Journal of Number Theory **67**, 215–228 (1997)
Article No. NT972185

Drinfeld Modular Forms of Weight One

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Communicated by K. Rubin

Received February 18, 1997; revised April 21, 1997

1. INTRODUCTION

Let $A = \mathbf{F}_q[T]$ be the polynomial ring over the finite field $\mathbf{F}_q$ of q elements. D. Goss remarks in [13, (2.1)] that the algebra of (Drinfeld) modular forms for $GL(2, A)$ is the free ring generated by the two Eisenstein series of weights $q-1$ and q^2-1. For a more general congruence subgroup, an abstract presentation of the corresponding ring of modular forms is not known, and is marked an open question by E.-U. Gekeler ([10, VIII.3.1]). The aim of this paper is to develop tools from algebraic geometry to study the case of a principal congruence subgroup $\Gamma(N)$ of level N in $GL(2, A)$.

The paper is organized as follows: Eisenstein series of weight one are defined, and it is shown that they generated the vector space of modular forms of weight one (which is in sharp contrast with the classical situation, where *e.g.* the square of Dedekind's eta-function η^2 is a cusp form of weight one for $\Gamma(12)$). Using Castelnuovo–Mumford regularity, we show that the ring of modular forms is generated by these Eisenstein series and the cusp forms of weight two, thus improving slightly a bound obtained by D. Goss. We then study embeddings of Drinfeld modular curves via Eisenstein series, and the normality of rings of modular forms. The next paragraph interprets Eisenstein series as torsion points of a generic Drinfeld module, to obtain a reduced set of equations for the image of the embedded modular curve under the group of permutations of the variables. This result will allow the use of computational commutative algebra for solving the original problem. In the final paragraph, we calculate explicitly the degrees of various embeddings of Drinfeld modular curves, and give applications to automorphisms of modular curves and the original problem

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0022-314X/97 $25.00

for linear and quadratic level. Finally, future applications to the computation of zeroes of modular forms are presented.

1. EISENSTEIN SERIES OF WEIGHT ONE

A theory of moduli objects for global function fields (analogous to *e.g.* elliptic curves over number fields) was introduced by Drinfeld [6]. Based on this, a theory of modular forms and curves was developed by Goss [13], [14] and Gekeler [10]. We refer to all these references for further information on the objects introduced in this paragraph; our notation will follow as closely as possible Gekeler & Reversat [11].

(1.1) Let $A=\mathbf{F}_q[T]$ be the polynomial ring over the finite field $\mathbf{F}_q$ of q elements, and normalize an absolute value by $|a|=q^{\deg(a)}$, $a\in A$. Let $K=\mathbf{F}_q(T)$ be the quotient field of A, whose completion w.r.t. $|.|$ is denoted by $K_\infty=\mathbf{F}_q((T^{-1}))$. Finally, let C be the completion of an algebraic closure of K_∞. The "Drinfeld upper half plane" is $\Omega=C-K_\infty$, which caries a structure as rigid analytic space.

(1.2) We will write $\Gamma(1)$ for the group $GL(2, A)$. To further simplify notation, we will fix *once and for all* a non-constant polynomial $N\in A$, and let Γ denote the corresponding principal congruence subgroup $\{\gamma\in\Gamma(1)\colon \gamma\equiv\mathbf{1} \bmod N\}$. Any group Γ' such that $\Gamma\subseteq\Gamma'\subseteq\Gamma(1)$ will be called a congruence group of level N, and its elements γ act on Ω via fractional linear transforms, which will be denoted by $z\mapsto\gamma.z$. The compactification of the corresponding orbit space by a finite set of cusps $\Gamma'\backslash\mathbf{P}^1(K)$ is denoted by $\bar{M}_{\Gamma'}$, which is either considered as a rigid analytic space, or as a smooth algebraic projective curve.

(1.3) The curves $\bar{M}_\Gamma$ correspond to a component of the higher moduli scheme for Drinfeld modules with level N-structure (of rank 2, over A, and of "infinite characteristic") $\phi\colon A\to\mathrm{End}(\mathbf{G}_a(C))$—where the latter space will be identified with additive polynomials in $C[X]$. An affine point of $\bar{M}_\Gamma$ corresponds to the isomorphism class of a Drinfeld module with a fixed isomorphism of the N-division points ${}_N\phi$ of ϕ (i.e., the zeroes of $\phi_N(X)$ in C) with $(A/N)^2$.

(1.4) A congruence group Γ' also acts on the set of rigid analytic functions $f\colon\Omega\to C$ via the rule $f\mapsto f|_{[\gamma]_{k,l}}(z)=(\det\gamma)^l\,(cz+d)^{-k}\,f(\gamma.z)$. If f is invariant under this action, and admits a converging Taylor series development in the analytic parameter at all cusps, it is called *a modular form of* *weight k and type l for Γ'*. The C vector space of such forms is denoted by $M_{k,l}(\Gamma')$, and the graded algebra

$$M(\Gamma') = \bigoplus_{\substack{k \in \mathbf{N} \\ l \in \mathbf{Z}/(q-1)}} M_{k,l}(\Gamma')$$

is the ring of modular forms for Γ'. If we omit l in these notations, $l=0$ is understood. If we add a superscript $M^n_{*,*}$, we mean the subspace of $M_{*,*}$ of modular forms that vanish of order $\geqslant n$ at all cusps.

As essential examples of modular forms, consider the coefficient forms g, Δ defined by $\phi^{(z)}_T = TX + g(z)\,X^q + \Delta(z)\,X^{q^2}$, where $\phi^{(z)}$ is the Drinfeld module corresponding to the lattice $zA \oplus A$. Then g, Δ are modular forms of weights $q-1$ and q^2-1, which generate the algebra of modular forms for $\Gamma(1)$ (Goss [13]). Also, $j : \bar{M}_{\Gamma(1)} \to \mathbf{P}^1 : z \mapsto g^{q+1}(z)/\Delta(z)$ is an isomorphism.

(1.5) Since $\Gamma \subseteq SL(2, A)$, the l plays no role, so we let $l=0$. As examples of modular forms for Γ, consider the *Eisenstein series of weight* k defined by

$$E^{(k)}_u(z) := \frac{1}{N} \sum_{\substack{(a,b) \equiv u(N) \\ (a,b) \in A^2}} \frac{1}{(az+b)^k} \in M_k(\Gamma),$$

for any $u \in U := (A/N)^2 - \{(0,0)\}$. If no superscript k is written, $k=1$ is understood. Note that by non-archimedian analysis, this series converges for *any* $k \in \mathbf{N} - \{0\}$. Also, it is easy to see that

$$(1.6) \quad E^{(k)}_u|_{[\gamma]_k} = E^{(k)}_{u.\gamma} \qquad \text{for any} \quad \gamma \in \Gamma(1)$$

(where $\Gamma(1)$ acts from the right on U in the obvious way). This fact allows one to "transport information" (series developments, etc.) from one cusp of Γ to another.

(1.7) Proposition (Gekeler, [9], 2.2).

1. *E_u is never zero at an affine point $z \in \Omega$.*
2. *${}_N\phi^{(z)} = \{[E_u(z)]^{-1} : u \in U\}$ is the set of N-division points of the Drinfeld module $\phi^{(z)}$.*
3. *The order of $E_{(u,v)}$ at the cusp ∞ is equal to $\mathrm{ord}_\infty E_{(u,v)} = |u|$, where u is reduced modulo N (i.e. $\deg(u) < \deg(N)$).*

We will fix *throughout* an (arbitrary) ordering $\{u_1, ..., u_n\}$, $n = q^{2\deg(N)} - 1$ of the elements in U, and index the Eisenstein series of weight $k=1$ accordingly: $E_i = E_{u_i}$.

(1.8) Let $\mathscr{M}$ be the line bundle of germs of modular forms of weight one on $\bar{M}_\Gamma$; Ω^1 the canonical bundle, and Σ the divisor of exact order one at each cusp. Then one can identify

(1.9) $\mathscr{M}^{\otimes 2} \cong \Omega^1 \otimes \mathcal{O}(2\Sigma)$

sending $f(z)$ to $f(z)\,dz$, i.e., differential forms are *double* cusp forms of weight two. Thus, $\mathscr{M}$ is approximately half-canonical. One can calculate the number of cusps, the genus of $\bar{M}_\Gamma$, and via Riemann–Roch one can determine the dimension of M_k as a C-vector space for $k>1$. This is done by Goss in [14], section 4. For $k=1$, the argument evidently fails. However, the following mysterious result holds:

(1.10) THEOREM (E.-U. Gekeler—see [3]). *There are no cusp forms of weight one for any congruence group* $\Gamma' \subseteq \Gamma(1)$, *i.e.* $M^1_{1,l}(\Gamma')=0$.

(1.11) The set of Eisenstein series $\{E_i\}$ we have considered is not always a basis of $M_1(\Gamma)$: there do exists linear relations. But choose (non-canonically) any section of the map

$$U \to \text{Cusps of } \Gamma = \Gamma \backslash \mathbf{P}^1(K) = \text{primitive pairs in } U \text{ modulo } \mathbf{F}_q^*,$$

i.e. a set $S \subseteq U$ of representatives for the cusps. Then, following Hecke [15], one defines the *restricted Eisenstein series* of weight k by

$$F_u^{(k)}(z) := \sum_{\substack{(a,\,b) \equiv u(N) \\ (a,\,b) \in A,\ (a;\,b)=1}} \frac{1}{(az+b)^k} \in M_1(\Gamma)$$

for $u \in S$. One can copy Hecke's proof of

(1.12) PROPOSITION. *For all k, the space* $\langle E_u^{(k)} : u \in S\rangle$ *is of dimension* $\#S$, *hence forms a vector space complement for the space of cusp forms of weight k in* $M_k(\Gamma)$. *In particular,* $M_1(\Gamma)$ *is generated by* $\{E_u : u \in S\}$.

Proof. It follows from Kapranov [18], 1.9 that $\langle E_u^{(k)}\rangle = \langle F_u^{(k)}\rangle$. On the other hand, it is easy to see that

(1.12.1) for each $u \in S$, there is a *unique* cusp $s(u)$ such that $F_u^{(k)}(s(u)) \neq 0$, and $s : S \to S$ is bijective.

Hence the F_u are linearly independent. Finally, one computes via Riemann–Roch that $\dim M_k(\Gamma) - \dim M_k^1(\Gamma) = \#S$ for $k>1$, and for $k=1$, one finds $\dim M_1(\Gamma) = \#S + h^1(\bar{M}_\Gamma, \mathscr{M}) = \#S + \dim M_1^2(\Gamma) = \#S$, the second equality by duality and (1.9), the third equality by (1.10). ▮

(1.13) *Remark.* Apparently, the contents of the above proposition is tacitly assumed by Teitelbaum, Corollary 17 in [22].

(1.14) One can go one step further, and ask whether the Eisenstein series of weight one algebraically generate $M(\Gamma)$. The answer to this question is not known in general, but one can invoke Castelnuovo–Mumford theory to show the following, which slightly improves a result by Goss ([14], 4.6.2):

(1.15) PROPOSITION. *The ring $M(\Gamma)$ is generated by $\{E_i\}$ and the cusp forms of weight two.*

Proof. Reinterpret the natural maps $M_k(\Gamma) \otimes M_1(\Gamma) \to M_{k+1}(\Gamma)$ as

$$H^0(\bar{M}_\Gamma, \mathscr{M}^{\otimes k}) \otimes H^0(\bar{M}_\Gamma, \mathscr{M}) \to H^0(\bar{M}_\Gamma, \mathscr{M}^{\otimes k+1}).$$

A result by Mumford ([19], Theorem 3) says this map is surjective for $k \geqslant 2$ if the following three conditions hold:

1. $\mathscr{M}$ is base-point free. This is true, because for every point of $\bar{M}_\Gamma$ there is an Eisenstein series E_i which is non-zero at that point: at affine points z, interpret $E_i^{-1}(z)$ as a (non-zero) N-torsion point of $\phi^{(z)}$; at cusps, use (1.12.1).
2. $\mathscr{M}$ is of positive degree, which of course holds.
3. $H^1(\bar{M}_\Gamma, \mathscr{M}^{\otimes j}) = 0$ for all $j \geqslant 1$: this follows from the arguments at the end of the proof of the previous proposition.

This shows that $M(\Gamma)$ is generated by forms of weight $\leqslant 2$. Now, $M_1(\Gamma)$ is generated by the Eisenstein series of weight one, and the *squares* of the restricted Eisenstein series $\{F_u^2 : u \in S\}$ (which are degree two polynomials in $\{E_u\}$) form a vector space complement to the space of cusp forms of weight two (—same proof as (1.12)). ▌

By this proposition, the question whether $\{E_i\}$ generates $M(\Gamma)$ algebraically is reduced to the question whether all cusp forms of weight two can be written as polynomials in $\{E_i\}$.

(1.16) We will denote by $E(\Gamma)$ the subring of $M(\Gamma)$ which is generated by the Eisenstein series of weight one. Note that these rings are domains, since they consist of analytic functions. One can wonder how far they are apart.

(1.17) PROPOSITION. *$\bar{M}_\Gamma$, Proj $M(\Gamma)$, and Proj $E(\Gamma)$ are isomorphic curves, and the ring $M(\Gamma)$ is the normalization of $E(\Gamma)$.*

Proof. The map

$$i : \bar{M}_\Gamma \to \mathbf{P}^{n-1} : z \mapsto (E_1(z) : \cdots : E_n(z))$$

is a closed embedding. This is shown by Kapranov ([18], 1.12 and 1.19, noting that one can take $m_0=1$ if $d=2$, in his notations), and independently stated by Pink ([20]). For another independent proof, see [5], IV.2.2.

The image of i is exactly Proj $E(\Gamma)$. On the other hand, Proj $M(\Gamma)$ and $\bar{M}_\Gamma$ are birational, since they have the same function field. But the ring $M(\Gamma)$ is normal (Igusa, [16], III.5), hence Proj $M(\Gamma)$ is smooth. Since birational smooth curves are isomorphic, the first claim follows.

Finally, $M(\Gamma)$ and $E(\Gamma)$ have the same quotient field, since multiplication by E_1^{-k} maps any $f\in M_k(\Gamma)$ into the function field of $\bar{M}_\Gamma=\operatorname{Proj} E(\Gamma)$, which is generated by elements of weight zero in the quotient field of $E(\Gamma)$. ∎

(1.18) For later use, we will even prove a slight generalization. We introduce the following notation: let $\mathbf{P}(Q)$ denote the weighted projective space of dimension $n+1$ over C, with weight vector $Q=(1, ..., 1, q-1, q^2-1)$. If $R=C[x_1, ..., x_n, \gamma, \delta]$ is the polynomial ring in $n+2$ variables, graded by weight$(x_i)=1$, weight$(\gamma)=q-1$ and weight$(\delta)=q^2-1$, then $\mathbf{P}(Q)=$ Proj R. For a permutation $\sigma\in S_n$ we let i_σ be the map

$$i_\sigma\colon \bar{M}_\Gamma\to \mathbf{P}(Q) : z\mapsto (E_{\sigma(1)}(z):\cdots : E_{\sigma(n)}(z) : g(z) : \Delta(z)),$$

(Note again that, although the value of the functions E_i, g, Δ are not defined on $\bar{M}_\Gamma$, their ratios in the weighted space $\mathbf{P}(Q)$ are.) Similar to (1.17), one has:

(1.19) Proposition. *For all $\sigma\in S_n$, the maps i_σ are embeddings.* ∎

(1.20) Corollary. *Let $f\colon\Omega\to C$ be a meromorphic function which satisfies $f|_{[\gamma]_1}=f$ for all $\gamma\in\Gamma$, and such that*

$$div(f)\geqslant 2\Sigma-P-Q,$$

for two arbitrary points P, $Q\in\bar{M}_\Gamma$. Then $f=0$.

Proof. $\mathcal{M}$ defines an embedding of $\bar{M}_\Gamma$ by (1.17), hence it is very ample, so $h^0(\bar{M}_\Gamma, \mathcal{M}\otimes\mathcal{O}(-P-Q))=h^0(\bar{M}_\Gamma, \mathcal{M})-2$. If we apply Riemann–Roch, the vanishing of H^1 (1.10) and duality (1.9) to both sides of this formula, we find that

$$h^0(\bar{M}_\Gamma, \mathcal{M}\otimes\mathcal{O}(-2\Sigma+P+Q))=0,$$

which is exactly the statement of the proposition. ∎

2. THE RING STRUCTURE OF $E(\Gamma)$

We conclude from the preceding section that $M(\Gamma)$ is the normalization of $E(\Gamma)$. The aim of this section is to determine the abstract structure of $E(\Gamma)$, up to a certain symmetry, *i.e.* the defining ideal of the embedding i. On the ring side, define an ideal $I^\sigma \subseteq R$ (where R is as in (1.18)) for any permuation $\sigma \in S_n$ as the kernel of the evaluation map

$$I^\sigma = \ker R \to E(\Gamma): (x_i, \gamma, \delta) \mapsto (E_{\sigma(i)}, g, \Delta).$$

Define an ideal $J \subseteq R$, by requiring that J is generated by the coefficients in the (formal) variable X, of the polynomial

$$(2.1)\ \phi_N^{\gamma,\delta}(X) - NX \prod_{i=1}^{n} (Xx_i - 1),$$

where $\phi^{\gamma,\delta}$ is the $C[\gamma, \delta]$-valued Drinfeld module determined by $\phi_T^{\gamma,\delta}(X) = TX + \gamma X^q + \delta X^{q^2}$. By $Z(I)$, we indicate the zero set of a homogeneous ideal I in weighted projective space $\mathbf{P}(Q)$.

(2.2) Theorem. *As ideals in* R, $J = \bigcap_{\sigma \in S_n} I^\sigma$.

Proof. The proof will consist of two steps: first we will show that the two ideals have the same zero set in $\mathbf{P}(Q)$, then we show that both are radical; so that finally, they will be equal by applying the Nullstellensatz in $\mathbf{P}(Q)$.

(2.2.1) The image of the map i_σ is exactly $Z(I^\sigma)$. One already has that $\operatorname{im}(i_\sigma) \subseteq Z(J)$, since $E_i(z)$, $g(z)$, $\Delta(z)$ satisfy the relations exhibited in J:

$$(*) \quad \forall z \in \Omega : \phi_N^{g(z), \Delta(z)}(X) = NX \prod_{i=1}^{n} (XE_i(z) - 1).$$

The identity $(*)$ even holds at the cusps, say $z = \infty$, after taking limits for

$$\inf\{ |z-k| : k \in K_\infty\} \to \infty.$$

Suppose on the other hand that $P = (x_i^0 : \gamma^0 : \delta^0)$ is a point on $Z(J)$. Since $\bar{M}_{\Gamma(1)} \to \mathbf{P}(q-1, q^2-1) : z \mapsto (g(z) : \Delta(z))$ is an isomorphism, there exists a $z \in \Omega \cup \mathbf{P}^1(K)$ and $\lambda \in C^*$ such that $\gamma^0 = \lambda^{q-1} g(z)$ and $\delta^0 = \lambda^{q^2-1} \Delta(z)$. The polynomial

$$\phi_N^{\gamma^0, \delta^0}(X) = \lambda^{-1} \phi_N^{(z)}(\lambda X)$$

has a reciprocal zeroes on the one hand $\{x_i^0\}$ since $P \in Z(J)$; on the other hand $\{\lambda E_i(z)\}$; so there is a permutation $\sigma \in S_n$ such that $x_i = \lambda E_{\sigma(i)}(z)$, i.e.

$P \in Z(I^\sigma)$. Remark again that if $\delta^0 = 0$, then $\gamma^0 \neq 0$ (since otherwise the relations in J would imply $x_i^0 = 0$, $\forall i$). We can choose $z = \infty$, and then $\{\lambda E_i(\infty)\}$ is the set of zeroes of $\phi_N^{g(\infty), 0}$, by the same limit argument following $(*)$ (for $\lambda = \sqrt[q-1]{\gamma^0/g(\infty)}$).

(2.2.2) Fix first of all some auxiliary notation: we will denote by $Z_R(L)$ the zero set of an ideal L in $n+2$-dimensional affine space Spec(R).

It is clear that I^σ is radical, since $E(\Gamma)$ is reduced. As far as J is concerned, observe that R/J is a complete intersection ring: the number of variables is $n+2$, the number of relations n, and the dimension $= 1 + \dim(Z(I^\sigma)) = 1 + \dim(\bar{M}_\Gamma) = 2$. In particular, R/J is Cohen–Macaulay (Eisenbud, [7] 18.13). Hence it is reduced if and only if it is regular in codimension zero (*loc. cit.*, 18.15). So we have to show that the set of singular points on $Z_R(J)$ is of dimension one. The jacobian ideal (*id.*, 16.20) of J consists of the $n \times n$ minors in the matrix $M := ((\partial f_j/\partial x_i)(\partial f_j/\partial \gamma)(\partial f_j/\partial \delta))$, $(1 \leqslant i,\ j \leqslant n)$, where f_j is the coefficient of X^j in (2.1). In particular, taking the first n columns of M, we find that the jacobian ideal of J contains the minor

$$m = \left| \frac{\partial}{\partial x_j} s_i(x_1, ..., x_n) \right|,$$

where s_i is the symmetric polynomial of weight i. It is easy to see that

$$m = \prod_{i<j} (x_i - x_j).$$

Singular points of $Z_R(J)$ therefore belong to a diagonal hyperplane $Z_R(x_i - x_j)$. It suffices to show that no component of $Z_R(J)$ belongs to such a hyperplane, since then the dimension of the singular locus of $Z_R(J)$ is $\leqslant \dim Z_R(x_i - x_j) \cap Z(J) = \dim Z(J) - 1 = 1$ (*id.* 13.11). But these components are of the form $Z_R(I^\sigma)$, since we already know by (2.2.1) that the prime factors of the ideal J are the ideals I^σ. Suppose that $Z_R(I^\sigma) \subseteq Z_R(x_i - x_j)$, then also $Z(I^\sigma) \subseteq Z(x_i - x_j)$, and hence $E_{\sigma(i)}(z) \equiv E_{\sigma(j)}(z)$ for all $z \in \bar{M}_\Gamma$: a contradiction, since the n N-division points $E_i^{-1}(z)$ are distinct for all affine z. This proves that J is radical. ▮

(2.3) Example/Algorithm. The above theorem implies that

$$\text{Spec } M(\Gamma) = \text{Normalization of (Component of Spec } R/J).$$

Thus, to study a property of the ring $M(\Gamma)$, one can study the corresponding property for R/J, and its behaviour under decomposition and normalization.

One can also try to find a presentation of $M(\Gamma)$ by calculating explicitly the factors of J. As an example, consider the case where $N = T$, $q = 3$, $\# S = 4$ ($g(\bar{M}_\Gamma) = 0$—normality is then trivial). It is computationally more efficient to eliminate the linear relations in $\{E_i\}$ (at the cost of enlarging the degree of J). To calculate a factor of J, one can apply *e.g.* the primary factorization algorithm of Gianni, Trager & Zacharias [12]. This was implemented by Hans Decker at Saarbrücken, to find that

$$M(\Gamma(T)) \cong C[t, x, y, z]/\langle ty - tz - yz, xy + xz - yz, tx + tz + xz \rangle,$$

where the variables represent primitive Eisenstein series. The general case will perhaps (?) require normalization, for which several algorithms are known (*e.g.* Vasconcelos, [23]).

(2.4) Finding an explicit presentation for the algebra of modular forms for any congruence group Γ' of level N, requires the additional study of invariants for the group $G := \Gamma'/\Gamma$ acting on M (—and semi-invariants for the character $\det^{\otimes l}$ if the type is l). This action is by permutation (1.6). But the characteristic of C can divide the order of the group G, thus creating a question of modular invariant theory.

3. DEGREES AND AUTOMORPHISMS

(3.1) Proposition. *The number of distinct prime components of the ideal J in* (2.2) *equals*

$$M = \frac{n!}{(q-1)(q^2-1) \deg \mathscr{M}}.$$

Proof. The degree of $\mathrm{im}(i_\sigma)$ in $\mathbf{P}(Q)$ is exactly the degree of the corresponding line bundle $\mathscr{M}$ (Iitaka [17], Prop. 6.2). On the other hand, the degree of $Z(J)$ is the value $\lim_{t \to 1} (1-t)^2 H_J(t)$, where H_J is the Hilbert function of R/J. Now the latter is a complete intersection, so one can use the result of Stanley [21], (3.3) to calculate its Hilbert function. One finds that

$$\deg Z(J) = \lim_{t \to 1} \prod_{\substack{i=1 \\ i \neq q-1,\, q^2-1}}^{n} \frac{1-t^i}{1-t} = \frac{n!}{(q-1)(q^2-1)}.$$

To find the value of M, note that $\deg Z(J) = M \deg Z(I)$ (this is clear if one interprets the degree as the number of intersection points with a generic hyperplane in $\mathbf{P}(Q)$). ∎

Let $I = I^{id}$. If $\tau \in S_n$ is a permutation which maps $Z(I)$ to itself, then $i_{id}^{-1} \circ \tau \circ i_{id}$ is an automoprhism of $\bar{M}_\Gamma$. The permutations that fix the component $Z(I)$ pointwise form a cyclic group $\mathrm{Fix}(I)$ of order $q-1$ (induced by $E_u \mapsto E_{\lambda u} = \lambda E_u$ for $\lambda \in \mathbf{F}_q^*$). We find an injection of groups $\mathrm{Stab}(I)/\mathrm{Fix}(I) \hookrightarrow \mathrm{Aut}(\bar{M}_\Gamma)$.

(3.2) Corollary. *Under this injection, Stab*(I)*/Fix*(I) *maps isomorphically onto the modular automorphisms* (*induced by* $z \mapsto \gamma \cdot z$, $\gamma \in \Gamma(1)/\Gamma \cdot \mathbf{F}_q^*$).

Proof. Any modular automorphism $z \mapsto \gamma \cdot z$ lies in the image of the map, since it is given on $Z(I)$ by $(E_u(z)) \mapsto (E_u(z)|_{[\gamma]_1}) = (E_{u\gamma}(z))$. On the other hand, $\#\mathrm{Stab}(I) = \#S_n/M$, where M is the number of components of $Z(J)$. Plugging in the value of M, we find that $\#\mathrm{Stab}(I) = \#\Gamma(1)/\Gamma$, which implies the result. ▮

One can wonder whether $\bar{M}_\Gamma$ has any non-modular automorphisms at all. The least one can say is that such an automorphism induces an automorphism on the linear system $|\mathscr{M}|$, and hence on $Z(I) \subseteq \mathbf{P}(Q)$. Is the action always through permutation?

(3.3) Example/Theorem. The ring $M(\Gamma(T))$ of (Drinfeld) modular forms of level T is isomorphic to

$$C_\infty[x_\beta : \beta \in \mathbf{F}_q \cup \{\infty\}]/\langle (\alpha - \beta)\, x_\alpha x_\beta + (x_\alpha - x_\beta)\, x_\infty : \alpha, \beta \in \mathbf{F}_q \rangle,$$

where the isomorphism is given by the identification $E_{(0,1)} \to x_\infty$, $E_{(1,\beta)} \to x_\beta$.

Proof. A geometric proof of this is given in [4], where an equivalent, more complicated set of equations was found. The following elegant ad-hoc argument is due to Don Zagier: $E_\beta^{-1}(z) = e_z((z+\beta)/T)$, where e_z is the $\mathbf{F}_q$-*linear* lattice function

$$e_z(\omega) := \omega \prod_{(0,0) \neq (a,b) \in A^2} \left(1 - \frac{\omega}{az+b}\right).$$

Hence for any $\beta \in \mathbf{F}_q$:

$$E_\beta^{-1}(z) - \beta E_\infty^{-1}(z) = e_z\left(\frac{z+\beta}{T}\right) - \beta e_z\left(\frac{1}{T}\right) = E_0^{-1}$$

is independent of β. This implies that the above relations hold. An easy count then gives that the Hilbert functions of both rings coincide, whence the result. ▮

(3.3.1) *Remark.* The same result of course holds for any level N of degree one. The result is also compatible with the example calculated in (2.3).

(3.4) PROPOSITION. *If* $(q, \deg N) = (2, 2)$, *then* $M(\Gamma)$ *is generated by* $\{E_i\}$.

Proof. The proof is based on Castelnuovo's bound ([1], p. 116): if the genus of an embedded projective curve attains this bound, then it is projectively normal, *i.e.* its coordinate ring is normal. We will show this to be true for the embedding

$$i' : \bar{M}_\Gamma \to \mathbf{P}^{\#S-1} : z \mapsto (E_1(z) : \cdots : E_{\#S}(z)),$$

where we have chosen a basis $\{E_i\}$ of $M_1(\Gamma)$. The coordinate ring is $E(\Gamma)$, hence the result will follow. There are three cases, summarized in the following table:

No. of Distinct Prime Divisors of Degree One of N	No. of S	$g(\bar{M}_\Gamma)$	$\deg \mathscr{M}$	Castelnuovo's Bound on $g(\bar{M}_\Gamma)$
0	15	6	20	6
1	12	5	16	5
2	9	4	12	4

∎

(3.5) COROLLARY. *If* $(q, \deg N) = (2, 2)$, *the ideal of relations between* $\{E_i\}$ *can be generated by polynomials of degree* $\leqslant 3$.

Proof. Let R_k be the kernel of the natural map $\mu_k : M_1(\Gamma) \otimes M_k(\Gamma) \to M_{k+1}(\Gamma)$. Then a result of Fujita ([8], 1.8.c) says that the natural map $\rho_k : R_k \otimes M_1(\Gamma) \to R_{k+1}$ is surjective if $\mathscr{M}$ is base point free, which we know by the proof of (1.15); if $H^1(\bar{M}_\Gamma, \mathscr{M}^{k-1}) = 0$, which is true for $k > 1$ by (1.10); and if μ_{k-1} is surjective, which is true for $k > 1$ by (4.4). Hence for $k \geqslant 2$, the map ρ_k is surjective, and from this the result follows. ∎

(3.6) *Concluding Remarks.* The following three topics seem to deserve future attention:

(3.6.1) *Projective Normality of Rings of Modular Forms.* Is it true that the line bundle $\mathscr{M}$ is normally generated, *i.e.* are all modular forms for Γ polynomials in the Eisenstein series of weight one? We have seen a non-trivial case of this listed in (3.4). The setup of (2.2) is meant to deal with such questions in greater generality; we have seen that $\mathscr{M}$ is very ample, and hence we can store its properties in the corresponding embedding. We have found a set of equations for it "up to permutation" and normalization then determines the ring of modular forms. Since the equations cut out a complete intersection, all information about its "normalization" seems to to lie hidden in its jacobian ideal (by Serre's criterium). Can one somehow calculate it, and is it in some sense "independent of the level N"? I hope to return to this point in the future.

(3.6.2) *Computing Zeroes of Modular Forms.* We know by (1.17) that any modular form f can be written as the quotient of two homogeneous polynomials in the Eisenstein series of weight one: $f = P(E_u)/Q(E_u)$. Assume for simplicity that $f \in E(\Gamma)$, *i.e.* $Q = 1$ (normal generation would imply that this holds in general). The modular form f is represented in $\mathbf{P}(Q)$-space by the hypersurface $Z(P)$, and the zeroes z of f correspond to the intersection points $(E_u(z))$ of $Z(P)$ with $Z(I)$. Since the functions E_u are locally $1-1$ on $\bar{M}_\Gamma$, the intersection multiplicity of $Z(P)$ and $Z(I)$ at $(E_u(z))$ is the multiplicity of the zero of f at z. We know that $Z(J)$ is smooth away from the cusps. Hence the non-cuspidal zeroes of f correspond to intersection points of $Z(P)$ with $Z(J)$, and multiplicity is "preserved". The equations of $Z(J)$ seem efficient from the point of view of computational geometry (small coefficients, complete intersection, ...), and *e.g.* the elimination of the variables $\{x_i\}$ from the equations $I + P$ in R (calculated via Groebner bases) produces a finite list of possible j-invariants of zeroes of f. This is a generalization to higher level of the study of the zeroes of Eisenstein series for the full modular group that was initiated in [2].

(3.6.3) *The "Classical" Case.* Division values of elliptic curves are modular forms of weight *two*. A theory similar to (2.2) can be developped for them, but these division values cannot be expected to generate all modular forms of even weight, due to the existence of cusp forms of weight two in general. On the other hand, there are (rather mysterious) modular forms of weight one for $\Gamma(N)$. So the classical theory diverges into (a) an explicity theory of rings of modular forms of *even* weight; and (b) the question whether all modular forms are generated by forms of weight *one*. To answer part (b) would require completely new techniques. A positive answer would imply a (crude) lower bound on $\dim M_1(\Gamma(N))$. Part (a) survives, and can be applied in the spirit of (3.6.2) to calculate information about zeroes of classical modular forms (if the weight is odd, just take squares). If f is a modular form for $\Gamma_0(N)$, one can alternatively do intersection theory on a plane model of $X_0(N)$. This is a two variable situation, but has the computational disadvantage of having large coefficients and high self intersection at complex multiplication points. We hope to compute a few interesting examples in the future.

ACKNOWLEDGMENTS

The author is supported as “aspirant” by the Belgian National Fund for Scientific Research (NFWO). He thanks E.-U. Gekeler for continuous interest in this work, and Don Zagier for a short cut in (3.3).

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