, and let $d : r \mapsto 1$ and $d' : s \mapsto 1$ then we have $d = d'\varphi$ as required. Now let $I = (r)$ and $J = (s)$ then $IJ = (rs)$. We note that $d_I d_J : rs \mapsto 1$ and $d_J d_I rs \mapsto 1$ so $d_I d_J = d_J d_I$ and both are a surjective module homomorphism from IJ to R. Now we have proved all that had to be proved.
□

The converse is also true.

Theorem 2.27 *Suppose* R *admits Collatz-functions then* R *is a principal ideal domain.*

Proof: The proof of this is rather elegant. Suppose that R admits Collatz-functions and let I be any ideal of R then there is a surjective module homomorphism d from I to R. So there is a $i \in I$ such that $d(i) = 1$, now suppose $\exists b \neq 0 \; d(b) = 0$, then $d(rb) = 0$, so $d : (b) \to 0$ i.e. d is the zeromap on the ideal (b) but we also have $d(ri) = r$ and $(i) \cap (b) \neq (0)$ because R is a domain, so d can't be the zeromap on (b) unless $b = 0$. So the kernel of d is $\{0\}$, so d is injective as well as surjective so d^{-1} is well defined and provides a basis for I as module and therefore I is a principal ideal. And since I was arbitrary we see that R is a principal ideal domain.
□

Now we are finally ready to define Collatz-functions.

Definition 2.28 *A Collatz-function* $f_{I,d}$ *over* I *is given by a pair* $(f_{I,I}, d)$ *where* $f_{I,I}$ *is a monideal Collatz set and* d *a basis for Collatz-functions over* I.

We see that a Collatz-function induces a function $f_{I,d} : R^n \to R$ by the following diagram:

$$\begin{array}{ccccc} R^n & & \xrightarrow{\quad f_{I,d} \quad} & & R \\ \Big\downarrow{\scriptstyle id\times\varphi} & & & & \Big\uparrow{\scriptstyle d} \\ R^n \times \left(\underset{i=1}{\overset{n}{\times}} R/I_i \right) & \xrightarrow{id\times f_{I,I}} & R^n \times R[x_1, \ldots, x_n] & \xrightarrow{\circ} & I \end{array} \tag{2.7}$$

In the language of a pre-Collatz set we can define the evaluation at $p \in R^n$ by $p \mapsto d(f_p(p))$, or in the language of a Collatz set $p \mapsto d(f_{i_1,\ldots,i_n}(p))$ with $p = (p_1, \ldots, p_n)$ and $\forall j\ p_j \equiv i_j \pmod{I}$.

We will end this chapter with a section that gives examples of all the concepts that were introduced in this chapter.

2.5 Examples

In this section we will provide examples to the concepts and try to explain why we defined them in this way. And of course we will give an answer to the question what does this all have to do with the $3n + 1$-problem?

We will start with an example of a pre-Collatz set.

Example 2.29 *The map* $p \mapsto x - p$ *is a pre-Collatz set for all* I. *This is rather obvious of course since* $\forall p, If_p(p) = p - p = 0 \in I$.

Lets give a somewhat more complicated example of a pre-Collatz set.

Example 2.30 *Let* $R = \mathbb{Z}$ *then the map* $(x, y) \mapsto 2x3y + 6x + 18y - 42$ *is a pre-Collatz set for* $I = (1), (2), (3), (6)$.

So far we have only given examples such that f_p was the same for all p but note that we don't have to, but it is rather difficult to write out an arbitrary pre-Collatz set because then we have to give a polynomial for each point and in the interesting case this would come down to giving infinitely many polynomials. But of course the only use we have for a pre-Collatz set is that we want a function that assigns to a point a polynomial with a certain property, which we need to define Collatz sets. In fact we are not interessed in general pre-Collatz sets but only those who are actually Collatz sets.

However it is sometimes useful to prove something at the level of a pre-Collatz set and afterward proof that in fact it is a Collatz set. For instance we could define addition of two Collatz sets over I by using the addition we defined on pre-Collatz sets (over I) and note that the result is a Collatz set over I.

Now we will look at examples of Collatz sets.

Example 2.31 $x \mapsto x$ *if* $x \equiv 0 \pmod 2$ *and* $x \mapsto 3x+1$ *if* $x \equiv 1 \pmod 2$. *Is an example of a Collatz set that might look familiar to you since this is the Collatz set that will give rise to the* $3n+1$*-function.*

Lets give another less trivial example.

Example 2.32 *Let* $R = \mathbb{Z}$ *and*

$$\begin{array}{lll} (x,y) \mapsto xy - x + y & x \equiv 0 \pmod 2 & y \equiv 0 \pmod 3 \\ (x,y) \mapsto 2xy + 3x - 2y - 3 & x \equiv 1 \pmod 2 & y \equiv 0 \pmod 3 \\ (x,y) \mapsto 3xy + 18x - 2y - 2 & x \equiv 0 \pmod 2 & y \equiv 1 \pmod 3 \\ (x,y) \mapsto 6xy - 3x - 2y + 5 & x \equiv 1 \pmod 2 & y \equiv 1 \pmod 3 \\ (x,y) \mapsto 3xy + 9x - 2y + 4 & x \equiv 0 \pmod 2 & y \equiv 2 \pmod 3 \\ (x,y) \mapsto 6xy - 36 & x \equiv 1 \pmod 2 & y \equiv 2 \pmod 3 \end{array}$$

This is an example of a Collatz set in two variables we see that all equations nicely map into $(2) \cap (3) = (6)$. *So in fact this Collatz set is equivalent to a monideal Collatz set over* (6) *but since that Collatz set has 36 branches I leave it as an exercise for the reader (note that all equations are known but you have to copy them six times).*

Now that we have seen some examples of (pre-)Collatz sets it is time to look at basis for Collatz-functions. Now these functions take over the role that division by 2 plays in the $3n+1$-problem. In the previous section we proved that any principal ideal domain admits Collatz-functions and the surjective module homomorphisms that we have provided to prove that are basically the same as division by the generator of the ideal. Now the reason that I wanted the d's to be module homomorphisms is because division distributes over addition (since we multiply by an multiplicative inverse and because multiplication distributes). Now the reason that I wanted it to be surjective is because I feel that if d is surjective than you have no information about in which residue class the answer lies (based on d alone that is), and because it will enable us to "restrict" Collatz-functions as we will see in the next section. The other requirements are needed to be able to define a ring structure on Collatz-functions. We will not look at explicit examples of basis for Collatz-functions since we implicitly named all possibilities for principal ideal domains.

Now all we have to do is give an example of a Collatz-function. We will start with the obvious example, namely the $3n+1$-function.

Example 2.33

$$f(n) = \begin{cases} \dfrac{3n+1}{2} & n \equiv 1 \pmod 2 \\ \dfrac{n}{2} & n \equiv 0 \pmod 2 \end{cases}$$

So we see that the $3n+1$-function is a special case of a Collatz-function. Now at this stage this has not many implications, but I have conjectures about so called semi (quasi) simple Collatz functions which probably included the $3n+1$-function (probably because I'm not completely sure about the definition yet). I plan to write an article on that as soon as I find the time (hopefully this summer). We will give another (obvious) example.

Example 2.34

$$f(x,y) = \begin{cases} \dfrac{xy - x + y}{6} & x \equiv 0 \pmod 2 \; y \equiv 0 \pmod 3 \\ \dfrac{2xy + 3x - 2y - 3}{6} & x \equiv 1 \pmod 2 \; y \equiv 0 \pmod 3 \\ \dfrac{3xy + 18x - 2y - 2}{6} & x \equiv 0 \pmod 2 \; y \equiv 1 \pmod 3 \\ \dfrac{6xy - 3x - 2y + 5}{6} & x \equiv 1 \pmod 2 \; y \equiv 1 \pmod 3 \\ \dfrac{3xy + 9x - 2y + 4}{6} & x \equiv 0 \pmod 2 \; y \equiv 2 \pmod 3 \\ \dfrac{6xy - 36}{6} & x \equiv 1 \pmod 2 \; y \equiv 2 \pmod 3 \end{cases}$$

Of course this is not a correct example of a Collatz-function because we should use a monideal Collatz set but since the associated Collatz set has 36 branches and therefore so does the corresponding Collatz-function. But the correct Collatz-function can trivially be constructed from this function.

We will give one last interesting example:

Example 2.35 $R = \mathbb{F}_2[t]$, $n = 1$, $I = (t)$

$$f(x) = \begin{cases} \dfrac{(t+1)x + 1}{t} & x \equiv 1 \pmod t \\ \dfrac{x}{t} & x \equiv 0 \pmod t \end{cases}$$

This particular example is quite interesting because it looks quite a lot like the $3n + 1$-problem in it's binary form but only the addition doesn't carry (since $1 + 1 \neq t$ etc.).

In the next chapter we will prove that Collatz-functions form a ring, furthermore we will define the concept of a Collatz map. We will also look at the endomorphisms of a Collatz-ring as well as some other properties of this ring.

Chapter 3

Collatz rings

In this chapter we will define the notion of a Collatz-ring $R\{[x_1, \dots, x_n]\}$. We will use the concepts that we have defined in the previous chapter an proof that all the Collatz-functions over all the ideals of R form a ring if R admits Collatz-functions. Once we have done this we will study some of the basic properties of these rings. Then we will work towards the definition of a generalized form of Collatz-problems, we will start with defining a more general type of function which forms a natural generalization of a Collatz-function namely Collatz-maps. We will then prove that these Collatz-maps induce homomorphisms between Collatz-rings.

In the next section we will give the last missing piece that stands in the way of defining a ring structure on Collatz-functions, namely the concept of restricting and lifting Collatz-functions.

3.1 Restrictions

In the previous chapter we saw that Collatz sets over I have a natural group structure (even a $R_R[x_1, \dots, x_n]$-module structure), we even were able to multiply two Collatz sets even if they where not defined over the same ideal. In fact we showed that this multiplication was a module-homomorphism, which basically means that this multiplication distributes over addition. So the only thing that is missing in order to define a ring structure on Collatz sets is the ability to add two arbitrary Collatz sets. But alas as we have seen in the previous section we cannot define a natural addition on Collatz sets. But however we will prove in this section that we can define an addition of two arbitrary Collatz-functions. But before we do that we will first state results that are similar to those we have found for Collatz sets.

Definition 3.1 *By a branch* i *of a Collatz-function* $f_{I,d}$ *we mean the function* $d(f_i)$.

Definition 3.2 *We will denote the set of Collatz-functions in* n *variables over* I *by* $R_I\{[x_1, \ldots, x_n]\}$.

Lemma 3.3 $R_R\{[x_1, \ldots, x_n]\}$ *forms a ring and* $R_R\{[x_1, \ldots, x_n]\} \cong R[x_1, \ldots, x_n]$.

Proof: We note that if $f_{R,d} \in R_R\{[x_1, \ldots, x_n]\}$ then its Collatz set has only one branch namely f_0, so we see that $f_{R,d}$ has only one branch namely $d(f_0)$. Now d is a surjective module homomorphism from R to R so d is in fact an module isomorphism. Now any such isomorphism would be multiplication by a unit u so we have $d(f_0) = uf_0 = f'_0$ for some unit u (proof: $\varphi(r) = r\varphi(1)$, $\exists a \varphi(a) = 1$ so $a\varphi(1) = 1$ so $\varphi(1) = a^{-1}$). So we see that Collatz-functions (in n variables) over R are just ordinary polynomials. And for polynomials we know that they form a ring. Now all we have to show is that all polynomials are also Collatz-functions e.a. that they induce a map from R^n to R but this is obvious since for this we can use the evaluation homomorphism.
□

Now that we have established a ring structure on $R_R\{[x_1, \ldots, x_n]\}$ we will look at what we can do with Collatz-functions over I. But before we do this we will first show give an explanation for the name of d (namely a basis for Collatz-functions over I).

Lemma 3.4 *Let* $f_{I,d}$ *be a Collatz-function over* I *and* d' *a basis for Collatz-functions over* I *then there is a Collatz-function* $g_{I,d'}$ *such that* $\forall p \in R^n f_{I,d}(p) = g_{I,d'}(p)$.

Proof: Since d and d' are both a basis for Collatz-functions over I there is a module isomorphism φ such that $d = d'\varphi$. But since I is a principal ideal domain and the generators must be map to the generators we see that φ can only be multiplication by a unit. So we see that $\forall i d(f_i) = d'(uf_i)$ for some unit u. But we know that Collatz sets form a R-module so $uf_{I,I}$ is also a monideal Collatz set over I so $g_{I,I} = uf_{I,I}$.
□

So we see that by picking a d_I for each I we can create all possible Collatz-functions and if we choose another set d'_I then we have a bijection between the

Collatz-functions define by that basis and the other basis. So in fact just like the matrix of a linear map depends on the choice of basis so does the Collatz set of a Collatz-function depend on the choice of d.

Definition 3.5 *We call the homomorphism φ as used in the proof of the previous lemma a change of basis.*

Definition 3.6 *We will call two Collatz functions $f_{I,d}$, $g_{I,d'}$ equivalent if they have the same Collatz set with respect to the same basis, i.e. $f_{I,d}$ has the same Collatz set as $g_{I,d}$.*

Lemma 3.7 *A change of basis induces an equivalence relation.*

It is obvious that the relation is reflexive (take $\varphi = \text{ID}$), and symmetry follows from the fact that φ has an inverse, and transitivity follows from the fact that the composition of two isomorphisms is again a isomorphism.

From now on we will look at Collatz functions over I up to equivalence, but we will still denote them by $R_I\{[x_1, \ldots, x_n]\}$. That means that we will fix a d for each ideal I and denote it by d_I or simply by d if the ideal is clear from the context, and of course we choose d_I and d_J such that $d_I d_J = d_{IJ}$.

Lemma 3.8 *The Collatz-functions over I form a $R_R\{[x_1, \ldots, x_n]\}$-module.*

Proof: First we must define a addition on the Collatz functions, and prove that this forms a group. Let $(f_{I,I}, d)$ and $(g_{I,I}, d)$ be two Collatz-functions (note that we can assume that they have the same d) then $(f_{I,I}, d) + (g_{I,I}, d) = (f_{I,I} + g_{I,I}, d) = ((f+g)_{I,I}, d)$. So we use the group structure that we defined on Collatz sets, also note that $d(f_i(p)) + d(g_i(p)) = d(f_i(p) + g_i(p)) = d((f+g)_i(p))$ i.e. we see that image of the sum is the sum of the images. Now we need to define a module structure on $R_I\{[x_1, \ldots, x_n]\}$, but this simple since we have already defined a module structure on the Collatz sets which we can transport to the Collatz-functions, i.e. $r \in R_R\{[x_1, \ldots, x_n]\}, (f_{I,I}, d) \mapsto (rf_{I,I}, d)$.
□

Lemma 3.9 *Let $(f_{I,I}, d_I)$ and $(g_{J,J}, d_J)$ be two Collatz-functions then we can define their product and each Collatz function $(f_{I,I}, d_I)$ induces a module homomorphism from $R_J\{[x_1, \ldots, x_n]\}$ to $R_{IJ}\{[x_1, \ldots, x_n]\}$.*

Proof: Let us first define this product: $\mu : ((f_{I,I}, d_I), (g_{J,J}, d_J)) \mapsto (f_{I,I}g_{J,J}, d_Id_J) = (fg_{IJ,IJ}, d_{IJ})$. Here we use again that we've already defined a product on the Collatz sets and that we proved that they form a Collatz set over IJ and since R admits Collatz-functions is there a d_{IJ} such that $d_{IJ} = d_Id_J = d_Jd_I$. Now $fg_{IJ,d_{IJ}}(p) = d_{IJ}((fg)_IJ(p)) = d_I(d_J(f_i(p)g_j(p))) = d_I(f_i(p)d_J(g_j(p))) = d_I(f_i(p))d_J(g_j(p)) = f_{I,d_I}(p)g_{J,d_J}(p)$. So we see that the product of the images is the image of the product. Now we have to prove that this product induces a module homomorphism. This is simple because we see that the product depends on two things first of all that the composition of two d's is a d for the product ideal and second the product of the two Collatz sets is a Collatz set over the product ideal. Now the first one is always true since R admits Collatz functions, the second part defines (in some sense because that is part where we have the addition) the actual product but for this product we already know that it induces a module homomorphism. $f(g+h)_{IJ,d_Id_J}(p) = d_I(d_J(f_i(p)(g_j(p) + h_j(p)))) = d_I(f_i(p)d_J(g_j(p) + h_j(p))) = d_I(f_i(p)d_J(g_j(p))) + f_i(p)d_J(h_j(p))) = d_I(f_i(p)d_J(g_j(p))))+d_I(f_i(p)d_J(h_j(p))) = d_I(f_i(p))d_J(g_j(p))+d_I(f_i(p))d_J(h_j(p)) = f_{I,d_I}g_{J,d_J}(p) + f_{I,d_I}h_{J,d_J}(p)$.
□

Now that we have found all lemmas for Collatz-functions that we've already found for Collatz sets it becomes time that we start comparing two Collatz-functions over two different ideals. In other words we want to expand our equivalence relation that we have defined in definition 3.6 and lemma 3.7.

Lemma 3.10 *Let* f_{I,d_I} *be a Collatz-function and* $J \subseteq I$ *then there is a Collatz-function* g_{J,d_J} *such that* $f_{I,d_I}(p) = g_{J,d_J}(p)$ *for all* p.

Proof: Suppose $I = a$ and $d_I(a) = 1$, we can assume this because R is a principal ideal domain. Now since $J \subseteq I$ we know that there is a r such that $J = (ra)$. Furthermore we can choose r so that $d_J(ra) = 1$, since one of the generators of J is mapped to 1 by d_J. Now we know that $f_{I,I}$ is not only symmetric with respect to I but also with respect to J since J requires less symmetry than I. However $f_{I,I}$ doesn't map to J but we can fix this by multiplying by r so we get $g_{J,J} = rf_{I,I}$ (that is we multiply the pre-Collatz set of $f_{I,I}$ by r and then notice that the result is a Collatz set with respect to J). Now we have $f_{I,d_I}(p) = g_{J,d_J}(p)$ for all p because for all s we have $d_I(sa) = d_J(sra) = s$.
□

Now we are ready to define restrictions.

Definition 3.11 *The construction in the proof of the previous lemma is called the restriction of* f_{I,d_I} *to* J *and is denoted by* $f_{I,d_I}|_J$.

Now we can extend our equivalence relation.

Definition 3.12 *We call two Collatz-functions* f_{I,d_I} *and* g_{J,d_J} *equivalent if there is an ideal* H *such that* $H \subset I$ *and* $H \subset J$ *and* $f_{I,d_I}|_H = g_{J,d_J}|_H$*, meaning that the Collatz sets of the restrictions to* H *are equal.*

This induces indeed an equivalence relation, it is obvious that f_{I,d_I} is equivalent to itself just take $H = I$, also symmetry is no problem because if $f_{I,d_I}|_H = g_{J,d_J}|_H$ then it is obvious that $g_{J,d_J}|_H = f_{I,d_I}|_H$. Transitivity is a bit more difficult but still quite simple. Let f_{I,d_I}, g_{J,d_J} and k_{L,d_L} three Collatz-functions and let f_{I,d_I} and g_{J,d_J} be equivalent and also let g_{J,d_J} and k_{L,d_L} be equivalent then there is an H_1 and an H_2 such that $f|_{H_1} = g|_{H_1}$ and $g|_{H_2} = k|_{H_2}$. Now it is obvious that $g|_{H_1}|_{H_1\cap H_2} = g|_{H_2}|_{H_1\cap H_2}$ and $H_1 \cap H_2 \subset H_1, H_2$ so the restrictions are allowed but then we also have $f|_{H_1\cap H_2} = k|_{H_1\cap H_2}$. So we see that we have an equivalence relation. Now let us look what this equivalence relation means at the level of (pre-)Collatz sets. We see that $(f_{I,I}, d_I)$ and $(g_{J,J}, d_J)$ are equivalent if there is an ideal L and numbers $a, b \in R$ such that $f_{I,I}|_L = af_{I,I} = bg_{J,J} = g_{J,J}|_L$ (as (pre-)Collatz sets) and $d_L(ad_I^{-1}(1)) = 1 = d_L(bd_J^{-1}(1))$.

So basically what you do is you want to multiply both the "numerator" and "denominator" by a constant a (and in case of g by b) and we require that both "numerators" and both "denominators" are equal. Like wise let $x, y \in \mathbb{Q}$ then the are equivalent if there are $a, b \in \mathbb{Z} - \{0\}$ such that $\frac{a}{a}x = \frac{b}{b}y$. Notice that in particular the equivalence of Collatz functions over I is also of this form with $a = 1$ and b a certain unit of R such that $d(bd'^{-1}(1)) = 1$.

Now we will state one final, interesting result on restrictions.

Lemma 3.13 *Let* $I \subseteq J$ *then* $\cdot|_I$ *induces a module homomorphism from* $R_J\{[x_1, \ldots, x_n]\}$ *to* $R_I\{[x_1, \ldots, x_n]\}$.

Proof: Since $I \subseteq J$ $\cdot|_I$ is nothing more then multiplying both "numerator" and "denominator" by the unique constant a such that $d_I(ad_J^{-1}(1)) = 1$, and $R_J\{[x_1, \ldots, x_n]\}$ is a $R[x_1, \ldots, x_n]$-module and $a \in R[x_1, \ldots, x_n]$ so the multiplication distributes over $+$ and respects the $R[x_1, \ldots, x_n]$ action.
□

We are now in the position to define a ring structure on the set of Collatz-functions. We will do will do this in the next section. We will will also study the basic properties of these rings.

3.2 Collatz rings

In this section we will finally give the definition of a Collatz-ring. We will give some examples of Collatz-rings and we will study some of their basic properties. We will learn that Collatz-functions form the "ideal" generalization of ordinary polynomials. But first we will look at what we have got so far.

1. If I is a non-zero ideal of R then $R_I[x_1, \ldots, x_n]$ is a $R[x_1, \ldots, x_n]$-module.
2. If $I \subseteq J$ then we have a restriction map (a module homomorphism) from $R_J\{[x_1, \ldots, x_n]\}$ to $R_I\{[x_1, \ldots, x_n]\}$, which is the identity if $J = I$.
3. If $L \subseteq I \subseteq J$ then $\cdot|_L \circ |_I = \cdot|_L$.
4. If $f|I = 0$ then $f = 0$.
5. If $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_m$ are ideals and $f_i \in R_{I_i}\{[x_1, \ldots, x_n]\}$ such that $f_i|_{I_i \cap I_j} = f_j|_{I_i \cap I_j}$ then there is a unique $f \in R_{I_m}\{[x_1, \ldots, x_n]\}$ such that $f|_{I_i} = f_i$.
6. If $f \in R_I\{[x_1, \ldots, x_n]\}$ then f induces a module homomorphism from $R_J\{[x_1, \ldots, x_n]\}$ to $R_{IJ}\{[x_1, \ldots, x_n]\}$.

Notice that the first three statements are pretty much in some twisted way the conditions of a pre-sheaf of modules (notice that the requirement that the module of the empty set is the trivial module is not fulfilled). And the first five statements look an awful lot like the conditions of a sheaf of modules. The biggest problem we have here is that how to define an open covering of a ideal with ideals is not as clear as the covering of an open set with open sets.

But we are wandering off from our goal here, all we need in order to be able to define a ring structure on Collatz functions in n variables of R is the ability to add two arbitrary Collatz-functions in a way that is compatible with the addition of two Collatz-functions over I as well as that it should be compatible with our previous defined multiplication. Now this is rather simple, let f_{I,d_I} and g_{J,d_J} be Collatz-functions, then we can represent them by $f|_{I \cap J}$ and $g|_{I \cap J}$ but these are Collatz-functions defined over the same ideal and we have already defined an addition for this situation. Now we have everything we need for a ring structure.

Theorem 3.14 *The Collatz-functions in* n *variables of* R *modulo the defined equivalence relation form a ring which we denote by* $R\{[x_1, \ldots, x_n]\}$.

Proof: This follows immediately from all previous results (in fact from a subset of all previous results). Notice that for all I the Collatz-function $\forall i f_i = 0$ are equivalent with one another, this equivalence class forms the 0 of the ring, all the function equivalent to 1 form the multiplicative identity. And we've already proved all other relations between addition and multiplication.
□

We will finally give the definition of a Collatz-ring.

Definition 3.15 *Let* R *be a principal ideal domain then we call* $R\{[x_1, \ldots, x_n]\}$ *the Collatz ring in* n *variables over* R.

Now this definition is a bit rough because I'd also like to call $(R\{[x_1, \ldots, x_n]\})^m$ a Collatz ring, so in fact $R\{[x_1, \ldots, x_n]\}$ is more of an example of a Collatz ring, but I don't have a clear picture yet of the most general definition of a Collatz ring at this moment. So far now we will assume that a Collatz ring is either of the form $R\{[x_1, \ldots, x_n]\}$ or $(R\{[x_1, \ldots, x_n]\})^m$. We will now give an example of a Collatz-ring.

Example 3.16 *The ring* $\mathbb{Q}[x_1, \ldots, x_n]$ *is a Collatz-ring. This obvious because* $\mathbb{Q}$ *has only one non-zero ideal, namely* (1) *and all Collatz-functions defined over this ideal are just (up to isomorphism) ordinary polynomials.*

In fact this holds for any field.

Lemma 3.17 *Let* F *be a field then* $F\{[x_1, \ldots, x_n]\} \cong F[x_1, \ldots, x_n]$.

Proof: The proof has already been given for $\mathbb{Q}$, the arguments for an arbitrary field are the same.
□

In fact we have the following result.

Lemma 3.18 *Let* R *be a ring that admits Collatz-functions then* $R[x_1, \ldots, x_n]$ *is a subring of* $R\{[x_1, \ldots, x_n]\}$.

Proof: $R_R\{[x_1, \ldots, x_n]\}$ is a subring of $R\{[x_1, \ldots, x_n]\}$ and we've already shown that $R_R\{[x_1, \ldots, x_n]\} \cong R[x_1, \ldots, x_n]$.
□

So we see that Collatz-functions form an "ideal" generalization of the ordinary polynomials in n variables, of course the adjective "ideal" is obvious. The following result is also quite obvious.

Lemma 3.19 *Let* R *be a ring that admits Collatz-functions, then* $R\{[x_1, \ldots, x_{n-1}]\}$ *is a sub-ring of* $R\{[x_1, \ldots, x_n]\}$.

Proof: Just consider the Collatz-functions whose branches are polynomials in $x_1, \ldots, x_{n-1}$, these are obviously closed under multiplication and addition.
□

And of course the following lemma is obvious to.

Lemma 3.20 *Let* σ *be a permutation of* n *elements then* $R\{[x_1, \ldots, x_n]\} \cong R\{[x_{\sigma(1)}, \ldots, x_{\sigma(n)}]\}$

Proof: This is a direct consequence of the fact that $R[x_1, \ldots, x_n] \cong R[x_{\sigma(1)}, \ldots, x_{\sigma(n)}]$ and that addition and multiplication is defined on the level of branches of a Collatz set which are just ordinary polynomials.
□

And last but not least.

Lemma 3.21 *Let* R *admit Collatz-functions then* $R\{[x_1, \ldots, x_{n-1}]\}[x_n]$ *is a sub-ring of* $R\{[x_1, \ldots, x_n]\}$

Proof: If we look at an element of the first ring we see that it is of the form: $\sum_{i=0}^{N} f_i x_n^i$ but since f_i and x_n^i are both Collatz-functions so is their product, and since $f_i x_n^i$ are all Collatz-functions so is their sum. It is obvious that both 0 and 1 are in the sub ring.
□

There is only just one more interesting subring.

Definition 3.22 *Each ideal* I *generates in a natural way an subring of* $R\{[x_1,\ldots,x_n]\}$ *which we denote by* $R^I\{[x_1,\ldots,x_n]\} := \bigcup_{i=1}^{\infty} R_{I^i}\{[x_1,\ldots,x_n]\}$.

This subring (well actually $(R^I\{[x_1,\ldots,x_n]\})^n$) plays an important role when we start looking at compositions of Collatz-functions (well Collatz-maps actually).

We will now study some of the other properties of Collatz rings. We will first show that in general a Collatz ring is no a domain.

Lemma 3.23 *Suppose that* R *admits Collatz-functions and* R *is not a field then* $R\{[x_1,\ldots,x_n]\}$ *is not a domain and the zero divisors are those Collatz-function with* $f_i = 0$ *for some (but not all)* i.

Proof: The proof is rather simple, since R is not a field there is an ideal I such that I has more than one residue class. Now we know that the multiplication of Collatz-functions is based on multiplication of polynomials (that is the product of the branches of the Collatz sets involved) and since R is a domain so are the polynomials in n variables. So if the product is ever to become zero then one of the two branches involved needs to be zero. Of course if for one of the two functions involved all the branches are zero then it is equivalent to zero and it is not a zero-divisor. But if a branch $(f_{I,d})_i$ is equal to zero when $(g_{I,d})_i$ is not and $g_{I,d}$ and $f_{I,d}$ have at least one non-zero branch then their product is zero (this can be easily seen if you look at their pre-Collatz sets). Now on the other hand suppose that $f_{I,d}$ only has non-zero branches then the only Collatz-function g such that $fg = 0$ is a restriction of 0, again this is obvious if you look at the pre-Collatz sets, since $(fg)_p = f_p g_p = 0$ but $f_p \neq 0$ so $g_p = 0$ for all p therefore $g = 0$.
□

So we see that $R\{[x_1,\ldots,x_n]\}$ is usually not a domain but also we can easily identify the zero-divisors. We could now study extensions of Collatz rings, this is a rather interesting subject since no element e that is algebraic over R can be added to $R\{[x_1,\ldots,x_n]\}$ without us getting in to trouble. To make this more precise in general there is no natural embedding of $R\{[x_1,\ldots,x_n]\}$ into $R\{[x_1,\ldots,x_n]\}[e]$ or $R[e]\{[x_1,\ldots,x_n]\}$, we will not prove this nor study extensions of $R\{[x_1,\ldots,x_n]\}$ but I had to mention this interesting fact. I do intend to study this in another article as soon as I have the time. But for now we will continue on our quest to define the most general form of Collatz problems.

In the next section we will make a short detour in our study of generalizations of Collatz-problems and take a closer look at the ideals of Collatz rings in order to better understand these interesting rings.

3.3 Ideals of $R\{[x_1,\ldots,x_n]\}$

In this section we will study the ideals of a Collatz ring. We will see that in general the zero divisors play an important role in the study of these ideals. But we will start with defining an interesting function Φ.

Definition 3.24 *Let* I *be an ideal of* $R\{[x_1,\ldots,x_n]\}$ *then we define the function* $\Phi : R\{[x_1,\ldots,x_n]\} \to \mathcal{P}(\{J \subseteq R\})$ *by* $\Phi(I) = \{J \subset R | \exists f \in I, 0 \neq f \in R_I\{[x_1,\ldots,x_n]\}\}$. *That is* $\Phi(I)$ *is the set of ideals* J_i *of* R *such that there is a non-zero function in* I *which is defined over* J_i.

It is obvious that $\Phi(I)$ is closed under intersection (because I is closed under addition) and under scalar multiplication by any ideal (since I is closed under scalar multiplication). Now we will state a lemma that shows some of the importance of the zero divisors with respect to the ideals of a Collatz-ring.

Lemma 3.25 *Let* $f \in I$ *be a non-zero divisor then* $(1) \in \Phi(I)$, *therefore we have for every ideal* J *of* R *that* $J \in \Phi(I)$.

Proof: The proof is quite simple after you realize that the fact that the product of a Collatz-function over L and a Collatz-function over M is a Collatz-function over LM doesn't mean that it can't be a restriction of a Collatz-function over $N \supset LM$. So let f be a non-zero divisor defined over L, all we have to do is to find a g over M such that fg is the restriction of a polynomial. This means that all the branches of fg have to be the same and of the form $(fg)_i = ah$ such that $a \in LM$, $h \in R[x_1,\ldots,x_n]$ because then it is equivalent to $d_{LM}(a)h$ which is a polynomial. Now this is quite easy, take $g_i = d_L^{-1}(1)^2 \frac{\prod_j f_j}{f_i}$. If we take the product of f and g which are both Collatz-functions over L we get a Collatz-function over L^2 with branches of he form $d_L^{-1}(1)^2 \prod_j f_j = d_{L^2}^{-1}(1) \prod_j f_j$, so we see that their product is the restriction of $\prod_j f_j$ to L^2 hence it is $\prod_j f_j$.
□

So we see that once there is a non-zero divisor over J in I there are actually non-zero divisors over all ideals, of R, in I. So far we have seen only the role of non-zero divisors, while I claimed that the zero divisors play an important role in the study of the ideals of a Collatz-ring. In order to be able to demonstrate that we will have to state a few definitions.

Definition 3.26 *Let* f *be a zero divisor over* J*, then we say it is of type* $(J,\mathfrak{J})$ *with* $\mathfrak{J} \in \mathcal{P}((R/J)^n)$ *if* $\forall j \in \mathfrak{J} f_j \neq 0$ *and* $f_j = 0$ *else.*

Now we will define a very special and very useful type of zero divisor.

Definition 3.27 *We call a zero divisor of simple type if it is of type* $(J,\mathfrak{J})$ *such that* $\#(\mathfrak{J}) = 1$.

So basically a zero divisor is of simple type if only one of the branches is non-zero. And within this special class of zero divisors there is a very special very useful class of Collatz-functions, which we will define now.

Definition 3.28 *We call a zero divisor* f *a branch-picker function of type* $(J,(j_1,\ldots,j_n))$ *if* $f(x_1,\ldots,x_n) = 1$ *if* $(x_1,\ldots,x_n) \equiv (j_1,\ldots,j_n) \pmod J$ *and* $f(x_1,\ldots,x_n) = 0$ *else.*

So basically this type of function is very powerful because if we multiply it with a function f over J then the product of f and a branch-picker of type (J,j) then (if $f_j \neq 0$) we end up with a zero divisor h of simple type (J,j) with $h_j = f_j$. We will now prove that all branch-picker functions exist and that they are Collatz-functions.

Lemma 3.29 *Let* $(J,\mathfrak{J})$ *be a simple type then there is a* $f \in R\{[x_1,\ldots,x_n]\}$ *such that* f *is the branch-picker function of this type.*

Proof: Note that $0, d_J^{-1}(1) \in J$ now take $f_i = 0$ if $i \neq j$ and $f_j = d_J^{-1}(1)$.
□

Now let us see what that means for an ideal I of $R\{[x_1,\ldots,x_n]\}$. We see that if $f \in I$ and it is of type $(J,\mathfrak{J})$ then so are all functions h_α with $\alpha \in \mathfrak{J}$ such that $(h_\alpha)_i = 0$ if $i \neq \alpha$ and $(h_\alpha)_\alpha = f_\alpha$. Now this is very useful, because let I be an ideal of $R\{[x_1,\ldots,x_n]\}$ then we can fix an ideal J of R and fix a residue class j of J and consider all functions in I of simple type (J,j), if we look at the non-zero branch of these functions, they form an ideal of $R[x_1,\ldots,x_n]$. But now we can use the theory of Gröbner bases to find the generators of this ideal. Now if we do this for all j and J then we get the generators of I of course this set of generators is probably too large because some of the generators are equivalent. Now this algorithm is nowhere near perfect but it shows us that we can use the well developed theory of Gröbner bases for ideals of Collatz-rings, at least to some extend. We will state another lemma without proof, because we will prove a stronger result later.

Lemma 3.30 *Let* I *be an ideal of* $R\{[x_1, \ldots, x_n]\}$ *and suppose* $\exists J$ *such that* $J \notin \Phi(I)$ *then if* $f \in I$ *and* $f \neq 0$ *then* f *is a zero divisor.*

The proof proof of this lemma is a direct consequence of lemma 3.25, but it is quite a weak result, but we can get a better result. In order to formulate this result we need yet a few more definitions.

Definition 3.31 *We call* $(J, \mathfrak{J}')$ *a subtype of* $(J, \mathfrak{J})$ *if* $\mathfrak{J}' \subseteq \mathfrak{J}$.

This definition is quite obvious and so is the next.

Definition 3.32 *We call a type* $(J', \mathfrak{J}')$ *equivalent to type* $(J, \mathfrak{J})$ *if* $(J', \mathfrak{J}')|_{J \cap J'} = (J, \mathfrak{J})|_{J \cap J'}$, *where the restriction is the obvious one.*

This definition is chosen such that f and $f|_J$ are of equivalent type.

Definition 3.33 *We call two types* $(J', \mathfrak{J}')$ *and* $(J, \mathfrak{J})$ *compatible if* $\mathfrak{J}'|_{J \cap J'} \cup \mathfrak{J}|_{J \cap J'}$ *is not the collection of all residue classes of* $J \cup J'$, *i.e. if their "union" is still a zero divisor type.*

Now we can formulate our stronger result.

Lemma 3.34 *If* I *is an ideal of* $R\{[x_1, \ldots, x_n]\}$, $\exists J \notin \Phi(I)$ *and* $f \in I$ *then* f *is a zero divisor and if* $g \in I$ *then* f *and* g *are of compatible types and for each subtype of* f*'s type there is a* $h \in I$ *of this type.*

Proof: Suppose f and g would not be of compatible type then their sum would be a nonzero divisor which is a contradiction to our hypothesis. The functions of any subtype of the type of f can be found by adding the products of f with the corresponding branch-picker functions.
□

Probably would could formulate an even stronger result but for now it is the best I've got. There are a lot of other interesting properties of ideals of Collatz-functions we could study, but won't study hem now. One interesting thing would be the quotient ring of a Collatz ring and one of it's ideals,. I'm particular curious about quotient rings of the from: $\mathbb{Z}\{[x]\}/(f)$ where f is a irreducible polynomial over $\mathbb{Z}$. For instance we know very well what $\mathbb{Z}[x_1, \ldots, x_n]/(3)$ looks like but it is not immediately clear to me what $\mathbb{Z}\{[x_1, \ldots, x_n]\}/(3)$ looks like, I just know that it is a lot more complicated. The other thing, that actually led to this section, is the question whether $R\{[x_1, \ldots, x_n]\}$ is Noetherian if R is.

A related question is the question is we could do algebraic geometry with these functions. I suspect that we can because if you look at the Collatz-functions over a (algebraically closed) field these are nothing more than ordinary polynomials, the usual objects of classical algebraic geometry. On an ordinary principal ideal domain R the Collatz-functions still behave pretty much like ordinary polynomials.

Another question that pops into my mind is, suppose we can do "classical" algebraic geometry with Collatz-functions then does the scheme of the coordinate ring of a Collatz variety correspond in the same way to that Collatz variety as the coordinate ring of a affine variety does to an affine variety?

In the next section we will look at very special Collatz rings namely the ring of Collatz-maps. And of course we will give a proper definition of Collatz-maps. That will bring us one step closer to the definition of a generalized Collatz problems.

3.4 Collatz-maps

So far we have only studied Collatz-functions that is maps form R^n to R. In this section we will study Collatz-maps. These maps form a generalization to polynomial maps just like Collatz-functions form a generalization of polynomials. But first we will give a proper definition of a Collatz-map.

Definition 3.35 *A map* $F : R^n \to R^m$ *is called a Collatz-map if* $\forall i\ \pi_i(F)$ *(that is the projection on the* i*-th coordinate) is a Collatz-function.*

We will show that Collatz-maps behave much like Collatz-functions. We now define what it means for a Collatz-map to be monideal.

Definition 3.36 *A Collatz-map is said to be monideal (over* I*) if* $\forall i\ F_i := \pi_i(F)$ *is a monideal Collatz-function over* I.

We will also extend our definition of equivalence from Collatz-functions to Collatz-maps.

Definition 3.37 *We call two Collatz maps* F *and* G *equivalent if* F_i *is equivalent (as a collatz-function) to* G_i *for all* i.

We will start our study of Collatz-maps with an easy lemma.

Lemma 3.38 *Let* F *be a Collatz-map then there is a Collatz-map* G *that is monideal over an ideal* I *and* F_i *is equivalent to* G_i *for all* i.

Proof: Let F_i be a monideal over I_i then we can take $I = \bigcap_{j=1}^{m} I_j$ and just take $G_i = F_i|_I$.
□

We could now do what we have done so far namely show that the Collatz-maps over I form a module etcetera and finally define a ring structure on them but we will skip all that and state the following theorem.

Theorem 3.39 *The Collatz-maps from* R^n *to* R^m *form a ring and is isomorphic to* $(R\{[x_1, \ldots, x_n]\})^m$*, modulo equivalence relation given in definition 3.37.*

Proof: First we note that by rules of basic algebra that $(R\{[x_1, \ldots, x_n]\})^m$ is a ring, so all we have to prove is that every element in $(R\{[x_1, \ldots, x_n]\})^m$ is a Collatz-map and that every Collatz-map is of this form. But this is obvious because $F = (F_1, \ldots, F_m)$ this is obvious an element of $(R\{[x_1, \ldots, x_n]\})^m$ on the other hand it is obvious that any element of $(R\{[x_1, \ldots, x_n]\})^m$ is of the required form.
□

We will now state the most important result so far.

Theorem 3.40 *Let* F *be a Collatz-map from* R^n *to* R^m *and* G *be a Collatz-map from* R^m *to* R^p *then* $G \circ F$ *is a Collatz-map from* R^n *to* R^p*. And* $(G + H) \circ F = G \circ F + H \circ F$ *and* $GH \circ F = (G \circ F)(H \circ F)$.

Proof: Let F be a Collatz-map from R^n to R^m and g a Collatz-map from R^m to R^p, it is of course obvious that we can take the composition of these two functions in the set-theoretical sense, but if we want this to be Collatz then we must factor it over an ideal and find the corresponding Collatz sets. This will be our greatest challenge, let us assume that F is defined over I and G is defined over J. Based on this we have to determine the result of $F(p)$ modulo J, so that we know which branch of F we have to compose with which branch of G. The solution is quite simple, we take representatives of the residue classes of IJ and name them $p_1, \dots, p_q$ and determine $r_i = f(p_i) \pmod{J}$, let $s_i \equiv p_i \pmod{I}$ then $G(F(p)) = d_J(G_{r_i}(d_I(F_{s_i}(p))))$ if $p \equiv p_i \pmod{IJ}$. The proof for this is quite simple, we will prove it for $m = 1$ so F is a simple Collatz-function, that makes it easier to talk about the branches, for $m > 1$ we can apply the same reasoning on $\pi_i(F)$ for all i and therefore we can apply it to F itself. Now if we evaluate F in p and we assume that $p \equiv p_i \pmod{IJ}$ then we note that $F_{s_i}(p)$ is determined up to an element of IJ, so $d_I(f_{s_i}(p))$ is determined up to an element of J as was required. Again if $m > 1$ we can apply this argument to $\pi_j(F(p))$ and therefore to $F(p)$. Now we are halfway there, but we're not quite there yet, since the composition of g_{r_i} with $F(p)$ is all but polynomial, however since we substitute Collatz-functions in polynomials we see that the result is a Collatz-function (or map) since these are closed under addition, multiplication and scalar multiplication by R. So we can find a Collatz-map H over $I^{\deg(g_{r_i})}$ and we know that $H(p) \in J^m$ by construction of H so d_J cans till be applied to it. So we see that $G \circ F$ is (certainly) a Collatz-map over $J(I)^{\max_i(\deg(g_i))}$.

Now we want to prove composition with F induces a endomorphism, this is obvious because for polynomials we have that $(g + h) \circ f = g \circ f + h \circ f$ and the same for multiplication. Now $\pi_i((G+H)\circ F) = (\pi_i(G)+\pi_i(H))\circ F = (d((G_i)_j) + d((H_i)_j))\circ F = d\circ((G_i)_j + (H_i)_j)\circ F = d\circ((G_i)_j\circ F + (H_i)_j\circ F) = \pi_i(G\circ F + H\circ F)$. The same holds for multiplication, but this is probably all a direct consequence of elementary category theory.
□

There is another way of proving that composition induces an endomorphism, notice that composition induces a map $\forall i \ x_i \mapsto \pi_i(F)$. Now notice that such maps always preserve multiplication and addition, so we immediately see that this is an endomorphism, since in particular we have $0 \circ F = 0$, $1 \circ F = 1$ and we know that the composition of two Collatz-maps is again a Collatz-map. We will now look at a simple example in which the composition of the two Collatz-maps is not simply the branch-wise composition.

Example 3.41

$$f(n) = \begin{cases} \frac{6n+6}{2} & n \equiv 1 \pmod 2 \\ \frac{6n}{2} & n \equiv 0 \pmod 2 \end{cases}$$

$$g(n) = \begin{cases} \frac{n+2}{3} & n \equiv 1 \pmod 3 \\ \frac{n+1}{3} & n \equiv 2 \pmod 3 \\ \frac{n}{3} & n \equiv 0 \pmod 3 \end{cases}$$

We first evaluate f *in the points* 0, 1, 2, 3, 4 *and* 5 *modulo* 3. *We see that* $f(0) \equiv 0 \pmod 3$, $f(1) \equiv 0 \pmod 3$, $f(2) \equiv 0 \pmod 3$, $f(3) \equiv 0 \pmod 3$, $f(4) \equiv 0 \pmod 3$ *and* $f(5) \equiv 0 \pmod 5$. *So we see that we only need the branch* g_0 *of* g. *If we determine* $g \circ f$ *we see that we get the Collatz-function that is equivalent to:*

$$h(n) = \begin{cases} \frac{2n+2}{2} & n \equiv 1 \pmod 2 \\ \frac{2n}{2} & n \equiv 0 \pmod 2 \end{cases}$$

Well actually you would get $h|_{(6)}$ *which you will recognize as the restriction of* h *to* (6) *and therefore replace it by the equivalent Collatz-function* h.

We are now in a position to study the behavior of the $3n+1$-problem and we can see how we can generalize this. This is the goal of the next chapter.

Chapter 4

Generalized Collatz-problems

So far we have only formalized and generalized the $3n+1$-function but we have totally ignored the problem part. That is with the $3n+1$-problem we look at the iterations of the $3n+1$-function. But the generalized version of the $3n+1$-function could not be iterated, fortunately in the last section of the previous chapter we developed the notion of a Collatz-map. Collatz-maps from R^n to R^n can be iterated and what is even better the iteration of such a Collatz-map F gives rise to a family of Collatz-maps from R^n to R^n, namely F, $F^{(2)}$, $F^{(3)}$,.... So in the first section we will formalize this observation. In the second section we will make some observations that will allow us to make the ultimate algebraic generalization of the $3n+1$-problem.

4.1 Iterating Collatz-maps

In this section we will formalize the $3n+1$-problem in terms of Collatz-maps. We know that we can iterate Collatz-maps by the last theorem of the previous chapter. In this section we will give the basic definitions.

We will start with the definition of a fixed point.

Definition 4.1 *Let* F *be a Collatz-map from* R^n *to* R^n *then we call* $p \in R^n$ *a fixed point of* F *if* $F(p) = p$.

When we study polynomials we are in most cases interested in the zeros and since Collatz-functions and Collatz-maps behave in a very similar way as polynomials we could wonder if we should study the zeros of these types of functions as well. Of course the answer is yes because we can define the translation map of F.

Definition 4.2 *We define the translation map* $\mathcal{T}(F,k)$ *(*$k \geq 1$*) of a Collatz-map* F *from* R^n *to* R^n *to be:* $\mathcal{T}(F,k)(p) := F^{(k)}(p) - p$. *For a fixed* k *we call* $\mathcal{T}(F,k)$ *the* k^{th}*-translation map of* F.

Notice that for all $k \geq 1$ the k^{th}-translation map of F is a Collatz-map since both $F^{(k)}$ and p are. Furthermore $\mathcal{T}(F,k)(p) = 0$ iff p is a fixed point of $F^{(k)}$. Another interesting observation is that if F is a Collatz-map over I then $F^{(k)} \in (R^I\{[x_1,\ldots,x_n]\})^n$. We will now define the cycle structure associated to a Collatz-map F.

Definition 4.3 *We define the cycle structure* $\mathfrak{C}(F)$ *associated to a Collatz-map* F *from* R^n *to* R^n *to be a countable collection of collections* $\mathcal{C}_k$ *of finite cyclic ordered collections* C *of points of* R^n *such that if* $\mathcal{C}_k \in \mathfrak{C}(F)$ *and* $C, C' \in \mathcal{C}_k$ *then* $\#(C) = \#(C') = k$ *and* $C \cap C' = \emptyset$ *or* $C = C'$. *And if* $c_1,\ldots,c_k \in C$ *then* $c_{i+1} = F(c_i)$ *if* $i < k$ *and* $c_1 = F(c_k)$.

So the cycle structure associated to F is exactly what you would expect, namely the collection of all the cycles induced by iterating F and in particular if $p \in C \in \mathcal{C}_k \in \mathfrak{C}(F)$ then $\mathcal{T}(F,k)(p) = 0$ which means that the point p is in a cycle of length k iff it is a fixed point of $F^{(k)}$.

Now basically these are all the definitions we need, the general problem associated to a Collatz-map F will be to determine the cycle structure associated to F. Now besides generalizing the $3n+1$-problem we have not made much of a difference because in general these problems are at least as complicated as the original. I believe that the solution to the original problem should not be sought in these generalizations but in the ring structure that we have defined on the Collatz-functions/Collatz-maps.

Even though we have come a long way from the original problem we are not yet at our destination. In fact we will give a definition of Collatz-problems which we could have given right at the beginning of this article. That is it will be formulated purely in the language of basic algebra and doesn't use anything that we thought up so far.

4.2 An abstract algebraic definition

As promised in this section we will give the most general and abstract definition of Collatz-problems that I could come up with. To get to this definition we will make a few simple observations based up on the theory we have developed so far, but the final definition will only use terms of basic algebra.

We will start with a simple observation, namely if $p \in R^n$ then $p \in (R\{[x_1, \ldots, x_n]\})^n$, this is completely obvious. Furthermore we observe if p as before and $G \in (R\{[x_1, \ldots, x_n]\})^n$ ten $p \circ G = p$. This statement is basically saying that the composition a function g with a constant function p is a constant map. So we can reformulate $F^{(k)}(p) = p$ to $F^{(k)}(p) = p(F^{(k)})$. This almost trivial restatement is very powerful and we now are almost ready to define a general Collatz-problem. Notice that in theorem 3.40 we proved that composition (from the right) with a Collatz map induces an endomorphism. Now we can give the general definition of a Collatz-problem associated to the endomorphism F.

Definition 4.4 *Let* R *be a ring and* $F \in End(R)$ *then the Collatz-problem associated* F *is the problem of finding all* $G \in End(R)$ *such that* $G(F^{(k)}) = F^{(k)}(G)$ *for some* $k \geq 1$.

So basically we look for all the endomorphisms G such that they commute with $F^{(k)}$ for certain k. It is obvious that this definition is compatible with the one we have given before if we take $R = S\{[x_1, \ldots, x_n]\}$ for some integral domain S and if we only look at the constant endomorphisms that commute with a iteration of F.

Now notice however that we no longer have such a beautiful cycle structure as we had for the problems in the previous section because we not longer need to have that $F^{(n)} \circ G = G$ if $G \circ F^{(n)} = F^{(n)} \circ G$, in fact that won't hold in general. However if $G \circ F^{(n)} = F^{(n)} \circ G$ then $F^{(i)} \circ G$ commutes with $F^{(n)}$ for all i.

So here we have it, the most general algebraic generalization of Collatz-problems that I could come up with, it's too bad I had to waist about forty pages to get there though. In the next chapter we will look back to what we did in these forty pages and we will also look at the things that still can be/should be done.

Chapter 5

The road from here

In this chapter we will discuss all that we have studied as well as where to go from here. We will start with a section that summarizes our findings so far, then we will discuss some of the conjectures/visions I have on the subject, after that we will discuss some interesting examples of Collatz-functions.

5.1 Where we are right now

We started with the original $3n+1$-function and looked how we could generalize it. We started of with defining pre-Collatz and Collatz sets which we needed in order to give the formal definition of a Collatz-function. We allowed the functions in the Collatz sets to be polynomials in several variables. We replaced the division part of the $3n+1$-function by a surjective module homomorphism. And that is how we got to the definition of a Collatz-function. We saw that these functions were the ideal generalizations of polynomials, and in many ways they behave like polynomials.

We constructed a ring structure of the Collatz-functions in n variables over a principal ideal domain R, and we saw that if R was a field then the ring of Collatz-functions was isomorphic to the ring of polynomials over R. We then we took it a step further and introduced Collatz-maps, we saw that these behaved very much like a polynomial map. But more important we used Collatz-maps to generalize the $3n+1$-function because (for suitable Collatz-maps) we can iterate them. We saw that these maps also induced an endomorphism, and all this showed us the most general and abstract definition of a Collatz-problem. So far we didn't notice that in the general context we no longer have a cycle structure, this is obvious because if $F^{(n)}G = GF^{(n)}$ then obvious $F^{(n)}F^{(i)}G = F^{(i)}GF^{(n)}$ for $0 \leq i \leq n$, but we do not longer need to have $F^{(n)}G = GF^{(n)} = G$.

5.2 *Untied threads*

We will now think a bit more about things we have swept under the rug so far. For instance we have defined a Collatz-function only over a non-zero ideal, but why wouldn't we define Collatz-functions over (0)? Of course when you look at functions over (0) then they are not very interesting since they are equivalent to 0 as function (since it factors over (0) and then is send to 0 by the module homomorphism), but by no means do they have to be the restriction of 0 to (0), furthermore we get into trouble if we apply our addition algorithm since the addition of a function over I and a function over J gives a function over $I \cap J$. If either I or J is (0) then the intersection is (0) but that would mean that the sum would be equivalent to 0 as a function while the sum of the images need not be (in fact probably not). Now of course we can easily solve this problem by only using the sum algorithm if both I and J are (not) (0) and consider $+$ to be purely formal if either I or J is (0). So then we would get functions of the form $f + f_0$ where f_0 is the part that is defined over (0) and f is a normal Collatz-function. If we do this we see that if R is a field then the Collatz ring in n variables over R is no longer isomorphic to the ring of polynomials.

Now that we have considered the problems with addition lets look at multiplication, this also brings complications, but can be solved in exactly the same way. We just consider the multiplication be be formal if the algorithm "fails to behave nicely". These considerations open the door for principal ideal rings, we could do the same in this case as we could do for the zero ideal for. One of the major problems in the past was that the intersection of two non-zero ideals could become zero hence their sum was not well defined. Of course one shouldn't forget that one can no longer assume the Collatz sets to be monideal, which further complicates matters. So one should look at if these generalizations lead to a sensible theory or not.

Another point of consideration is non-principal ideal domains/rings, by our strict definitions we have excluded them from consideration. However we could ask ourselves if we have to, because we could take a domain/ring R such that if I is principal and J is principal then $I \cap J$ is principal, then we could still define a ring structure on all Collatz-functions over the principal ideals. Or another option is to no longer require the images of the functions to be in R but rather in its field/ring of fractions.

Furthermore we should give a nice formal definition of what a Collatz ring is. When it comes to that I think that the observations in the beginning of section 3.2 will play a very important role. But in order for it to work properly, we should set up a theory that treats the ideals of a ring R like the open sets of a topological space. Now this of course possible, but one should notice that when it comes to unions of ideals that the set theoretic union is NOT an ideal in general, so of course you would want it to be the ideal generated by the set theoretic union. So one should take this into mind when looking "open coverings" in this theory because of course you want that if you cover I by $I_1, I_2, \ldots$ then $\forall i \in I \exists j\ i \in I_j$, just like in the topological sense. A nice name for this theory would be ideology theory, since ideals play such an important role and it looks so much like topology. Once one has established this theory then I think that it will be relatively simple to give a more general definition of a Collatz ring. It might even be a interesting question whether schemes could be formulated in this language, although right now I think not because in a scheme a prime ideal is seen as a point while in ideology theory it would be seen as an open set.

We could also try to develop Collatz-functions by using formal power series or skew polynomials instead of polynomials. Now I don't think that formal power series would give any trouble, but I don't know for sure whether there even are skew polynomials in several variables. Once one has set up the theory of formal Collatz series one could try to develop some analysis, although this will probably be not as straight forward, but possibly with the right valuation on R this might be possible.

Another interesting angle would be to try to setup a algebraic geometry theory based on Collatz-functions, in particular all the cycles of length n would be associated with an algebraic variety (namely the one associated to the ideal $(x - f^{(n)})$, and the full cycle set would of course be the union of these varieties. We have already given a rough sketch of how a Gröbner basis theory would look like for Collatz-functions, but of course that sketch will have to be formalized.

However I do not expect the solution for the $3n+1$-problem to come from this part of the theory, but fortunately I do have some ideas where it might come from. These ideas will make up most of the next section.

5.3 *Back to number theory*

Now so far we have caused a lot of problems but hardly solved any at all. First of all we should wonder why are these Collatz-functions even worth our time, if they are worth it at all. Because the definition is rather exotic, so we will probably never run in to them any way. For instance take the $3n+1$-function, who thinks up such a thing? Of course the $3n+1$-problem is all but a natural function, nor does it have (as far as I know) very little purpose outside the $3n+1$-problem, but we are mistaken if we think we never saw any other Collatz-functions before. Trust me you have but you probably didn't recognize them. But let's back up that claim.

Example 5.1 *Notice that* $\lfloor \frac{x}{N} \rfloor = d_{(N)}(f_{(N)}$, *where* $(f_{(N)})_i = x-i$ *for* $0 \le i < N$ *for all* $N\mathbb{Z}$. *Likewise* $\lceil \frac{x}{N} \rceil = d_{(N)}(g_{(n)})$ *where* $(g_{(N)})_i = x+(N-i)$ *for* $0 < i < N$ *and* $(g_{(N)})_0 = x$, *for all* $N \in \mathbb{Z}$.

We will give some more examples of number-theoretic functions that are quite natural as a Collatz-function.

Example 5.2 *The function* $m|n$ *is a natural Collatz-function, just take* $I = (m)$ *and* $f_0 = m$ *and* $f_i = 0$ *for* $0 < i < m$. *So* $m \not| n$ *is also a Collatz-function with* $g_0 = 0$ *and* $g_i = m$ *for* $0 < i < m$.

Example 5.3 *The function* $\texttt{isprime}(n)$ *is a Collatz-function since* $\texttt{isprime}(n) = \prod_{i=2}^{n-1} i \not| n$, $i \not| n$ *are all Collatz-functions and a finite product of Collatz-functions is a Collatz-function.*

Example 5.4 *The total number of divisors of* n *(*$d(n)$*) is a Collatz-function:* $d(n) = \sum_{i=1}^{n} i|n$, *and so is sum of all divisors* $s(n) = \sum_{i=1}^{n} n(i|n)$. *Also the number of prime numbers smaller or equal to* n *is a Collatz-function* $\pi(n) = \sum_{i=2}^{n} \mathit{isprime}(i)$.

Example 5.5 *The function* $\texttt{relativeprime}(n, m)$ *is a Collatz-function,* $\mathit{relativeprime}(n, m) = \prod_{i=2}^{n} \left\lceil \frac{i \not| n + i \not| m}{2} \right\rceil$. *So so is the Euler-phi function,* $\phi(n) = \sum_{i=1}^{n} \mathit{relativeprime}(i, n)$.

Notice that is construction is a bit more complicated because we had to use the composition of Collatz-functions in order to normalize the function if we left out the ceiling part of the function, it would have been either zero or non-zero, now it is either zero or one. The following two functions may be a bit of surprise.

Example 5.6 *The least common multiple and the greatest common divisor of* n *and* m *are Collatz-functions (that is if not both* n *and* m *are equal to one).*

$$gcd(n,m) = \sum_{i=1}^{n}\left(i \prod_{j=i+1}^{n} \left\lceil \frac{j \nmid n + j \nmid m}{2} \right\rceil\right), \text{ and } lcm(n,m) = \sum_{i=2}^{nm}\left(i \prod_{j=1}^{i-1} \left\lceil \frac{n \nmid j + m \nmid j}{2} \right\rceil\right).$$

The examples we have seen in this section are a bit dull. Another class of more interesting examples comes from modulo calculations. For instance consider the function in example 5.7. If we evaluate this function at 0, 1, 2, 3, 4, 5, 6, 7 and 8 we see that this function behaves exactly like division by 5 modulo 9, first of all we notice that if $n > 9$ or $n < 0$ then $f(n)$ is closer to the interval $[0, 9]$ than n, furthermore if we look modulo 9 we see that we add 0 to n until it becomes divisible by 5, then we divide by 5. That is the algorithm that I usually apply if I do that kind of thing without a calculator. So on the one hand we have that n converges to the interval $[0, 9]$ on the other hand we know how it behaves on this interval so we can give the full cycle structure of this function.

Example 5.7

$$f(n) = \begin{cases} \frac{n}{5} & x \equiv 0 \pmod 5 \\ \frac{n+9}{5} & x \equiv 3 \pmod 5 \\ \frac{n+18}{5} & x \equiv 1 \pmod 5 \\ \frac{n+27}{5} & x \equiv 4 \pmod 5 \\ \frac{n+36}{5} & x \equiv 2 \pmod 5 \end{cases}$$

But more interestingly we can construct a function g such that $f(g(n)) = n = g(f(n))$ for all cycle elements of f. The function g is given in example 5.8.

Example 5.8

$$f(n) = \begin{cases} \frac{n}{2} & x \equiv 0 \pmod 2 \\ \frac{n+9}{2} & x \equiv 1 \pmod 2 \end{cases}$$

Now these constructions can be made for all a, N with $0 < a < N$ and a, N relative prime, by taking $f_i = x + jN$ if $i \equiv -jN \pmod{a}$ for $0 \le i, j < N$ (notice the absolute value) it is obvious that the cycles of f and g in this construction are related (in fact the same but opposite in direction). Now we could do the same for $-N < a < 0$, only this time the representatives will lie in the interval $[-N, N]$ and the cycles are those of division by $-a$ modulo N. The function looks like $f_i = -x + jN$. Now one of the big questions is can we find a g if f is of the form $f_i = \pm x + jN$. Personally I think we can do this. Furthermore I conjecture that if we look at the cycles of the sums of such functions and at the sums of their duals (their gs so to speak) then we will find a relation between them, that is for all the cycles of the original function of length k we can find a cycle of the dual function with length n such that $n|k$ or $k|n$, but above that if $\#(n)$ is the number of cycles of length n of the original function and $\#(k)$ the number cycles of length k of the dual function then $\#(n)|\#(k)$ or $\#(k)|\#(n)$. Personally I think this relation will solve the problem, that is if the relation is there at all, because there is more than one way to write as a sum of simple functions, and all these will give restrictions on the possibilities for the cycles and hopefully enough to prove the $3n+1$-problem.

I propose to call functions of the form $f_i = x + jN$, $f_i = -x + jN$ or $f_i = \pm x + jN$ over (a), with a and N relative prime, simple Collatz-functions. Furthermore I'd like to call sums of simple functions semi simple Collatz-functions, and products of simple Collatz-functions, quasi simple Collatz-functions, and last but not least sums of products of Collatz-functions semi quasi simple Collatz-functions. Notice that the set of semi quasi simple Collatz-functions forms a subring of the Collatz ring. Of course these functions are very specific for $\mathbb{Z}$, so we should try to generalize these functions for any ring R over which we can define Collatz-functions, as well as to multiple variables.

The last thing that we still should study is the relation between the Collatz ring over S and R where R is a ring extension of S. Also we know that the $3n+1$-function behaves very natural on the ring of 2-addic numbers (that is it is a continuous function on the 2-addic numbers). I expect that the functions in the (sub)ring $R^I\{[x_1, \ldots, x_n]\}$ will behave very natural with respect to $\left(\hat{R}_I\right)^n$, where $\hat{R}_I$ is the completion of R with respect to I.

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