EQUIVARIANT DEFORMATION OF MUMFORD CURVES AND OF ORDINARY CURVES IN POSITIVE CHARACTERISTIC

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Abstract

We compute the dimension of the tangent space to, and the Krull dimension of, the prorepresentable hull of two deformation functors. The first one is the "algebraic" deformation functor of an ordinary curve X over a field of positive characteristic with prescribed action of a finite group G, and the data are computed in terms of the ramification behaviour of $X \to G\backslash X$. The second one is the "analytic" deformation functor of a fixed embedding of a finitely generated discrete group N in $\mathrm{PGL}(2, K)$ over a nonarchimedean-valued field K, and the data are computed in terms of the Bass-Serre representation of N via a graph of groups. Finally, if Γ is a free subgroup of N such that N is contained in the normalizer of Γ in $\mathrm{PGL}(2, K)$, then the Mumford curve associated to Γ becomes equipped with an action of N/Γ, and we show that the algebraic functor deforming the latter action coincides with the analytic functor deforming the embedding of N.

Introduction

Equivariant deformation theory is the correct framework for formulating and answering questions such as the following: Given a curve X of genus g over a field k and a finite group of automorphisms $\rho : G \hookrightarrow \mathrm{Aut}(X)$ of X, in how many ways can X be deformed into another curve of the same genus on which the same group of automorphisms still acts? The precise meaning of this question (at least infinitesimally) is related to the deformation functor $D_{X,\rho}$ of the pair (X, ρ), which associates to any element A of the category $\mathscr{C}_k$ of local Artinian k-algebras with residue field k the set of isomorphism classes of liftings $(X^\sim, \rho^\sim, \phi^\sim)$, where $X^\sim$ is a smooth scheme of finite type over A, $\phi^\sim$ is an isomorphism of $X^\sim \otimes k$ with X, and $\rho^\sim : G \to \mathrm{Aut}_A(X)$ lifts ρ via $\phi^\sim$. In general, $D_{X,\rho}$ has a prorepresentable hull $H_{X,\rho}$ in the sense of M. Schlessinger [18]. This means that there is a smooth map of functors $\mathrm{Hom}(H_{X,\rho}, -) \to D_{X,\rho}$ which induces an isomorphism on the level of

DUKE MATHEMATICAL JOURNAL
Vol. 116, No. 3, © 2003
Received 29 March 2001. Revision received 5 November 2001.
2000 *Mathematics Subject Classification*. Primary 14G22, 14D15, 14H37.

tangent spaces, where $H_{X,\rho}$ is a Noetherian complete local k-algebra with residue field k. The above question can then be reformulated as the computation of the Krull dimension of $H_{X,\rho}$. Two remarks are in order. First, if $g \geq 2$, then the functor is even prorepresentable by $H_{X,\rho}$, so $\operatorname{Spf} H_{X,\rho}$ is a formal scheme that is, so to speak, the universal basis of a family of curves that have the same automorphism group as X. Second, $H_{X,\rho}$ is even algebraizable (cf. A. Grothendieck [11, Section 3]), and the underlying algebraic scheme over k might be considered as the genuine "universal basis" scheme.

If the characteristic of the ground field k is zero, the dimension of $H_{X,\rho}$ is easy to compute. All obstructions and group cohomology (cf. Section 3.2) disappear, and we find that

$$\dim H_{X,\rho} = 3g_Y - 3 + n, \tag{1}$$

where $Y := G\backslash X$ is the quotient of X, g_Y is its genus, and n is the number of branch points on Y. (Note that $3g_Y - 3$ is the degree of freedom of varying the moduli of Y and that one extra degree of freedom comes in for every branch point.) This result can be found in any classical text on Riemann surfaces (see, e.g., [7, Section V.2.2]); the moral is that ramification data of $X \to Y$ provide all the necessary information for computing $H_{X,\rho}$.

In this work we are interested in the corresponding question in positive characteristic. Let us first present the motivating example for our studies: moduli schemes for rank 2 Drinfeld modules with principal level structure (see, e.g., E.-U. Gekeler and M. Reversat [8]).

Example

Let $q = p^t$, $F = \mathbf{F}_q(T)$, and $A = \mathbf{F}_q[T]$; let $F_\infty = \mathbf{F}_q((T^{-1}))$ be the completion of F, and let C be a completion of the algebraic closure of F_∞. On Drinfeld's "upper half-plane" $\Omega := \mathbf{P}^1_C - \mathbf{P}^1_{F_\infty}$ (which is a rigid analytic space over C), the group $\mathrm{GL}(2, A)$ acts by fractional transformations. Let $\mathscr{Z} \cong \mathbf{F}_q^*$ be its center. For $\mathfrak{n} \in A$, the quotients of Ω by congruence subgroups $\Gamma(\mathfrak{n}) = \{\gamma \in \mathrm{GL}(2, A) : \gamma = \mathbf{1} \bmod \mathfrak{n}\}$ are open analytic curves that can be compactified to projective curves $X(\mathfrak{n})$ by adding finitely many cusps. These curves are analogues in the function field setting of classical modular curves $X(n)$ for $n \in \mathbf{Z}$. Clearly, elements from $G(\mathfrak{n}) := \Gamma(1)/\Gamma(\mathfrak{n})\mathscr{Z}$ induce automorphisms of $X(\mathfrak{n})$. It is even known that $G(\mathfrak{n})$ is the full automorphism group of $X(\mathfrak{n})$ if $p \neq 2, q \neq 3$ (cf. [5, Proposition 4]). It follows from (1) that a classical modular curve does not admit equivariant deformations since $X(n) \to X(1) = \mathbf{P}^1$ is ramified above three points. (The most famous such curve is probably $X(7)$, which is isomorphic to Klein's quartic of genus 3 with $\mathrm{PSL}(2, 7)$ as the automorphism group.) What is the analogous result for the Drinfeld modular curves $X(\mathfrak{n})$? Note that $X(\mathfrak{n}) \to X(1) = \mathbf{P}^1$ is ramified above two points with ramification groups

$\mathbf{Z}/(q+1)\mathbf{Z}$ and $(\mathbf{Z}/p\mathbf{Z})^{td} \rtimes \mathbf{Z}/(q-1)\mathbf{Z}$, where $d = \deg(\mathfrak{n})$ (cf. [5, Proposition 4]), which already shows that something goes wrong when naively applying (1).

The correct framework for carrying out such computations is that of equivariant cohomology (see Grothendieck [10]). It predicts that in positive characteristic the group cohomology of the ramification groups with values in the tangent sheaf at branch points contribute to the deformation space. J. Bertin and A. Mézard have considered the case of a cyclic group of prime order $G = \mathbf{Z}/p\mathbf{Z}$ in [1], mainly concentrating on mixed characteristic lifting. In this paper, we work in the equicharacteristic case and we do not want to impose direct restrictions on the group G, but rather on the curve X, which we require to be *ordinary*. This means that the p-rank of its Jacobian satisfies

$$\dim_{\mathbf{F}_p} \mathrm{Jac}(X)[p] = g.$$

The property of being ordinary is open and dense in the moduli space of curves of genus g, so our calculations do apply to a large portion of that moduli space. (There is some cheating here since curves without automorphisms are also dense; but notice that the analytic constructions in part B of the paper at least show the existence of lots of ordinary curves with automorphisms.) The main advantage of working with ordinary curves is that their ramification groups are of a very specific form, so the Galois cohomological computation becomes feasible (cf. Proposition 1.4): If P_i is a point on Y branched in $X \to Y$, then

$$G_{P_i} \cong (\mathbf{Z}/p\mathbf{Z})^{t_i} \rtimes \mathbf{Z}/n_i\mathbf{Z} \tag{2}$$

for some integers (t_i, n_i) satisfying $n_i | p^{t_i} - 1$. If $t_i > 0$, we say that P_i is wildly branched. Our main algebraic result is the following (which we state here in a form that excludes a few anomalous cases); one can think of it giving the “positive characteristic error term” to the Riemann surface computation in (1).

MAIN ALGEBRAIC THEOREM (cf. Theorem 5.1)
Assume that $p \neq 2, 3$, *that* X *is an ordinary curve of genus* $g \geq 2$ *over a field of characteristic* $p > 0$, *and that* G *is a finite group acting via* $\rho : G \to \mathrm{Aut}(X)$ *on* X, *such that* $X \to Y := G\backslash X$ *is branched above n points, of which the first s are wildly branched. Then the equicharacteristic deformation functor* $D_{X,\rho}$ *is prorepresentable by a ring* $H_{X,\rho}$ *whose Krull dimension is given by*

$$\dim H_{X,\rho} = 3g_Y - 3 + n + \sum_{i=1}^{s} \frac{t_i}{s(n_i)},$$

where $s(n_i) := [\mathbf{F}_p(\zeta_{n_i}) : \mathbf{F}_p] = \min\{s' > 0 : n_i | p^{s'} - 1\}$, g_Y *is the genus of* Y, *and* (t_i, n_i) *are the data corresponding to the wild ramification points via (2).*

The theorem is proved by first computing the first-order local deformation functors, then studying their liftings to all of $\mathscr{C}_k$, and putting the results together via the localization theorem [1, Théorème 3.3.4]. In the end, we can even describe the ring $H_{X,\rho}$ explicitly (cf. Proposition 1.12 and Section 4.4). It turns out to be a formal polydisc to which, for each ramification point with $n_i \leq 2$, a $((p-1)/2)$-nilpotent zero-dimensional scheme is attached. (In characteristic $p = 2$ even more weird things can happen.) These nilpotent schemes are related to the lifting of a specific first-order deformation via what we call *formal truncated Chebyshev polynomials*.

Example (*continued*)
For the Drinfeld modular curve $X(\mathfrak{n})$, we find a $(d-1)$-dimensional reduced deformation space.

The second part of this paper is concerned with analytic equivariant deformation theory. Let $(K, |\cdot|)$ be a nonarchimedean-valued field of positive characteristic $p > 0$ with residue field k. Recall that a *Mumford curve* over K is a curve X whose stable reduction is isomorphic to a union of rational curves intersecting in k-rational points. D. Mumford has shown that this is equivalent to its analytification X^{an} being isomorphic to an analytic space of the form $\Gamma\backslash(\mathbf{P}_K^{1,\text{an}} - \mathscr{L}_\Gamma)$, where Γ is a discontinuous group in $\mathrm{PGL}(2, K)$ with $\mathscr{L}_\Gamma$ as the set of limit points. It is known that Mumford curves are ordinary, so the above algebraic theory applies. But what interests us most in this second part is to find out where these deformations "live" in the realm of discrete groups.

The setup is as follows: Let X be a Mumford curve, and let N be a group contained in the normalizer $N(\Gamma)$ of the corresponding so-called Schottky group Γ such that $\Gamma \subseteq N$. Then there is an injection $\rho : G := N/\Gamma \hookrightarrow \mathrm{Aut}(X)$ (an isomorphism if $N = N(\Gamma)$). By rigidity, two Mumford curves are isomorphic if and only if their Schottky groups are conjugate in $\mathrm{PGL}(2, K)$. Hence it is natural to consider the analytic deformation functor $D_{N,\phi} : \mathscr{C}_K \to \mathsf{Set}$ that associates to $A \in \mathscr{C}_K$ the set of homomorphisms $N \to \mathrm{PGL}(2, A)$ that lift the given morphism ϕ in the obvious sense. This functor comes with a natural action of conjugation by the group functor $\mathrm{PGL}(2)^\wedge : \mathscr{C}_K \to \mathsf{Groups}$ given by

$$\mathrm{PGL}(2)^\wedge(A) := \ker\big[\mathrm{PGL}(2, A) \to \mathrm{PGL}(2, K)\big],$$

and we denote by $D_{N,\phi}^{\sim}$ the quotient functor $D_{N,\phi}^{\sim} = \mathrm{PGL}(2)^\wedge\backslash D_{N,\phi}$. We now set out to compute the hulls corresponding to these functors.

For this we use the fact that N can be described by the theorem of H. Bass and J.-P. Serre, which states that there is a graph of groups $(T_N, N_\bullet)$ such that N is a semidirect product of a free group (of rank the cyclomatic number of T_N) and the tree (amalgamation) product associated to a lifting of stabilizer groups N_* of vertices and edges $*$ from T_N to the universal covering of T_N as as graph. In a technical proposition, we show that this decomposition of the group N induces a decomposition of functors

$$D_{N,\phi} \cong \varprojlim_{T_N} D_{N_\bullet,\phi|_{N_\bullet}} \times D_{F_c,\phi\circ s},$$

where s is a section of the semidirect product. (Note that this is a *direct product* of functors, that the inverse limit is over T_N, *not* over its universal covering, and that we are using $D_{N,\phi}$, not $D^{\sim}_{N,\phi}$.) This reduces everything to the computation of $D_{N,\phi}$ for free N (which is easy) or finite N acting on $\mathbf{P}^1$. The latter can be done by using the classification of finite subgroups of $\mathbf{P}^1$ and the algebraic results from the first part in the particular case of $\mathbf{P}^1$. Modulo a few anomalous cases, the result is the following.

MAIN ANALYTIC THEOREM (cf. Theorem 8.4)
If X is a Mumford curve of genus $g \geq 2$ over a nonarchimedean field K of characteristic $p > 3$ with Schottky group Γ, then for a given discrete group N contained in the normalizer of Γ in $\mathrm{PGL}(2, K)$ with corresponding graph of groups $(T_N, N_\bullet)$, the equicharacteristic analytic deformation functor $D^{\sim}_{N,\phi}$ is prorepresentable by a ring $H^{\sim}_{N,\phi}$ whose dimension satisfies

$$\dim H^{\sim}_{N,\phi} = 3c(T_N) - 3 + \sum_{\substack{v=\mathit{vertex}\\ \mathit{of}\ T_N}} d(v) - \sum_{\substack{e=\mathit{edge}\\ \mathit{of}\ T_N}} d(e),$$

where $c(*)$ denotes the cyclomatic number of a graph $*$ and

$$d(*) = \begin{cases} 2 & \text{if } N_* = \mathbf{Z}/n\mathbf{Z}, \\ 3 & \text{if } N_* = A_4, S_4, A_5, D_n, \mathrm{PGL}(2, p^t), \mathrm{PSL}(2, p^t), \\ t & \text{if } N_* = (\mathbf{Z}/p\mathbf{Z})^t, \\ t/s(n) + 2 & \text{if } N_* = (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}, \end{cases}$$

where n is coprime to p.

Example (*continued*)
The Drinfeld modular curve $X(\mathfrak{n})$ is known to be a Mumford curve, and the normalizer of its Schottky group is isomorphic to an amalgam (cf. [5, Proposition 4])

$$N(\mathfrak{n}) = \mathrm{PGL}(2, q) *_{(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(q-1)\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z})^{td} \rtimes \mathbf{Z}/(q-1)\mathbf{Z}. \tag{3}$$

The above formula again gives a $(d-1)$-dimensional deformation space. We can see these deformations explicitly as follows. By conjugation with $\mathrm{PGL}(2, C)$, we can assume that the embedding of $\mathrm{PGL}(2, p^t)$ in (3) into $\mathrm{PGL}(2, C)$ is induced by the standard $\mathbf{F}_q \subseteq C$. Then a little matrix computation shows that the freedom of choice left is that of a d-dimensional $\mathbf{F}_q$-vector space V of dimension d in C which contains the standard $\mathbf{F}_q$ in order to embed the order-p elements from the second group involved in the amalgam as $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ for $x \in V$.

In the final section of the paper, we compare algebraic and analytic deformation functors. What comes out is an isomorphism of functors

$$D^{\sim}_{N,\phi} \cong D_{X,\rho}.$$

To prove this, we remark that the whole construction of Mumford can be carried out in the category $\mathscr{C}_K$ (by fixing a lifting of N to $\mathrm{PGL}(2, A)$) to produce a scheme X_Γ over Spec A whose central fiber is isomorphic to X, and such that X_Γ carries an action ρ of N/Γ which reduces to the given action on X.

Remark. We did not take the more global road of “equivariant Teichmüller space” to analytic deformation, as F. Herrlich does in [12] and [13]; one can consider the space

$$\mathscr{M}(N, \Gamma) = \mathrm{PGL}(2, K)\backslash \mathrm{Hom}^*\big(N, \mathrm{PGL}(2, K)\big)/\,\mathrm{Aut}_\Gamma(N),$$

where Hom^* means the space of injective morphisms with discrete image, the action of $\mathrm{PGL}(2, K)$ from the left is by conjugation, and the action of the group of automorphisms of N which fix Γ is on the right. The relations in N impose a natural structure on $\mathscr{M}(N, \Gamma)$ as an analytic space, but in positive characteristic this structure might be nonreduced due to the presence of parabolic elements; for example, if N contains $\mathbf{Z}/p\mathbf{Z}$, $p > 3$, then the condition that a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be of order p leads to

$$\big(\mathrm{tr}^2(\gamma) - 4\det(\gamma)\big)^{(p-1)/2} \cdot b = 0.$$

Therefore in [13] global considerations are restricted to the case where N does not contain parabolic elements (see [13, p. 148]), whereas our local calculations are independent of such restrictions.

Let us note that if K has characteristic zero, then in [13] a formula is given for the dimension of $\mathscr{M}(N, \Gamma)$ (which turns out to be an equidimensional space) compatible with our main analytical theorem.

Convention. Throughout this paper, if R is a local ring, we let $\dim R$ denote its Krull dimension, and if V is a k-vector space, we let $\dim_k V$ denote its dimension as a k-vector space.

Part A: Algebraic theory

1. Deformation of Galois covers

1.1. The global deformation functor

We start by recalling the definition of an equivariant deformation of a curve with a given group of automorphisms (see [1] for an excellent survey).

Let k be a field, and let X be a smooth projective curve over k. We fix a finite subgroup G in the automorphism group $\mathrm{Aut}_k(X)$ of X, and we denote by ρ the inclusion

$$\rho : G \hookrightarrow \mathrm{Aut}_k(X) : \sigma \to \rho_\sigma .$$

Let $Y = G\backslash X$ be the quotient curve, and denote by π the quotient map $X \to Y$.

Let $\mathscr{C}_k$ be the category of Artinian local k-algebras with the residue field k. A *lifting* of (X, ρ) to A is a triple $(X^\sim, \rho^\sim, \phi^\sim)$ consisting of a scheme $X^\sim$ which is smooth of finite type over A, an injective group homomorphism

$$\rho^\sim : G \hookrightarrow \mathrm{Aut}_A(X^\sim) : \sigma \to \rho^\sim_\sigma ,$$

and an isomorphism

$$\phi^\sim : X^\sim \otimes_{\mathrm{Spec}\, A} \mathrm{Spec}\, k \to X$$

of schemes over k such that $\overline{\rho^\sim} = \rho$. Here, $\overline{\rho^\sim}$ denotes the composite of $\rho^\sim$ and the restriction onto the central fiber, identified with X by $\phi^\sim$.

Two liftings $(X^\sim, \rho^\sim, \phi^\sim)$ and $(X^\approx, \rho^\approx, \phi^\approx)$ are said to be *isomorphic* if there exists an isomorphism $\psi : X^\approx \to X^\sim$ of schemes over A such that $\phi^\sim \circ (\psi \otimes_A k) = \phi^\approx$ and, for any $\sigma \in G$, $\psi \circ \rho^\approx_\sigma = \rho^\sim_\sigma \circ \psi$.

We arrive at the *deformation functor*

$$D_{X,\rho} : \mathscr{C}_k \to \mathsf{Set}$$

that assigns to any $A \in \mathscr{C}_k$ the set of isomorphism classes of liftings of (X, ρ).

1.2. The functor π_*^G on tangents

In order to compute the tangent space to this deformation functor, we need to recall the equivariant cohomology theory of Grothendieck.

The morphism π induces a tangent map $\mathscr{T}_X \to \pi^*\mathscr{T}_Y$, which is a monomorphism since π is generically étale. (Dually, $\Omega_{X/Y}$ is a torsion sheaf.) Note that both $\mathscr{T}_X$ and $\pi^*\mathscr{T}_Y$ are G-$\mathscr{O}_X$-modules (cf. [10, Section 5.1]) and that the tangent map $\mathscr{T}_X \to \pi^*\mathscr{T}_Y$ is a morphism of G-$\mathscr{O}_X$-modules; the G-structure of $\pi^*\mathscr{T}_Y = \mathscr{O}_X \bigotimes_{\mathscr{O}_Y} \pi^{-1}\mathscr{T}_Y$ is the tensor product of the usual G-structure on $\mathscr{O}_X$ and the trivial G-structure on $\pi^{-1}\mathscr{T}_Y$ (cf. [10, Section 5.1]). We apply the functor π_*^G to the tangent map; recall that, by definition, for a G-sheaf $\mathscr{F}$ on X and an open U of Y, $\pi_*^G(\mathscr{F})(U)$ is the set of sections of $\mathscr{F}$ on $\pi^{-1}(U)$ invariant under G. Since $\pi_*^G\pi^*\mathscr{T}_Y \cong \mathscr{T}_Y$ (cf. [10, (5.1.1)]), we get

$$0 \to \pi_*^G\mathscr{T}_X \longrightarrow \mathscr{T}_Y. \tag{1.2.1}$$

The fact that this map is a monomorphism is due to the left exactness of π_*^G. Note that $\pi_*^G\mathscr{T}_X$ is an invertible sheaf on Y; indeed, it can be described by formula (1.6.1).

1.3. Ordinary curves

From now on, we assume that the field k is of characteristic $p > 0$ and that the curve X is *ordinary*. This means that

$$\dim_{\mathbf{F}_p} \mathrm{Jac}(X)[p] = g.$$

The property of being ordinary is open and dense on the moduli space of curves of genus g (see [17, Section 4], [14]) since it is essentially a maximal rank condition on the Hasse-Witt matrix of the curve (hence open), and ordinary curves exist for all g.

PROPOSITION 1.4

Let X be an ordinary curve, let G be a finite group of automorphisms of X, let $\pi : X \to G\backslash X$, and let $P \in Y$ be a branch point of π. Then the ramification filtration for P stops at $G_2 = \{1\}$. The ramification group at P is of the form $(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$ for $n|p^t - 1$. Actually, letting $s := [\mathbf{F}_p(\zeta) : \mathbf{F}_p]$, where ζ is a primitive nth root of unity, one can consider $(\mathbf{Z}/p\mathbf{Z})^t$ as a vector space of dimension t/s over $\mathbf{F}_q$, where $q = p^s$, and the action of $\mathbf{Z}/n\mathbf{Z}$ is exactly by multiplication with ζ. Locally at P, the cover can be decomposed as a tower

$$k\big((x_{t/s})\big) - \cdots - k\big((x_2)\big) - k\big((x_1)\big) - k\big((x_0)\big),$$

where the first step is a Kummer extension of degree n ($x_0 = x_1^n$) and all others are Artin-Schreier extensions of degree q ($x_i^{-q} - x_i^{-1} = c_i x_{i-1}^{-1}$).

Proof

Let P be a ramified point in that cover, and let $G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ be the standard filtration on the ramification group G_0 at P. In [17, Theorem 2(i)], S. Nakajima has shown that for an ordinary curve, $G_2 = \{1\}$. It is also known that G_i/G_{i+1} for $i \geq 1$ are elementary abelian p-groups and that G_0 is the semidirect product of a group of order prime to p and G_1 with the action given above (see [19, Section IV.2, Corollaries 1–4]). The results follows from this and standard field theory. □

Remark 1.4.1
The following formula for s holds (see [15, Section 2.47(ii)]):

$$s = \min\{s' \in \mathbf{Z}_{>0} : n \mid p^{s'} - 1\}.$$

Remark 1.4.2
The calculations that follow actually depend only upon the special form of the ramification filtration given in Proposition 1.4, so they also apply to (not necessarily ordinary) curves for which the action of G is as described here.

Notation 1.5
We break up the branch points P of π, which we enumerate as $\{P_1, \ldots, P_n\}$, into two sets T and W; let $\mathbf{Z}_p^t \rtimes \mathbf{Z}_n$ be the ramification group at P. Then

$$P \in T \iff t = 0 \text{ or } (p = 2 \text{ and } t = 1),$$

and $P \in W$ otherwise. Note that for $p \neq 2$, points in T are called *tame* and points in W are called *wild*; but for $p = 2$ the distinction is more subtle. Observe that $p = 2$ and $t = 1$ implies $n = 1$. We define the divisor Δ on Y by

$$\Delta = \sum_{P \in T} P + 2 \sum_{Q \in W} Q \quad \text{of degree } \delta := \deg(\Delta) = |T| + 2|W|.$$

PROPOSITION 1.6
We have $\pi_*^G \mathscr{T}_X \cong \mathscr{T}_Y(-\Delta)$.

Proof
As in [1, proof of Proposition 5.3.2], we have

$$\pi_*^G \mathscr{T}_X \cong \mathscr{T}_Y \otimes \big[\mathscr{O}_Y \cap \pi_*\big(\mathscr{O}_X(-\mathfrak{r})\big)\big], \tag{1.6.1}$$

where $\mathfrak{r}$ is the (global) different of $k(X)/k(Y)$. The divisor $\mathfrak{r}$ is supported at branch points P. The exponent of the local different at P is $\sum_{i=0}^{\infty}(|G_i| - 1)$, where G_i are the higher ramification groups at P. Then $\mathfrak{r}_P = (np^t - 1 + p^t - 1) \cdot P$ follows from Proposition 1.4. Hence, upon intersecting with $\mathscr{O}_Y$ in (1.6.1), we get (recall that the cover is totally ramified)

$$\begin{aligned}\big[\mathscr{O}_Y \cap \pi_*\big(\mathscr{O}_X(-\mathfrak{r})\big)\big]_P &= \mathscr{O}\Big(-\Big\lceil 1 + \frac{p^t - 2}{np^t}\Big\rceil P\Big)\\ &= \begin{cases} \mathscr{O}(-P) & \text{if } t = 0 \text{ or } (p = 2, t = 1),\\ \mathscr{O}(-2P) & \text{otherwise,}\end{cases}\end{aligned}$$

where $\lceil x \rceil = \min\{n \in \mathbf{Z}_{>0} : n \geq x\}$. If we now collect the local terms, the result comes out. □

Remark 1.6.2
In [1, pp. 235–236], $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ have to be interchanged everywhere.

1.7. Cohomology of π_*^G

We now recall the two spectral sequences from [10, Section 5.2]. Let $\mathscr{F}$ be a coherent G-$\mathscr{O}_X$-module, and write

$$\begin{aligned} \mathrm{H}^i(X; G, \mathscr{F}) &= \mathrm{R}^i\Gamma_X^G\mathscr{F}, \\ \mathscr{H}^i(G, \mathscr{F}) &= \mathrm{R}^i\pi_*^G\mathscr{F}, \end{aligned}$$

where $\Gamma_X^G = \Gamma_Y \circ \pi_*^G$, that is, where $\Gamma_X^G\mathscr{F} = (\Gamma_X\mathscr{F})^G$. There are two cohomological spectral sequences

$$\begin{aligned} {}^{I}E_2^{p,q} &= \mathrm{H}^p\big(Y, \mathscr{H}^q(G, \mathscr{F})\big) &\Longrightarrow\quad & \mathrm{H}^{p+q}(X; G, \mathscr{F}), \\ {}^{II}E_2^{p,q} &= \mathrm{H}^p\big(G, \mathrm{H}^q(X, \mathscr{F})\big) &\Longrightarrow\quad & \mathrm{H}^{p+q}(X; G, \mathscr{F}). \end{aligned}$$

The first one gives rise to the edge sequence

$$0 \longrightarrow \mathrm{H}^1(Y, \pi_*^G\mathscr{F}) \longrightarrow \mathrm{H}^1(X; G, \mathscr{F}) \longrightarrow \mathrm{H}^0\big(Y, \mathscr{H}^1(G, \mathscr{F})\big), \longrightarrow 0 \tag{1.7.1}$$

as we are on a curve Y.

1.8. Tangent space to the global deformation functor

We can use this equivariant cohomology to compute the tangent space to our deformation functor. Recall that the ring of dual numbers is defined as $k[\epsilon] = k[E]/(E^2)$; clearly, it belongs to $\mathscr{C}_k$. The tangent space to the deformation functor $D_{X,\rho}$ is by definition its value on the ring of dual numbers with its natural k-linear structure (cf. [18]).

PROPOSITION 1.9 (see [1, (3.2.1), (3.3.1)])
We have $D_{X,\rho}(k[\epsilon]) \cong \mathrm{H}^1(X; G, \mathscr{T}_X)$.

1.10. Localization

We now describe localization for our deformation functor. Let $Q_j \in X$ be a point lying over P_j for $1 \le j \le n$, and let G_j be the stabilizer of Q_j, which acts on the local ring $\mathscr{O}_{X,Q_j}$ by k-algebra automorphisms; we denote it by $\rho_j : G_j \to \mathrm{Aut}_k(\mathscr{O}_{X,Q_j})$. Changing the choice of Q_j does not affect, up to a suitable equivalence, the action ρ_j. Every lifting $(X^\sim, \rho^\sim, \phi^\sim)$ induces a *lifting of the local representation* ρ_j for any $1 \le j \le n$ in the sense defined in Section 1.11.

1.11. The local deformation functor

Let R be a k-algebra, and let G be a group acting via $\rho : G \to \mathrm{Aut}_k(R)$ on R by k-algebra automorphisms. Let $A \in \mathscr{C}_k$, and let $R_A = R \otimes_k A$. Let $\overline{\cdot}$ denote the reduction map $R_A \to R$ modulo the maximal ideal m_A of A.

We define a lifting of ρ to A as a group homomorphism $\rho^{\sim} \colon G \to \mathrm{Aut}_A(R_A)$ such that $\overline{\rho^{\sim}} = \rho$. Two liftings $\rho^{\sim}$ and $\rho^{\approx}$ are said to be *isomorphic* (over ρ) if there exists $\psi \in \mathrm{Aut}_A(R_A)$ with $\overline{\psi} = \mathrm{Id}_R$ such that for any $\sigma \in G$, $\psi \circ \rho^{\approx}_{\sigma} = \rho^{\sim}_{\sigma} \circ \psi$. We arrive at a *local deformation functor*

$$D_{\rho} \colon \mathscr{C}_k \longrightarrow \mathsf{Set}$$

such that $D_{\rho}(A)$ is the set of all isomorphism classes of liftings of ρ to A. It is but a formal check using Schlessinger's criterion to see that D_{ρ} has a prorepresentable hull (which we denote by H_{ρ}) if G is a finite group (cf. [1, Théorème 2.2]).

In our situation we get a transformation of functors

$$D_{X,\rho} \longrightarrow D_{\rho_1} \times \cdots \times D_{\rho_n}. \tag{1.11.1}$$

It turns out that this morphism is formally smooth (see [1, Théorème 3.3.4]). Hence we get the following localization result.

PROPOSITION 1.12 ([1, Lemme 3.3.1 and Corollaire 3.3.5])
The functor $D_{X,\rho}$ has a prorepresentable hull $H_{X,\rho}$; in fact,

$$H_{X,\rho} \cong H_{\rho_1} \hat{\otimes} \cdots \hat{\otimes} H_{\rho_n}[[u_1, \ldots, u_N]],$$

where $N = \dim_k \mathrm{H}^1(Y, \pi_*^G \mathscr{T}_X)$.

Remark 1.12.1
As soon as $g \geq 2$, the curve X does not have infinitesimal automorphisms, so the functor $D_{X,\rho}$ is in fact *prorepresentable* by $H_{X,\rho}$ (cf. [1, Théorème 2.1]).

2. Lifting of group actions and group cohomology

The aim of Sections 2–4 is to compute the tangent space and the prorepresentable hull of the local deformation functors for the representations of the branch groups in quotients of ordinary curves as automorphisms of the local stalks of the tangent sheaf.

2.1. Action on derivations

Let k be a field, and let R be a k-algebra. We denote by $\mathscr{T}_R$ the R-module of k-derivations, that is, the set of all maps $\delta \colon R \to R$ such that $\delta(xy) = \delta(x)y + x\delta(y)$ for $x, y \in R$ and such that $\delta(a) = 0$ for $a \in k$. Let $\varphi \in \mathrm{Aut}_k(R)$ be an automorphism of R over k. Then it induces a k-module automorphism of $\mathscr{T}_R$, denoted by Ad_φ, by

$$\mathrm{Ad}_\varphi(\delta) = \varphi \circ \delta \circ \varphi^{-1}.$$

Note that Ad_φ is not an R-module automorphism but rather is equivariant; for $x \in R$ and $\delta \in \mathscr{T}_R$, we have $\mathrm{Ad}_\varphi(x\delta) = \varphi(x)\,\mathrm{Ad}_\varphi(\delta)$. Let G be a finite group, and suppose it acts on R by k-algebra automorphisms; that is, suppose a group homomorphism

$$\rho \colon G \to \mathrm{Aut}_k(R) \ : \ \sigma \mapsto \rho_\sigma$$

is given. The induced action of G on $\mathscr{T}_R$ by k-module automorphisms is denoted by

$$\mathrm{Ad}_\rho \ : \ \sigma \mapsto \mathrm{Ad}_{\rho,\sigma}\,.$$

2.2. Tangent space to the local deformation functor

Let $k[\epsilon]$ be the ring of dual numbers. Then $R[\epsilon] = R \otimes_k k[\epsilon]$ is the $k[\epsilon]$-algebra of elements $x + y\epsilon$ with $x, y \in R$. Let $\overline{\cdot} \ : R[\epsilon] \to R$ denote the reduction map modulo ϵ. In the following proposition, we make an explicit identification between the tangent space $D_\rho(k[\epsilon])$ to the deformation functor D_ρ and the first group cohomology with values in the derivations.

PROPOSITION 2.3

There exists a bijection (depending on the deformation parameter ϵ) $d \ : \ D_\rho(k[\epsilon]) \xrightarrow{\sim} \mathrm{H}^1(G, \mathscr{T}_R)$ described as follows: The 1*-cocycle $d_{\rho^\sim}$ associated to a lifting $\rho^\sim$ is given by the formula*

$$d_{\rho^\sim}\sigma = \frac{\rho^\sim_\sigma \circ \rho_\sigma^{-1} - \mathrm{Id}}{\epsilon} \quad \Big(= \frac{d}{d\epsilon}(\rho^\sim_\sigma \circ \rho_\sigma^{-1})|_{\epsilon=0}\Big)$$

for any $\sigma \in G$.

Proof

For $\sigma \in G$ we set $\rho^\sim_\sigma(x) = \rho_\sigma(x) + \rho'_\sigma(x)\epsilon$ ($x \in R$). Then for $x + y\epsilon \in R[\epsilon]$, we have

$$\rho^\sim_\sigma(x + y\epsilon) = \rho^\sim_\sigma(x) + \rho^\sim_\sigma(y)\epsilon = \rho_\sigma(x) + \big[\rho'_\sigma(x) + \rho_\sigma(y)\big]\epsilon;$$

that is, ρ'_σ determines the lifting $\rho^\sim$. The 1-cocycle $d_{\rho^\sim}$ is given by $d_{\rho^\sim}\sigma = \rho'_\sigma \circ \rho_\sigma^{-1}$. The following two formulas are straightforward:

(i) $\rho'_\sigma(xy) = \rho_\sigma(x)\rho'_\sigma(y) + \rho'_\sigma(x)\rho_\sigma(y)$ for $x, y \in R$,

(ii) $\rho'_{\sigma\tau} = \rho'_\sigma \circ \rho_\tau + \rho_\sigma \circ \rho'_\tau$ for $\sigma, \tau \in G$.

From (i) it follows that $d_{\rho^\sim}\sigma \in \mathscr{T}_R$, and from (ii) it follows that $d_{\rho^\sim}$ is a cocycle; that is, $d_{\rho^\sim}\sigma\tau = d_{\rho^\sim}\sigma + \mathrm{Ad}_{\rho,\sigma}(d_{\rho^\sim}\tau)$.

Suppose that two liftings $\rho^{\sim}$ and $\rho^{\approx}$ are isomorphic by $\psi \in \mathrm{Aut}_{k[\epsilon]}(R[\epsilon])$. By the condition $\overline{\psi} = \mathrm{Id}_R$, we can write $\psi(x) = x + \delta(x)\epsilon$ ($x \in R$), where $\delta \in \mathscr{T}_R$. The equality $\psi \circ \rho_\sigma^{\approx} = \rho_\sigma^{\sim} \circ \psi$ implies

$$\rho_\sigma'' - \rho_\sigma' = \rho_\sigma \circ \delta - \delta \circ \rho_\sigma, \tag{2.3.1}$$

where ρ_σ'' is the ϵ-part in $\rho^{\approx}$ (as ρ_σ' was the ϵ-part of $\rho^{\sim}$). Hence

$$d_{\rho^{\approx}}\sigma - d_{\rho^{\sim}}\sigma = \mathrm{Ad}_{\rho,\sigma}(\delta) - \delta, \tag{2.3.2}$$

which implies that $(d_{\rho^{\approx}}) - (d_{\rho^{\sim}})$ is a coboundary. Conversely, if we have (2.3.2) for some $\delta \in \mathscr{T}_R$, then one can define $\psi \in \mathrm{Aut}_{k[\epsilon]}(R[\epsilon])$ by the obvious formula $\psi(x) = x + \delta(x)\epsilon$ (or, equivalently, $\psi(x + y\epsilon) = x + [y + \delta(x)]\epsilon$), which gives an isomorphism between the liftings $\rho^{\sim}$ and $\rho^{\approx}$. Therefore the map d is well defined and is injective.

We now show surjectivity. A given cocycle $d : G \to \mathscr{T}_R$ induces an automorphism $\rho_\sigma^{\sim}$ for any $\sigma \in G$ by the formula $\rho_\sigma^{\sim}(x) = \rho_\sigma(x) + (d\sigma) \circ \rho_\sigma(x)\epsilon$ for $x \in R$. One can easily check that $\sigma \mapsto \rho_\sigma^{\sim}$ gives a lifting of ρ whose associated 1-cocycle is exactly d. □

3. Computation of group cohomology

3.1

In this section, k denotes a field of characteristic $p > 0$, and $\mathscr{O}$ denotes a discrete valuation ring over k with the residue field k. We fix a regular parameter x for $\mathscr{O}$. The $\mathscr{O}$-module $\mathscr{T}_{\mathscr{O}}$ (defined in (2.1)) is free of rank 1 since every $\delta \in \mathscr{T}_{\mathscr{O}}$ is determined by $\delta(x)$. Let $\frac{d}{dx}$ be the unique k-derivation such that $\frac{d}{dx}(x) = 1$. Then $\mathscr{T}_{\mathscr{O}} = \mathscr{O}\frac{d}{dx}$. Let a group G act on $\mathscr{O}$ by k-algebra automorphisms. In this section we write the action of ρ exponentially ($f \to f^\sigma$ for $f \in \mathscr{O}$ and $\sigma \in G$), omitting ρ from the notation—we do the same for the induced action Ad on the tangent space $\mathscr{T}_{\mathscr{O}}$. The G-equivariancy condition now becomes $(f\delta)^\sigma = f^\sigma\delta^\sigma$ for $f \in \mathscr{O}$. In this section we compute the group cohomology $\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}})$ in the following three situations, which are exactly the ones that arise for the local action of the ramification group of a branch point on an ordinary curve.

Case 1. We have the tame case, so $G = \langle\tau\rangle \cong \mathbf{Z}/n\mathbf{Z}$ with $(n, p) = 1$ acting on $\mathscr{O}$ by $x^\tau = \zeta x$, where ζ is a primitive nth root of unity.

Case 2. We have the p-group case, so $G = \prod_{i=1}^t \langle\sigma_i\rangle \cong (\mathbf{Z}/p\mathbf{Z})^t$ acting on $\mathscr{O}$ by $x^{\sigma_i} = x/(1 - u_i x)$, where $u_1, \ldots, u_t \in k$ are linearly independent over $\mathbf{F}_p$. Let V be the t-dimensional $\mathbf{F}_p$-vector subspace in k spanned by $u_1, \ldots, u_t$. The group G and its action are isomorphic to the vector group V, acting on $\mathscr{O}$ by $x^u = x/(1-ux)$ for $u \in V$; by this, we can (and do) pretend that $G = V$.

Case 3. We have the mixed case, so $G = N \rtimes H$, where $N = \prod_{i=1}^{t} \langle \sigma_i \rangle \cong (\mathbf{Z}/p\mathbf{Z})^t$ and $H = \langle \tau \rangle \cong \mathbf{Z}/n\mathbf{Z}$ with $n > 1$ and $n | p^t - 1$. Let ζ be a primitive nth root of unity in k, and let $s = [\mathbf{F}_p(\zeta) : \mathbf{F}_p]$ and $q = p^s$ as in Proposition 1.4. Similarly to the previous case, we consider N as as vector group V of dimension t/s over $\mathbf{F}_q$ acting on $\mathscr{O}$ by $x^u = x/(1-ux)$ for $u \in V$, and the action of H is scalar multiplication by $\zeta \in \mathbf{F}_q^*$ on x.

3.2. Case 1

Since the G-module $\mathscr{T}_{\mathscr{O}}$ is killed by p but p is prime to the order of G, all higher group cohomology vanishes: $\mathrm{H}^n(G, \mathscr{T}_{\mathscr{O}}) = 0$ for $n > 0$ (see [4, Section III, Corollary 10.2]).

3.3. Case 2

Since $\left(\frac{d}{dx}\right)^u = (1-ux)^2 \frac{d}{dx}$, the G-module $\mathscr{T}_{\mathscr{O}}$ is isomorphic to $\mathscr{O}$ with the G-action given by

$$f^u(x) = f\left(\frac{x}{1-ux}\right)(1-ux)^2 \tag{3.3.1}$$

for $f \in \mathscr{O}, u \in V$.

LEMMA 3.4

The G-module $\mathscr{O}$ with G-action (3.3.1) is isomorphic to $M \oplus x\mathscr{O}$, where $M = k \oplus kx \oplus kx^2$ such that we have the following.

(1) *The G-module structure of M is given by*

$$(a_0 + a_1x + a_2x^2)^u = a_0 + (a_1 - 2ua_0)x + (a_2 - ua_1 + u^2a_0)x^2;$$

that is, with respect to the basis $\{1, x, x^2\}$,

$$u \longleftrightarrow \Phi(u) = \begin{pmatrix} 1 & 0 & 0 \\ -2u & 1 & 0 \\ u^2 & -u & 1 \end{pmatrix}.$$

(2) *The G-module structure on the second factor $x\mathscr{O}$ is the original G-action on $\mathscr{O}$; that is, $f^u(x) = f(x/(1-ux))$ for $f(x) \in x\mathscr{O}$.*

Proof

Note that $\mathscr{O} = M \oplus x^3\mathscr{O}$ is a G-stable direct decomposition. The action of G on the first factor is as stated in (1). For $x^2 f(x) \in x^3\mathscr{O}$ (i.e., $f(x) \in x\mathscr{O}$), the action (3.3.1) gives $(x^2 f(x))^u = x^2 f(x/(1-ux))$. □

3.5

Let G act on $x\mathscr{O}$ as in Lemma 3.4(2), and let $F = \mathrm{Frac}(\mathscr{O})$ be the fraction field of $\mathscr{O}$. Since this action comes from that on $\mathscr{O}$ by ring automorphisms, it extends to F, respecting the decomposition $F = x\mathscr{O} \oplus k[x^{-1}]$. We have

$$\mathrm{H}^1(G, x\mathscr{O}) \oplus \mathrm{H}^1(G, k[x^{-1}]) = \mathrm{H}^1(G, F) = 0$$

by standard facts (see [19, Section X.1]), and hence $\mathrm{H}^1(G, x\mathscr{O}) = 0$. Thus $\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong \mathrm{H}^1(G, M)$.

We now compute $\mathrm{H}^1(G, M)$. Since G is a commutative p-group, the condition for a map $d : V \to M$ to be a cocycle ($d(u+v) = du + (dv)^u$) implies

$$\begin{cases} d(pu) = 0 \iff du + (du)^u + (du)^{2u} + \cdots + (du)^{(p-1)u} = 0, & \text{(i)} \\ d(u+v) = d(v+u) \iff du + dv^u = dv + du^v. & \text{(ii)} \end{cases}$$

Let us write a cocycle d as $du = a_0(u) + a_1(u)x + a_2(u)x^2$ for $u \in V$.

3.5.1

Calculating the matrix $1 + \Phi(u) + \Phi(2u) + \cdots + \Phi((p-1)u)$, we deduce that condition (i) is

(a) empty unless $p = 2$ or 3,

(b) equivalent to $a_0(u) = 0$ if $p = 3$,

(c) equivalent to $ua_0(u) + a_1(u) = 0$ if $p = 2$.

3.5.2

Condition (ii) is equivalent to $2ua_0(u) = 2va_0(v)$ and $u^2a_0(v) - ua_1(v) = v^2a_0(u) - va_1(u)$. Hence we have the following.

(a) If $p \neq 2$, (ii) is equivalent to $a_0(u) = ua_0$ and $a_1(u) = u(a_1 - ua_0)$, where a_0 and a_1 are constants independent of u.

(b) If $p = 2$, (ii) together with (i) is equivalent to $a_0(u) = ua_0$ and $a_1(u) = u^2a_0$, where a_0 is a constant independent of u.

Thus we get

$$\mathrm{Z}^1(G, M) = \begin{cases} \left\{du = ua_0 + u(a_1 - ua_0)x + a_2(u)x^2\right\} & (p \neq 2, 3), \\ \left\{du = ua_1x + a_2(u)x^2\right\} & (p = 3), \\ \left\{du = ua_0 + u^2a_0x + a_2(u)x^2\right\} & (p = 2). \end{cases}$$

In each case, the cocycle condition is equivalent to the fact that the function a_2 satisfies

$$a_2(u+v) = a_2(u) + a_2(v) + uv\big[(u+v)a_0 - a_1\big]$$

(with $a_0 = 0$ if $p = 3$ and $a_1 = 0$ if $p = 2$). In particular, whenever $a_0 = a_1 = 0$, the function a_2 is $\mathbf{F}_p$-linear. Hence $\mathrm{Z}^1(G, M)$ contains the t-dimensional k-subspace $\mathrm{Hom}_{\mathbf{F}_p}(V, k) = V^* \otimes k$. The dimension over k of $\mathrm{Z}^1(G, M)$ therefore is $t + 2$ if $p \neq 2, 3$ or is $t + 1$ otherwise.

3.6

Let $g = b_0 + b_1 x + b_2 x^2$. Then a coboundary is of the form

$$g^u - g = -2ub_0 x + (-ub_1 + u^2 b_0)x^2.$$

We can then compute a k-basis for $\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}})$ as follows.

3.6.1

If $p \neq 2, 3$, a nontrivial such cohomology class, $[d_0]$, is given by the cocycle d_0 with

$$d_0 u = -u + (u^2 + u)x - \Big(\frac{1}{3}u^3 + \frac{1}{2}u^2 + \frac{1}{6}u\Big)x^2.$$

The other cohomology classes come from the subspace $\{a_0 = a_1 = 0\} \cong \mathrm{Hom}_{\mathbf{F}_p}(V, k)$ in $\mathrm{Z}^1(G, M)$, in which the coboundary classes are spanned by $du = ux^2$. Hence the part of the cohomology coming from $\{a_0 = a_1 = 0\}$ is isomorphic to

$$\mathrm{Hom}_{\mathbf{F}_p}(V, k)/k \cdot \iota,$$

where $\iota\colon V \hookrightarrow k$ is the natural inclusion. Hence

$$\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong k \cdot [d_0] \oplus \mathrm{Hom}_{\mathbf{F}_p}(V, k)/k \cdot \iota.$$

In particular, $\dim_k \mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) = t$.

3.6.2

If $p = 3$, then only the cocycles coming from $\{a_0 = a_1 = 0\}$ survive. Hence

$$\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong \mathrm{Hom}_{\mathbf{F}_3}(V, k)/k \cdot \iota,$$

and we have $\dim_k \mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) = t - 1$.

3.6.3

If $p = 2$, we use an ad hoc construction. Let $\{u_1, \ldots, u_t\}$ be a basis of V, define a cocycle $\widetilde{d}_0$ by $\widetilde{d}_0(u_i) := u_i - u_i^2 x$ on basis elements, and use $\widetilde{d}_0(u+v) := \widetilde{d}_0 u + (\widetilde{d}_0 v)^u$ inductively to define $\widetilde{d}_0 u$ for all u. Notice that this requirement is compatible with conditions (i) ($\widetilde{d}_0(2u) = 0$) and (ii) (commutativity) from Section 3.5. Thus we get a well-defined cocycle on all of V since $p = 2$.

The remaining part, $\{a_0 = a_1 = 0\} \cong \mathrm{Hom}_{\mathbf{F}_2}(V, k)$, contains two-dimensional coboundaries spanned by ι and the Frobenius embedding

$$\mathrm{Frob}\colon V \hookrightarrow k \;:\; u \mapsto u^2.$$

Hence

$$\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong k \cdot [\widetilde{d}_0] \oplus \mathrm{Hom}_{\mathbf{F}_2}(V, k)/(k \cdot \iota + k \cdot \mathrm{Frob}),$$

and, as the sum $k \cdot \iota + k \cdot \mathrm{Frob}$ is direct precisely when $t > 1$, we get that $\dim_k \mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}})$ is $t - 1$ if $t > 1$ or is 1 if $t = 1$.

3.7. Case 3

Since $|H|$ is coprime to the characteristic of k, we have

$$\mathrm{H}^1(G, M) = \mathrm{H}^1(N, M)^H \tag{3.7.1}$$

(see, e.g., [4, Section III, Proposition 10.4]). The calculation of $\mathrm{H}^1(N, M)$ is similar to the one in Sections 3.5 and 3.6, but $\mathbf{F}_p$ is replaced by $\mathbf{F}_q$ (for $q = p^s$) everywhere. The only difference is for $p = 2$ because then the subspace $\mathrm{Hom}_{\mathbf{F}_q}(V, k)$ in $\mathrm{Hom}_{\mathbf{F}_p}(V, k)$ has a trivial intersection with the one-dimensional subspace spanned by the Frobenius embedding $u \mapsto u^2$.

We now describe the action of H on the cohomology $\mathrm{H}^1(N, M)$. First, recall that the action of H on $\mathscr{T}_{\mathscr{O}}$ is given by

$$\left(x^r \frac{d}{dx}\right)^{\tau} = \zeta^{r-1} x^r \frac{d}{dx}.$$

It stabilizes M, on which it acts by

$$(a_0 + a_1 x + a_2 x^2)^{\tau} = \zeta^{-1} a_0 + a_1 x + \zeta a_2 x^2.$$

The action of H on the cohomology $\mathrm{H}^1(N, M)$ is induced from the action on the space of cocycles given by $d^{\tau} u = (d u^{\tau})^{\tau^{-1}}$ (cf. [4, Section III.8]). Hence if $p \neq 2$, we get $d_0^{\tau} u = (d_0 \zeta u)^{\tau^{-1}} = \zeta^2 d_0 u + (\zeta - \zeta^2) u x - ((1/2)(\zeta - \zeta^2) u^2 + (1/6)(1 - \zeta^2) u) x^2$, where the last two terms form a coboundary; thus

$$d_0^{\tau} = \zeta^2 d_0,$$

and if $p = 2$, a similar result holds for the cocycle $\widetilde{d}_0$ introduced in Section 3.6.3. This implies that the class d_0 is not H-invariant as long as $n \neq 2$, and $\widetilde{d}_0$ is not H-invariant at all. Next we look at the remaining part $\{a_0 = a_1 = 0\} \cong \mathrm{Hom}_{\mathbf{F}_q}(V, k)$. An element $a_2 \in \mathrm{Hom}_{\mathbf{F}_q}(V, k)$ (corresponding to the cocycle $du = a_2(u) x^2$) is H-invariant if and only if $d^{\tau} u = \zeta^{-1} a_2(\zeta u) x^2 = a_2(u) x^2$ or, equivalently, $a_2(\zeta u) = \zeta a_2(u)$. Since ζ generates $\mathbf{F}_q$, it is equivalent to the $\mathbf{F}_q$-linearity of a_2. Summing up, we have the following.

3.7.2
Suppose $p \neq 2, 3$. If $n \neq 2$, then

$$\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong \mathrm{H}^1(N, M)^H \cong \mathrm{Hom}_{\mathbf{F}_q}(V, k)/k \cdot \iota$$

and hence is of dimension $t/s - 1$. If $n = 2$ (which implies $s = 1$), it is of dimension t since d_0 is also H-invariant.

3.7.3
If $p = 3$, then we always have

$$\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong \mathrm{H}^1(N, M)^H \cong \mathrm{Hom}_{\mathbf{F}_q}(V, k)/k \cdot \iota,$$

and $\dim_k \mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) = t/s - 1$.

3.7.4
If $p = 2$, then $n \neq 2$ and $s > 1$ (since $1 \neq n | 2^s - 1$). Hence

$$\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) \cong \mathrm{H}^1(N, M)^H \cong \mathrm{Hom}_{\mathbf{F}_q}(V, k)/k \cdot \iota$$

(recall that the Frobenius embedding is no longer present), and $\dim_k \mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}}) = t/s - 1$.

Remark 3.8
Fix a subgroup $V_1 = (\mathbf{Z}/p\mathbf{Z})$ of V, and consider the inflation restriction sequence for this subgroup (with $V' := V/V_1$):

$$0 \to H^1(V', M^{V_1}) \xrightarrow{\mathrm{inf}} H^1(V, M) \xrightarrow{\mathrm{res}} H^1(V_1, M)^{V'}.$$

The left cohomology group is the following (for $p \neq 2$): The invariants M^{V_1} are just kx^2, on which V' acts trivially; so the group is isomorphic to $\mathrm{Hom}(V', k)$, which maps via inflation to the part $\mathrm{Hom}_{\mathbf{F}_p}(V, k)/k \cdot \iota$ of the cohomology $H^1(G, \mathscr{T}_{\mathscr{O}})$. On the other hand, the part $[d_0]$ is precisely the one that is nonzero when mapped under restriction to $H^1(V_1, M) = H^1(\mathbf{Z}/p\mathbf{Z}, \mathscr{T}_{\mathscr{O}})$. This group is the one studied in [1] (see, e.g., [1, Lemme 4.2.2]), where an intrinsic characterization of the class $[d_0]$ is provided.

4. The local prorepresentable hull

In this section we calculate the hull H_ρ of D_ρ, where the group G and its action on $R = \mathscr{O}$ are as in Cases 1, 2, and 3.

The first one is a trivial case; that is, the tame cyclic action is rigid (cf. [1, Section 4.3]).

To analyze Case 2, we have to look at lifting obstructions, much as is the case in [1, Section 4.2]. In our case, liftings are related to what we call formal truncated Chebyshev polynomials, which we now introduce.

4.1. Formal truncated Chebyshev polynomials

For a positive integer N, we set

$$M_{\alpha,\beta}^{[N]}(u)=\begin{bmatrix} \sum_{k=0}^{N}\binom{u+k-1}{2k}\alpha^k & \alpha\sum_{k=0}^{N-1}\binom{u+k}{2k+1}\alpha^k \\ \sum_{k=0}^{N-1}\binom{u+k}{2k+1}\alpha^k+\beta(u) & \sum_{k=0}^{N}\binom{u+k}{2k}\alpha^k \end{bmatrix},$$

where u, α, and $\beta(u)$ are indeterminates. The entries are considered to be formal power series of these indeterminates with coefficients in $\mathbf{Q}$. (This means in particular that the binomial coefficients evaluate to polynomials in u.) We denote by $M_{\alpha}^{[N]}(u)$ $(1\le N\le\infty)$ the matrix $M_{\alpha,\beta}^{[N]}(u)$ with $\beta(u)$ replaced by 0.

As a matrix of formal power series, we have an identity

$$M_{\alpha}^{[\infty]}(u)=\begin{bmatrix} S_u(1+\frac{\alpha}{2})-(1+\alpha)S_{u-1}(1+\frac{\alpha}{2}) & \alpha S_{u-1}(1+\frac{\alpha}{2}) \\ S_{u-1}(1+\frac{\alpha}{2}) & S_u(1+\frac{\alpha}{2})-S_{u-1}(1+\frac{\alpha}{2}) \end{bmatrix},$$

where $S_u(x)$ is the Chebyshev polynomial of second kind,

$$S_u(x)=(u+1)_2F_1\Big(-u,u+2,\frac{3}{2};-\frac{x}{4}\Big).$$

Recall that if $u\in\mathbf{Z}_{\ge 0}$, then $S_u(x)$ is defined by the generating series

$$\frac{1}{1-2xr+r^2}=\sum_{u=0}^{\infty}S_u(x)r^u$$

or by

$$S_u(x)=\frac{\sin(u+1)\theta}{\sin\theta}.$$

The above identity of matrices follows from the following easy fact:

$$S_u\Big(1+\frac{x}{2}\Big)=\sum_{k=0}^{\infty}\binom{u+k+1}{2k+1}x^k.$$

For $1\le N\le\infty$, let us write

$$M_{\alpha}^{[N]}(u)=\begin{bmatrix} A_{\alpha}^{[N]}(u) & B_{\alpha}^{[N]}(u) \\ C_{\alpha}^{[N]}(u) & D_{\alpha}^{[N]}(u) \end{bmatrix}.$$

Then it follows easily from the basic recursion between binomial coefficients that

$$B_{\alpha}^{[N]}(u)=\alpha C_{\alpha}^{[N]}(u) \qquad \text{and} \qquad A_{\alpha}^{[N]}(u)+B_{\alpha}^{[N]}(u)=D_{\alpha}^{[N]}(u);$$

that is, the relations between the coefficients that hold for $N=\infty$ are also true for the truncated versions. It follows formally from these identities between entries that

$$M_\alpha^{[N]}(u)M_\alpha^{[N]}(v)=M_\alpha^{[N]}(v)M_\alpha^{[N]}(u) \tag{4.1.1}$$

as identities of matrices of formal power series in u, v, and α.

Next, we observe the following trigonometric identities:

(i) $S_{u+v}(x)+S_{u-1}(x)S_{v-1}(x)=S_u(x)S_v(x)$,
(ii) $S_{u+v-1}(x)+2xS_{u-1}(x)S_{v-1}(x)=S_{u-1}(x)S_v(x)+S_u(x)S_{v-1}(x)$,
(iii) $S_u(x)^2-2xS_u(x)S_{u-1}(x)+S_{u-1}(x)^2=1$.

The first two of these imply

$$M_\alpha^{[\infty]}(u)M_\alpha^{[\infty]}(v)=M_\alpha^{[\infty]}(u+v), \tag{4.1.2}$$

and the third one implies

$$\det M_\alpha^{[\infty]}(u)=1. \tag{4.1.3}$$

4.2. Lifting obstructions

Now assume $p\neq 2$, and let A be an artinian local k algebra with $A/\mathfrak{m}_A=k$. Let $\alpha\in\mathfrak{m}_A$. Let $V\subset k$ be a finite-dimensional $\mathbf{F}_p$-vector space, and let $\beta\colon V\to\mathfrak{m}_A$ be an $\mathbf{F}_p$-linear map. For $u\in V\hookrightarrow k$ we use the notation

$$\widetilde{M}_{\alpha,\beta}(u)=M_{\alpha,\beta}^{[(p-1)/2]}(u) \qquad \text{and} \qquad \widetilde{M}_\alpha(u)=M_\alpha^{[(p-1)/2]}(u)$$

(where multiplication of u's takes place inside k). By (4.1.1), we have

$$\widetilde{M}_\alpha(u)\widetilde{M}_\alpha(v)=\widetilde{M}_\alpha(v)\widetilde{M}_\alpha(u). \tag{4.2.1}$$

Also, by (4.1.2) and (4.1.3), we get

$$\widetilde{M}_\alpha(u)\widetilde{M}_\alpha(v)\equiv\widetilde{M}_\alpha(u+v) \bmod (\alpha^{(p-1)/2}) \tag{4.2.2}$$

and

$$\det\widetilde{M}_\alpha(u)\equiv 1 \bmod (\alpha^{(p-1)/2+1}). \tag{4.2.3}$$

We now look at what happens if $\beta\not\equiv 0$. A small calculation shows that the commutation relation $\widetilde{M}_{\alpha,\beta}(u)\widetilde{M}_{\alpha,\beta}(v)=\widetilde{M}_{\alpha,\beta}(v)\widetilde{M}_{\alpha,\beta}(u)$ is equivalent to $\alpha\beta(u)C(v)=\alpha\beta(v)C(u)$ (where $C=C_\alpha$ depends on α). Putting $v=1$ (where we tacitly assume to have conjugated the action of V such that $\mathbf{F}_p\subseteq V$ (see [1, Lemme 4.2.1])), we get

$$\alpha\beta(u)C(1)=\alpha\beta(1)C(u).$$

Thus $\alpha\beta(1)C(u)$ is a linear map in u. Looking at the constant term using the explicit form of $C(u)(=1+\binom{u+1}{2}\alpha+\cdots)$, we get $\alpha\beta(1)=0$. If we substitute this back into the previous display, we find $\alpha\beta(u)C(1) = 0$, but since $C(1) = 1 + \alpha$ is invertible in A (recall that $\alpha \in \mathfrak{m}_A$), we finally get $\alpha\beta(u) = 0$ for all u. Conversely, if $\alpha\beta = 0$, the commutation relation clearly holds. Hence

$$\widetilde{M}_{\alpha,\beta}(u)\widetilde{M}_{\alpha,\beta}(v) = \widetilde{M}_{\alpha,\beta}(v)\widetilde{M}_{\alpha,\beta}(u) \iff \alpha\beta = 0. \tag{4.2.4}$$

Looking at (4.2.3), we get the following.

$$\text{If } \alpha\beta = 0, \text{ then } \det \widetilde{M}_{\alpha,\beta}(u) \equiv 1 \bmod (\alpha^{(p+1)/2}). \tag{4.2.5}$$

LEMMA 4.2.6

The following conditions are equivalent:

(i) *the formula*

$$x^u := \frac{ax+b}{cx+d}, \quad \textit{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \widetilde{M}_{\alpha,\beta}(-u), \tag{4.2.6.1}$$

defines a lift of the action of V on $\mathscr{T}_{\mathscr{O}}$ to A;

(ii) $\widetilde{M}_{\alpha,\beta}(u)\widetilde{M}_{\alpha,\beta}(v) = \gamma\widetilde{M}_{\alpha,\beta}(u+v)$ *for all $u, v \in V$ and a $\gamma \in A$ (possibly depending on u, v, α, β);*

(iii) $\alpha^{(p-1)/2} = \alpha\beta = 0$.

Actually, if this holds, we have $\gamma = 1$.

Proof

Looking at the relations in V, we see the equivalence of (i) and (ii).

(ii) $\Longrightarrow$ (iii). The relation $\alpha\beta = 0$ follows from (4.2.4). Since the identity in (i) obviously holds with $\gamma = 1$ if $A = k$, we can set $\gamma = 1 + \delta$ with $\delta \in \mathfrak{m}_A$. Taking determinants, it follows from (4.2.3) that $\gamma^2 (= 1 + \delta(2+\delta)) = 1 + P\alpha^{(p+1)/2}$ for some $P \in A$. Since $\delta + 2$ is invertible in A,

$$\gamma \equiv 1 \bmod \alpha^{N+1} \tag{4.2.6.2}$$

for $N = (p-1)/2$. The coefficient of $\alpha^{(p-1)/2}$ in the lower-left entry of $\widetilde{M}_{\alpha,\beta}(u)\widetilde{M}_{\alpha,\beta}(v)$ is

$$\sum_{k=0}^{N-1} \binom{u+k}{2k+1}\binom{v+N-k-1}{2(N-k)} + \sum_{k=0}^{N-1} \binom{v+k}{2k+1}\binom{u+N-k}{2(N-k)}. \tag{4.2.6.3}$$

(Note that we have used the fact that we know already that $\alpha\beta = 0$; so there is no contribution from β in this calculation.) We now put $u = N$ and $v = 2$ (again assuming tacitly that $\mathbf{F}_p \subseteq V$) into (4.2.6.3). The result is 1, so the above coefficient (4.2.6.3) is nonzero. Since the coefficient of $\alpha^{(p-1)/2}$ in the lower-left entry of $\widetilde{M}_{\alpha,\beta}(u+v)$ is zero, we conclude from this and (4.2.6.2) that $\alpha^N = 0$, as desired.

(iii) $\Longrightarrow$ (ii). If $\beta \equiv 0$, this is already in (4.2.2); so let $\beta \not\equiv 0$. Using $\alpha\beta = 0$ and $A(u) \equiv 1 \bmod \alpha$, we calculate the entries of the matrix $M := \widetilde{M}_{\alpha,\beta}(u)\widetilde{M}_{\alpha,\beta}(v)$ as follows:

$$\begin{aligned}
M_{1,1} &= A(u)A(v) + \alpha C(u)C(v),\\
M_{1,2} &= \alpha A(u)C(v) + \alpha A(v)C(u) + \alpha^2 C(u)C(v),\\
M_{2,1} &= A(v)C(u) + A(u)C(v) + \alpha C(u)C(v) + \beta(u+v),\\
M_{2,2} &= A(u)A(v) + \alpha A(u)C(v) + \alpha A(v)C(u) + \alpha(1+\alpha)C(u)C(v).
\end{aligned}$$

Note that, except for the third one, these entries are independent of β. Since we already have (4.2.2), we find that these entries are equal to $A(u+v)$, $\alpha C(u+v)$, $C(u+v) + \beta(u+v)$, and $A(u+v) + \alpha C(u+v)$, respectively. Thus we have the relation in (i) with $\gamma = 1$. □

Remarks 4.2.6.4
Notice the sign change from u to $-u$ in (i) since we are lifting $x^u = x/(1-x)$.

We see that the condition $\alpha^{(p-1)/2} = 0$ corresponds to the order-p condition on the lift (as in [1]), whereas $\alpha\beta = 0$ gives the obstruction to lifting the commutativity of V.

4.3. Explicit infinitesimal liftings

We continue for the time being to assume that $p \neq 2$. At the infinitesimal level $A = k[\epsilon]$ with $\epsilon^2 = 0$, we let $\alpha = a_0\epsilon$ and $\beta(u) = -\epsilon \cdot \phi(u)$ for a linear map $\phi \in \operatorname{Hom}_{\mathbf{F}_p}(V,k)$. Lemma 4.2.6 assures us of the fact that (4.2.6.1) does define a first-order lifting as long as $\alpha = 0$ for $p = 3$. It is explicitly given as

$$x^u = \frac{(1 + (1/2)u(u+1)a_0\epsilon)x - ua_0\epsilon}{1 - (u + (1/6)u(u^2-1)a_0\epsilon - \epsilon\phi(u))x + (1/2)u(u-1)a_0\epsilon}.$$

By the formula in Proposition 2.3, the corresponding cocycle d is given as

$$\begin{aligned}
du &= \frac{d}{d\epsilon}\Big(\frac{x^u}{1+ux^u}\Big)\Big|_{\epsilon=0}\\
&= -a_0u + a_0(u^2+u)x - \Big(a_0\Big(\frac{1}{3}u^3 + \frac{1}{2}u^2 + \frac{1}{6}u\Big) - \phi(u)\Big)x^2\\
&= a_0d_0 + \phi
\end{aligned}$$

in $Z^1(V,M) = kd_0 + \operatorname{Hom}_{\mathbf{F}_p}(V,k)$, where d_0 is as in Section 3.6.1. This means that (4.2.6.1) for $\alpha = 0$ defines a lifting in the direction of ϕ (which is unobstructed) and for $\beta = 0$ in the direction of $a_0[d_0]$ (obstructed by $\alpha^{(p-1)/2} = 0$). If both α and β are nonzero, a lifting in the direction of $a_0[d_0] + \phi$ is obstructed by the equations in Lemma 4.2.6.

4.4. Calculating the hull

4.4.1. The case $p \neq 2, 3$

If we let R be the ring

$$R = k[[x_0, x_1, \ldots, x_t]]/\langle x_0^{(p-1)/2}, x_0x_1, x_0x_2, \ldots, x_0x_t, x_1 + \cdots + x_t\rangle,$$

then Lemma 4.2.6 shows that (4.2.6.1) defines a lifting of ρ to R, and hence there is a morphism of functors

$$\mathrm{Hom}(R, -) \to D_\rho.$$

To prove that R is actually the hull H_ρ of D_ρ, we argue as in [1, p. 217]. It suffices to prove that R is a versal deformation, that is, that the above morphism is smooth and is an isomorphism on the level of tangent spaces. The latter is clear from our computation of the tangent space to D_ρ from group cohomology and the above explicit form of R. To prove the former, let $A' \to A$ be a small extension in $\mathscr{C}_k$ with kernel I. We have to show that

$$\mathrm{Hom}(R, A') \to D(A') \times_{D(A)} \mathrm{Hom}(R, A)$$

is surjective. So assume that $\rho^{\sim} \in \mathrm{im}(\mathrm{Hom}(R, A) \to D(A))$ lifts to $\rho^{\approx}$ in $D(A')$. This means the corresponding obstruction in $\mathrm{H}^2(G, \mathscr{T}_{\mathscr{O}}) \otimes I$ is zero. Since $G = (\mathbf{Z}/p\mathbf{Z})^t$, this obstruction measures exactly the possible failure of the commutation relation $\rho^{\approx}(u+v) = \rho^{\approx}(u)\rho^{\approx}(v),\ \forall u, v \in V$, which we know by Lemma 4.2.6 is given by the equations in R. So we can lift via $\widetilde{M}_{\alpha',\beta'}(u)$ to $\mathrm{Hom}(R, A')$, and we can adjust this lifting in such a way that its image in $D(A')$ coincides with the given one since the tangent spaces to the two functors are isomorphic. (Note that the fibers are $H^1(G, \mathscr{T}_{\mathscr{O}}) \otimes I$-torsors.) This finishes the proof of smoothness.

4.4.2. The case $p = 3$

The same argument works, except that the class $[d_0]$ does not occur so that all liftings in the direction of $\mathrm{Hom}_{\mathbf{F}_p}(V, k)$ are unobstructed. The result is

$$H_\rho = k[[x_1, \ldots, x_t]]/\langle x_1 + \cdots + x_t\rangle.$$

4.4.3. The case $p = 2$

We deal with this case by an ad hoc construction. Recall that in this case $G \cong V \subseteq k$ is a t-dimensional $\mathbf{F}_2$-vector space for which we pick a basis $\{u_1, \ldots, u_t\}$. As before, let $\alpha \in A$, and let $\beta : V \to \mathfrak{m}_A$ be a linear map. We lift the action of the *basis* u_i by

$$x^{u_i} := \frac{x + \alpha u_i}{(u_i + \beta(u_i))x + 1}.$$

We have $(x^{u_i})^{u_i} = x$, and we observe that $(x^{u_i})^{u_j} = (x^{u_j})^{u_i}$ is equivalent to

$$\alpha u_i \beta(u_j) = \alpha u_j \beta(u_i) \tag{4.4.4}$$

for all i, j. We now *define* a lift to any element of V by

$$x^{u_{i_1}+\cdots+u_{i_l}} := x^{u_{i_1}\circ\cdots\circ u_{i_l}}$$

for $i_1, \dots, i_l \in \{1, \dots, t\}$. This forces commutativity to hold. Notice that this only gives a well-defined lift to all of V since $p = 2$. If we set $A = K[\epsilon], \alpha = a_0\epsilon, \beta = \epsilon\phi(u)$, for a linear map $\phi \in \mathrm{Hom}_{\mathbf{F}_p}(V, k)$, then we find that this lift corresponds to the cocycle $a_0 d_0 + \phi \in Z^1(V, M)$, so that we are indeed lifting in all directions of the tangent space. One can now reason as before to find that the hull of D_ρ is given by

$$H_\rho = k[[x_0, x_1, \dots, x_t]]/\langle x_1 + \cdots + x_t, u_1 x_1 + \cdots + u_t x_t, x_0(x_i u_j - x_j u_i)_{i,j=1,\dots,t}\rangle.$$

In Figure 1, one sees a schematic representation of the geometrical structure of these hulls. For $p \neq 2$, a nilpotent zero-dimensional scheme is attached to a polydisc in the $[d_0]$-tangential direction.

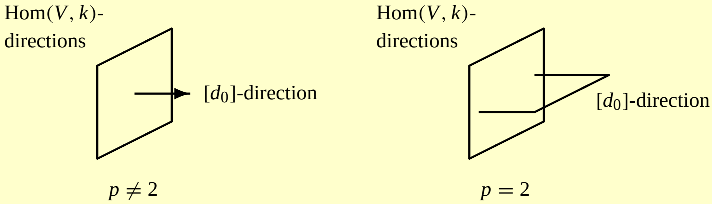

Figure 1. Pictorial representation of the local versal deformation ring

4.4.5. The case $n \neq 1$

Finally, in Case 3 of Section 3.1, the arguments are completely similar to those of the preceding paragraphs; the results are as follows.

(1) If $n \neq 2$ or $p = 2, 3$, then only first-order deformations coming from $\mathrm{Hom}_{\mathbf{F}_q}(V, k)$ occur, and these can be lifted without obstruction:

$$H_\rho \cong k[[x_1, \dots, x_{t/s}]]/\langle x_1 + \cdots + x_{t/s}\rangle.$$

(2) If $p \neq 2, 3$ and $n = 2$ (hence $s = 1$), then the action of H extends to the lifting (4.2.6.1) (just replacing u by ζu) and the obstructions do not change. Hence we have

$$H_\rho \cong k[[x_0, x_1, \dots, x_t]]/\langle x_0^{(p-1)/2}, x_0 x_1, \dots, x_0 x_t, x_1 + \cdots + x_t\rangle.$$

THEOREM 4.5

Let $\rho : G \to \mathrm{Aut}(\mathscr{T}_{\mathscr{O}})$ be a local representation of a finite group G, where $\mathscr{O}$ is of characteristic p. Let n be an integer coprime to p, define $s := \min\{s' : n | p^{s'} - 1\}$, and let $[d_0]$ be the cohomology class defined in Section 3.6. Table 1 lists the dimension *of the group cohomology* $H^1(G, \mathscr{T}_{\mathscr{O}})$*; whether* $[d_0]$ *is trivial (—), unobstructed (unobs.), or leads to obstructions (obs.); and the Krull dimension* $\dim H_\rho$ *of the prorepresentable hull* H_ρ *of the local deformation functor* D_ρ*.*

Table 1

G	(p, t, n)	$h^1(G, \mathscr{T}_{\mathscr{O}})$	$[d_0]$	$\dim H_\rho$
$\mathbf{Z}/n$		0	—	0
$(\mathbf{Z}/p)^t$	$p \neq 2, 3$	t	obs.	$t-1$
	$p = 3$	$t-1$	—	$t-1$
	$p = 2, t > 1$	$t-1$	obs.	$t-2$
	$p = 2, t = 1$	1	unobs.	1
$(\mathbf{Z}/p)^t \rtimes \mathbf{Z}/n$	$n \neq 2$ or $p = 2, 3$	$t/s - 1$	—	$t/s - 1$
	$n = 2$	t	obs.	$t-1$

Remark 4.5.1

For $n = t = 1$, this result agrees with [1, Proposition 4.1.1], where it is shown that

$$h^1(G, \mathscr{T}_{\mathscr{O}}) = \left\lfloor \frac{2\beta}{p} \right\rfloor - \left\lceil \frac{\beta}{p} \right\rceil$$

for $G = \mathbf{Z}/p$ a cyclic p-group and $\beta = \sum_{j=0}^{\infty}(|G_i| - 1)$. (Recall that G_i are the higher ramification groups.) Indeed, in our case, $G_0 = G_1 = G$ and $G_i = 0$ for $i > 1$, so that $\beta = 2p - 2$ and we get

$$h^1(G, \mathscr{T}_{\mathscr{O}}) = \begin{cases} 3 - 2 = 1 & \text{if } p > 3, \\ 2 - 2 = 0 & \text{if } p = 3, \\ 2 - 1 = 1 & \text{if } p = 2. \end{cases}$$

Similarly, our calculation of the hull and its Krull dimension is compatible with the results from Bertin and Mézard [1] if we observe that (in their notation, cf. [1, p. 215]) $\psi(X) = X^{(p-1)/2} \bmod p$ $(p > 2)$.

Remark 4.5.2

Obstructions to lifting first-order deformations are certain second cohomology classes in $\mathrm{H}^2(G, \mathscr{T}_{\mathscr{O}})$, but these can form a strict subset of this cohomology group. If $t = 1$, then the group is easy to calculate directly (cf. [1]) or is seen to be isomorphic to $\mathrm{H}^1(G, \mathscr{T}_{\mathscr{O}})$ by Herbrand's theorem since G is cyclic. This already shows that for $t = 1$, obstructions can form only a small part of $\mathrm{H}^2(G, \mathscr{T}_{\mathscr{O}})$. The computation of this second cohomology group in the general case $t > 1$ (so G is no longer cyclic) seems rather tedious.

5. Main theorem on algebraic equivariant deformation

We have now collected all information needed to prove our main algebraic theorem.

THEOREM 5.1

Let X be an ordinary curve over a field of characteristic $p > 0$, and let G be a finite group acting on X via $\rho : G \to \mathrm{Aut}(X)$.

(a) *The Krull dimension of the prorepresentable hull of the deformation functor $D_{X,\rho}$ is given by*

$$\dim H_{X,\rho} = 3g_Y - 3 + \delta + \sum_{i=1}^{s} \dim H_{\rho_i},$$

where g_Y is the genus of $Y := G\backslash X$, δ is given in Notation 1.5, and H_{ρ_i} is the prorepresentable hull of the local deformation functor associated to the representation of the ramification group G_i at the branch point w_i in $X \to Y$, whose dimension was given in Theorem 4.5, except *in the following four cases:*

(1) *when $p = 2, Y = \mathbf{P}^1$, and $X \to Y$ is branched above two points, then* $\dim H_{X,\rho} = \dim H_{\rho_1} + \dim H_{\rho_2}$;

(2) *when $X = \mathbf{P}^1 \to Y = \mathbf{P}^1$ is tamely branched above two points, then* $\dim H_{X,\rho} = 0$;

(3) *when $X = \mathbf{P}^1 \to Y = \mathbf{P}^1$ is wildly branched above a unique point, then* $\dim H_{X,\rho} = \dim H_{\rho_1}$ *(in this case, G is a pure-p group and $X \to Y$ is an Artin-Schreier cover);*

(4) *when $X \to Y$ is an unramified cover of elliptic curves, then* $\dim H_{X,\rho} = 1$.

Furthermore, if the genus g of X is greater than or equal to 2, then $D_{X,\rho}$ is prorepresentable by $H_{X,\rho}$.

(b) *The dimension of the tangent space to the functor $D_{X,\rho}$ as a k-vector space satisfies*

$$\dim_k D_{X,\rho}(k[\epsilon]) = \dim H_{X,\rho} + \begin{cases} \#\{i : n_i \leq 2\} & \text{if } p \neq 2, 3, \\ 0 & \text{if } p = 3, \\ \#\{i : n_i = 1 \text{ and } t_i > 1\} & \text{if } p = 2. \end{cases}$$

Proof

Let Y be the quotient $G\backslash X$. From Proposition 1.12 we find that

$$\dim H_{X,\rho} = \sum_{i=1}^{s} \dim H_{\rho_i} + h^1(Y, \pi_*^G \mathscr{T}_X),$$

where H_{ρ_i} are the hulls of the local deformation functors associated to the action of the ramification groups G_i at wild ramifications points $w_1, \dots, w_s$ on the space of local derivations, which was computed in Theorem 4.5.

To compute the h^1-term, recall from Proposition 1.6 that $\pi_*^G \mathscr{T}_X = \mathscr{T}_Y(-\Delta)$, where Δ is defined in Notation 1.5. By the Riemann-Roch theorem, we find that

$$h^1(\pi_*^G \mathscr{T}_X) = 3g_Y - 3 + \delta + h^0\big(\mathscr{T}_Y(-\Delta)\big),$$

where g_Y is the genus of Y.

Since $\deg(\mathscr{T}_Y(-\Delta)) = 2 - 2g_Y - \delta$, the last term vanishes if $g_Y > 1$ or $g_Y = 1$ and $\delta > 0$ or $g_Y = 0$ and $\delta > 2$.

If $g_Y = 1$ and $\delta = 0$, X is an unramified cover of an elliptic curve and hence is an elliptic curve itself.

Assume that $g_Y = 0$, that there are at least two branch points on Y, and that $\delta \leq 2$. If $p \neq 2$, then these branch points have to be tame, so $\delta = 2$, and the Hurwitz formula implies that $g_X = 0$ too. If $p = 2$, they can be wild, but both ramification groups have to be $\mathbf{Z}/2\mathbf{Z}$, so we still have $\delta = 2$. In both cases, $h^0(\mathscr{T}_Y(-\delta)) = h^0(\mathscr{O}_{\mathbf{P}^1}) = 1$.

If $g_Y = 0$ and only one point on Y is branched, then it follows from Hurwitz's formula (using the fact that second ramification groups vanish in the ordinary case; cf. [17]) that $g_X = 0$ too, and the ramification has to be wild at this point. So if $p \neq 2$ or $p = 2$, $t > 1$, we find that $\delta + h^0(\mathscr{T}_Y(-\delta)) = 2 + h^0(\mathscr{O}_{\mathbf{P}^1}) = 3$. On the other hand, if $p = 2$ and $t = 1$, then $\delta + h^0(\mathscr{T}_Y(-\delta)) = 1 + h^0(\mathscr{O}_{\mathbf{P}^1}(1)) = 3$. Let np^t (with n coprime to p) be the order of the ramification group at that unique point. Hurwitz's formula gives in particular that $(n-1)p^t + 2$ divides $2np^t$, and this (together with $n | p^t - 1$) implies $n = 1$. Hence we do get a $(\mathbf{Z}/p\mathbf{Z})$-cover. This finishes the proof of part (a).

For part (b), we apply the formula from Proposition 1.9 in combination with (1.7.1). It thus suffices to compute $h^0(Y, \mathscr{H}^1(G, \mathscr{T}_X))$, but $\mathscr{H}^1(G, \mathscr{T}_X)$ is concentrated in the branch points w_i, where it equals the group cohomology $H^1(G_i, \mathscr{T}_{\mathscr{O}_{w_i}})$ (see [1, Section 3.3]), so $\dim H_{X,\rho}$ and $\dim_k D_{X,\rho}(k[\epsilon])$ differ only at places where $[d_0]$ is obstructed in Table 1. □

Remark 5.1.1

The main algebraic theorem stated in the introduction follows from Theorem 5.1 by excluding the cases $g \leq 2$, $p = 2, 3$.

Example 5.1.2 (*Artin-Schreier curves*)

The Artin-Schreier curve whose affine equation is given by $(y^{p^t} - y)(x^{p^t} - x) = c$ for some constant $c \in k^*$ has automorphism group

$$G = (\mathbf{Z}/p\mathbf{Z})^{2t} \rtimes D_{p^t-1},$$

where D_* denotes a dihedral group of order $2*$. The quotient $Y = G\backslash X$ is a projective line, and the branching groups are $\mathbf{Z}/2\mathbf{Z}$ (twice if $p \neq 2$ and once if $p = 2$) and $(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(p^t - 1)\mathbf{Z}$ (once). This curve with its full automorphism groups hence allows for a one-dimensional deformation space (for different reasons if $p \neq 2$ and $p = 2$). This deformation is given exactly by varying c.

Example 5.1.3 (*Drinfeld modular curves*)
The Drinfeld modular curves $X(\mathfrak{n})$ from the introduction have automorphism group $G := \Gamma(1)/\Gamma(\mathfrak{n})\mathbf{F}_q^*$ for $d := \deg(\mathfrak{n}) > 1$. The quotient $Y := G\backslash X(\mathfrak{n})$ is a projective line, over which X is branched at two points with ramification groups $\mathbf{Z}/(p+1)\mathbf{Z}$ and $\mathbf{F}_q^d \rtimes \mathbf{F}_q^*$, respectively. Hence $X(\mathfrak{n})$ can be deformed in $d - 1$ ways (regardless of p, but for different reasons if $p = 2$—in that case we are in exceptional case (1) from Theorem 5.1).

Part B: Analytic theory

6. Equivariant deformation of Mumford curves

6.1. Mumford curves

Let $(K, |\cdot|)$ be a complete discrete valuation field with valuation ring $\mathscr{O}_K$ and residue field $\mathscr{O}_K/\mathfrak{m}_{\mathscr{O}_K} = k$. Recall that a projective curve X over K is called a *Mumford curve* if it is uniformized over K by a Schottky group. This means that there exists a free subgroup Γ in $\mathrm{PGL}(2, K)$ of rank g, acting on $\mathbf{P}_K^1$ with limit set $\mathscr{L}_\Gamma$ such that X satisfies $X^{\mathrm{an}} \cong \Gamma\backslash(\mathbf{P}_K^{1,\mathrm{an}} - \mathscr{L}_\Gamma)$ as rigid analytic spaces. Mumford [16] has shown that these conditions are equivalent to the existence of a stable model of X over $\mathscr{O}_K$ whose special fiber consists only of rational components with k-rational double points. Because of the GAGA-correspondence for one-dimensional rigid analytic spaces, we do not have to (and do not) distinguish between analytic and algebraic curves. It is well known that Mumford curves are ordinary. (This is basically because their Jacobian is uniformized by $(\mathbf{G}_{m,K}^{\mathrm{an}})^g/\Gamma^{\mathrm{ab}}$, where g is the genus of X; cf. [5, Lemma 1.2].) Thus the results from the previous section essentially solve the equivariant deformation problem for Mumford curves in a cohomological way. In this part, however, we want to develop an independent theory of analytic deformation of Mumford curves based on the groups that uniformize them. This makes the liftings and obstructions whose cohomological existence was proven in the previous part more visible as actual deformations of (2×2)-matrices over K.

6.2. Automorphisms

It is well known (see [5, Theorem 1.3]) that for a Mumford curve X of genus $g \geq 2$ with Schottky group Γ, $\mathrm{Aut}(X) = N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{PGL}(2, K)$. Conversely, if N is a discrete subgroup of $\mathrm{PGL}(2, K)$ containing Γ and contained in $N(\Gamma)$, then it induces a group of automorphisms

$$\rho : N/\Gamma \hookrightarrow \mathrm{Aut}(X).$$

Notation 6.3

If a finitely generated discrete subgroup N of $\mathrm{PGL}(2, K)$ is given, let $\mathrm{Hom}^*(N, \mathrm{PGL}(2, K))$ denote the set of *injective* homomorphisms $\phi : N \to \mathrm{PGL}(2, K)$ *with discrete image*. Then such an N contains a finite-index normal free subgroup of finite rank Γ (see [9, Section I.3]), and if Γ is nontrivial, the pair (N, Γ) gives rise to a Mumford curve with an action of N/Γ, as is being considered here.

6.4. Rigidity

Two Mumford curves X, X' with Schottky groups Γ, Γ' are isomorphic if and only if Γ and Γ' are conjugate in $\mathrm{PGL}(2, K)$ (see [16, Corollary 4.11]).

Remark 6.4.1

Note that this is very different from the situation in the uniformization theory of Riemann surfaces S, where, in a representation $S = \Gamma\backslash\Omega$ with Γ a Schottky subgroup of $\mathrm{PGL}(2, \mathbf{C})$, the domain of discontinuity Ω of Γ is not the universal topological covering space of S; this does hold for Mumford curves.

6.5. Analytic deformation functors

Recall that $\mathscr{C}_K$ is the category of local Artinian K-algebras. If N and $\phi \in \mathrm{Hom}^*(N, \mathrm{PGL}(2, K))$ are given, we consider the *analytic deformation functor*

$$D_{N,\phi} : \mathscr{C}_K \to \mathsf{Set}$$

of the pair (N, ϕ), which sends $A \in \mathscr{C}_K$ to the set of liftings of (N, ϕ) to A. Here, a lifting is a morphism $\phi^\sim \in \mathrm{Hom}(N, \mathrm{PGL}(2, A))$ which, when composed with reduction modulo the maximal ideal $\mathfrak{m}_A$ of A, equals the original embedding ϕ. (In particular, $\phi^\sim$ is injective.) Note that we do not consider classes of liftings modulo conjugacy by $\mathrm{PGL}(2)$—this implies that $D_{N,\phi}$ is naturally equipped with an action of group functor $\mathrm{PGL}(2)^\wedge$ given by

$$\mathrm{PGL}(2)^\wedge : \mathscr{C}_K \to \mathsf{Groups} : A \mapsto \ker\big[\mathrm{PGL}(2, A) \to \mathrm{PGL}(2, K)\big],$$

and we denote the quotient by

$$D^\sim_{N,\phi} := \mathrm{PGL}(2)^\wedge \backslash D_{N,\phi}.$$

Since N is finitely generated, it is not difficult to show that the functors $D_{N,\phi}$ and $D^{\sim}_{N,\phi}$ have prorepresentable hulls $H_{N,\phi}$ and $H^{\sim}_{N,\phi}$. We want to compute the dimension of the tangent spaces to these functors, as well as the Krull dimension of the hulls $H_{N,\phi}$ and $H^{\sim}_{N,\phi}$, and this is done by decomposing N using its structure as a group acting on a tree to give a decomposition of $D_{N,\phi}$. Note that such a decomposition is *not* given on the level of $D^{\sim}_{N,\phi}$.

7. Structure of N as a group acting on a tree

We fix a finitely generated discrete subgroup N of $\mathrm{PGL}(2, K)$, and we now recall how the structure of N can be seen from its action on the Bruhat-Tits tree (cf. [5, Section 2]).

7.1. The Bruhat-Tits tree

Let $\mathscr{T}$ denote the Bruhat-Tits tree of $\mathrm{PGL}(2, K)$ (i.e., its vertices are similarity classes Λ of rank two $\mathscr{O}_K$-lattices in K^2, and two vertices are connected by an edge if the corresponding quotient module has length one; see Serre [20] and L. Gerritzen and M. van der Put [9]). We assume K to be large enough so that all fixed points of N are defined over K; then N acts without inversion on $\mathscr{T}$. It is a regular tree in which the edges emanating from a given vertex are in one-to-one correspondence with $\mathbf{P}^1(k)$. The tree $\mathscr{T}$ admits a left action by $\mathrm{PGL}(2, K)$.

7.2. Notation on trees

For any subtree T of $\mathscr{T}$, let $\mathrm{Ends}(T)$ denote its set of ends (i.e., equivalence classes of half-lines differing by a finite segment). There is a natural correspondence between $\mathbf{P}^1(K)$ and $\mathrm{Ends}(\mathscr{T})$. Let $V(T)$ and $E(T)$ denote, respectively, the set of vertices and edges of T. For $\sigma \in E(T)$, let $o(\sigma)$ (resp., $t(\sigma)$) denote the origin (resp., terminal) vertex of σ. Let N_x denote the stabilizer of a vertex or edge x of T for the action of N. The maps o, t induce maps $N_\sigma \to N_\Lambda$ for $\Lambda = o(\sigma)$ or $\Lambda = t(\sigma)$, which are denoted by the same letter. For any $u, v \in \mathbf{P}^1(K)$, let $]u, v[$ denote the apartment in $\mathscr{T}$ connecting u and v (seen as ends of $\mathscr{T}$).

7.3. The trees associated to N

We can construct a locally finite tree $\mathscr{T}(\mathscr{L})$ (possibly empty) from any compact subset $\mathscr{L}$ of $\mathbf{P}^1(K)$; it is the minimal subtree of $\mathscr{T}$ whose set of ends coincides with $\mathscr{L}$ or, equivalently, the minimal subtree of $\mathscr{T}$ containing $\bigcup_{u,v\in\mathscr{L}}\,]u, v[$.

We define $\mathscr{T}_N$ to be the tree associated to the subset $\mathscr{L}_N$ consisting of the limit points of N in $\mathbf{P}^1(K)$. Since N is a finitely generated discrete group, $\mathscr{T}_N$ coincides with the tree of N as it is defined in Gerritzen and van der Put [9].

7.4. The graph associated to N

$\mathscr{T}_N$ admits a natural action of N, and we denote the quotient graph by $T_N := N\backslash\mathscr{T}_N$; the corresponding quotient map is denoted by π_N. The graph T_N is finite and connected.

We turn T_N into a graph of groups as follows. Let T be a spanning tree (maximal subtree) of T_N, which we can see as a subtree of $\mathscr{T}_N$ by a fixed section $\iota : T \to \mathscr{T}_N$ of π_N. Let $c = c(T_N)$ denote the cyclomatic number of T_N (equal to the number of edges outside T), and fix $2c$ lifts $e_i^{\pm}$ of these edges outside T to $\mathscr{T}_N$ which satisfy $t(e_i^+) \in V(\iota(T))$, $o(e_i^-) \in V(\iota(T))$. Fix c hyperbolic elements $\{\gamma_i\}_{i=1}^c$ in N such that $\gamma_i e_i^+ = e_i^-$. Then $\iota(T) \cup \{e_i^+\}_{i=1}^c$ is a fundamental domain for the action of N on $\mathscr{T}_N$.

For any vertex $\Lambda \in V(\mathscr{T}_N)$ and edge $\sigma = [\Lambda, M] \in E(\mathscr{T}_N)$, we denote by N_Λ and $N_\sigma = N_\Lambda \cap N_M$ their respective stabilizers for the action of N. Note that these groups are finite since N is discrete.

For a vertex $v \in V(T_N) = V(T)$, we let $N_v = N_{\iota(v)}$. For edges $e \in E(T_N)$, either $e \in E(T)$ and we let $N_e = N_{\iota(e)}$, or else there is a unique i such that $\pi_N(e_i^{\pm}) = e$ and we let $N_e = N_{e_i^+}$.

The morphisms between these groups are defined as follows: If $e \in E(T)$, then $N_e \hookrightarrow N_{t(e)}$ and $N_e \hookrightarrow N_{o(e)}$ are the natural inclusions; if, on the other hand, $e = \pi_N(e_i^{\pm})$, then $N_e \hookrightarrow N_{t(e)}$ is the natural inclusion but $N_e \hookrightarrow N_{o(e)}$ is given by $s \mapsto \gamma_i^{-1} s \gamma_i$.

We then have the following description of the group N.

THEOREM 7.5 (Bass-Serre theorem; see [6, Theorem I.4.1, Proposition I.4.4]; cf. [20]) *For any spanning tree T of T_N, N equals the fundamental group of the graph of groups T_N at T. This means that N is generated by the amalgam of N_v over N_e for all $e \in E(T_N)$, $v \in V(T_N)$ together with the fundamental group of T_N at T as a plain graph, namely, the free group F_c on c generators $\{n_i\}_{i=1}^c$, where $c = c(T_N)$ is the cyclomatic number of T_N. The further relations in N are of the form $n_i t(\gamma) n_i^{-1} = o(\gamma)$ for every $i = 1, \dots, c$ and for every $\gamma \in N_e$, $e \in T_N - T$. In particular, there is a split exact sequence of groups*

$$0 \to \varinjlim_{T_N^\sim} N_\bullet^\sim \to N \to F_{c(T_N)} \to 0,$$

where $\pi : T_N^\sim \to T_N$ is the universal covering of T_N as a plain graph, which has been made into a graph of groups by setting $N_\bullet^\sim$ for $\bullet \in V(T_N^\sim) \cup E(T_N^\sim)$ equal to $N_{\pi(\bullet)}$.

8. Decomposition of the functor $D_{N,\phi}$

PROPOSITION 8.1

Let $s : F_{c(T_N)} \to N$ be a splitting of the sequence in Theorem 7.5. Then there is an isomorphism of functors

$$D_{N,\phi} \cong \varprojlim_{T_N} D_{N_\bullet, \phi|_{N_\bullet}} \times D_{F_c, \phi \circ s},$$

where the inverse limit is in the category of functors. (Note that morphisms between $N_\bullet$ naturally induce morphisms of functors between $D_{N_\bullet}$.)

Remark 8.1.1

Note that we get a direct product of functors, but a limit of functors over T_N (instead of the obvious semidirect product and limit over $T_N^\sim$). We also note that there is no such decomposition on the level of the functors $D^\sim_{N,\phi}$.

Proof

Let $A \in \mathscr{C}_K$. By restriction, a deformation of N to A trivially gives rise to deformations of $N_\bullet$ and F_c.

For the rest of the proof, we imitate the construction of T_N as a graph of groups, but we lift to $T_N^\sim$ instead of $\mathscr{T}_N$. So choose a fixed maximal spanning tree $\iota : T \hookrightarrow T_N^\sim$ and a basis $\{\gamma_1, \ldots, \gamma_c\}$ of $s(F_c)$, where $c = c(T_N)$. Take, as before, $2c$ edges $e_i^\pm \in E(T_N^\sim)$ such that $t(e_i^+) \in V(T)$, $o(e_i^-) \in V(T)$, $\gamma_i e_i^+ = e_i^-$. Thus $T \cup \{e_i^+\}$ is a fundamental domain for $T_N^\sim \to T_N$.

To give elements in

$$\varprojlim_{T_N} D_{N_\bullet, \phi|_{N_\bullet}}(A) \qquad \text{and} \qquad D_{F_c, \phi \circ s}(A)$$

means precisely to give a compatible collection of $\phi_v : N_v \hookrightarrow \mathrm{PGL}(2, A)$ and $\phi_c : F_c \hookrightarrow \mathrm{PGL}(2, A)$. Compatibility means that for $e \in E(T_N)$, the following diagram is commutative:

$$\begin{tikzcd}
N_e \arrow[r] \arrow[d] & N_{o(e)} \arrow[d, "\phi_{t(e)}"] \\
N_{t(e)} \arrow[r, "\phi_{o(e)}"] & \mathrm{PGL}(2, A)
\end{tikzcd}$$

We want to extend this to an embedding of N.

By the construction of the fundamental domain, there exists for any $v \in V(T_N^\sim)$ a unique $\gamma \in s(F_c)$ such that $v \in \gamma T$, and this allows us to define $N_v \to \mathrm{PGL}(2, A)$ to be $\sigma \mapsto \phi_{\gamma^{-1} v}(\gamma^{-1} \sigma \gamma)$. For edges e, we similarly get γ such that $e \in \gamma \cdot (T \cup \{e_i^+\}_{i=1}^c)$, and the same works.

By the compatibility of ϕ_v and ϕ_c expressed in the commutative diagram above, we thus get an embedding of

$$\varinjlim_{T_N^\sim} N_\bullet \text{ into } \mathrm{PGL}(2, A),$$

which by construction is compatible with the conjugation action of F_c, so that we finally get an embedding $N \hookrightarrow \mathrm{PGL}(2, A)$, namely, an element of $D_{N,\phi}(A)$. Since this construction is functorial in A, we get the desired inverse map of functors. □

8.2. Computing the functor $D_{F_c,\phi\circ s}$

The set of morphisms

$$\mathrm{Hom}\big(F_c, \mathrm{PGL}(2, K)\big)$$

is a smooth algebraic variety over K; by choosing a basis of F_c, it is isomorphic to $\mathrm{PGL}(2, K)^c$ over K. We can take its formal completion at the K-rational point $\phi \circ s$ and then

$$D_{F_c,\phi\circ s} \cong \mathrm{Hom}\big(F_c, \mathrm{PGL}(2, K)\big)^\wedge_{\phi\circ s}$$

as formal functors. In particular,

$$\dim_K D_{F_c,\phi\circ s}\big(K[\epsilon]\big) = \dim H_{F_c,\phi\circ s} = 3c,$$

where the first expression is the dimension of the tangent space and the second expression is the Krull dimension of the prorepresentable hull of the functor.

8.3. Computing the functor $D_{N,\phi}$ for finite N

Here the argument is based on the simple observation that an injective element ϕ of $\mathrm{Hom}(N, \mathrm{PGL}(2, K))$ corresponds to a cover $\mathbf{P}^1 \to \mathbf{P}^1$ with Galois group N; hence it is related to the algebraic deformation functor (Section 1.1) of the pair $(\mathbf{P}^1, \phi)$ (regarding ϕ as a representation of N into $\mathrm{Aut}(\mathbf{P}^1)$). The functor $D_{\mathbf{P}^1,\phi}$ is defined modulo conjugation by $\mathrm{PGL}(2)$, whereas the analytic deformation functor $D_{N,\phi}$ carries a natural action of $\mathrm{PGL}(2)^\wedge$. However, it is easy to see that

$$D^\sim_{N,\phi} = D_{\mathbf{P}^1,\phi}.$$

From this formula we get in particular that

$$\dim H_{N,\phi} = \dim H_{\mathbf{P}^1,\phi} + 3 - \nu\big(\phi(N)\big), \tag{8.3.1}$$

where for a finite subgroup $G \subseteq \mathrm{PGL}(2, K)$,

$$\nu(G) = \dim \mathrm{Nor}_{\mathrm{PGL}(2,K)}(G)$$

is the dimension of the normalizer of G in $\mathrm{PGL}(2, K)$ as an algebraic group. Formula (8.3.1) continues to hold when hulls are replaced by tangent spaces.

By Dickson's *Hauptsatz* (see Theorem 8.3.3), the finite groups N acting on $\mathbf{P}^1$ in positive characteristic are known. Let us first set up the notation.

Notation 8.3.2

We let D_n denote the dihedral group of order $2n$. We write $P(2, q)$ to denote either $\mathrm{PGL}(2, q)$ or $\mathrm{PSL}(2, q)$ by slight abuse of notation, with the convention that any related numerical quantities that appear between set delimiters {} are only to be considered for $\mathrm{PSL}(2, q)$.

We now recall this classification in the version as it is given in R. Valentini and M. Madan [21], as this more geometrical form immediately allows us to compute $D_{N,\phi}$ using the results from Section 5.

THEOREM 8.3.3 (Dickson's *Hauptsatz*; see [21, Theorem 1])

Any finite subgroup of $\mathrm{PGL}(2, K)$ *is isomorphic to a finite subgroup of* $\mathrm{PGL}(2, p^m)$ *for some* $m > 0$. *The group* $\mathrm{PGL}(2, p^m)$ *has the following finite subgroups* G, *such that* π_G *is branched over* d *points with ramification groups isomorphic to* $G_1, \dots, G_d$:

(i) $G = \mathbf{Z}/n\mathbf{Z}$ *for* $(n; p) = 1$, $d = 2$, *and* $G_1 = G_2 = \mathbf{Z}/n\mathbf{Z}$;

(ii) $G = D_n$ *with* $p \neq 2$, $n | p^m \pm 1$, $d = 3$, *and* $G_1 = G_2 = \mathbf{Z}/2\mathbf{Z}$, $G_3 = \mathbf{Z}/n\mathbf{Z}$ *or, also,* $p = 2$, $(n; 2) = 1$, $d = 2$, *and* $G_1 = \mathbf{Z}/2\mathbf{Z}$, $G_2 = \mathbf{Z}/n\mathbf{Z}$;

(iii) $G = (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$ *for* $t \leq m$ *and* $n | p^m - 1$, $n | p^t - 1$ *with* $d = 2$, *and* $G_1 = G$, $G_2 = \mathbf{Z}/n\mathbf{Z}$ *if* $n > 1$, *and* $d = 1$, $G_1 = G$ *otherwise;*

(iv) $G = P(2, p^t)$ *with* $d = 2$, *and* $G_1 = (\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/\{1/2\}(p^t - 1)\mathbf{Z}$, $G_2 = \mathbf{Z}/\{1/2\}(p^t + 1)\mathbf{Z}$;

(v) $G = A_4$ *if* $p \neq 2, 3$, $d = 3$, *and* $G_1 = \mathbf{Z}/2\mathbf{Z}$, $G_2 = G_3 = \mathbf{Z}/3\mathbf{Z}$;

(vi) $G = S_4$ *if* $p \neq 2, 3$, $d = 3$, *and* $G_1 = \mathbf{Z}/2\mathbf{Z}$, $G_2 = G_3 = \mathbf{Z}/4\mathbf{Z}$;

(vii) $G = A_5$ *if* $5 | p^{2m} - 1$, $p \neq 2, 3, 5$ *with* $d = 3$, *and* $G_1 = \mathbf{Z}/2\mathbf{Z}$, $G_2 = \mathbf{Z}/3\mathbf{Z}$, $G_3 = \mathbf{Z}/5\mathbf{Z}$, *or* $p = 3$, $d = 2$, *and* $G_1 = \mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$, $G_2 = \mathbf{Z}/5\mathbf{Z}$.

LEMMA 8.3.4

The normalizer of a finite subgroup N *of* $\mathrm{PGL}(2, K)$ *has dimension* $\nu(N) = 0$ *unless* N *is cyclic of order prime to* p, *in which case* $\nu(N) = 1$ *or, if* N *is a pure* p*-group,* $\nu(N) = 2$.

Proof

Any group from the above list which does not belong to the mentioned exceptions has

Table 2

G	(p, t, n)	$h(G)$	$t(G)$
$\mathbf{Z}/n\mathbf{Z}$	$(n; p) = 1$	2	2
D_n	$p \neq 2$	3	3
	$p = 2$	4	4
$(\mathbf{Z}/p\mathbf{Z})^t$	$p \neq 2, 3$	t	$t + 1$
	$p = 3$	t	t
	$p = 2, t > 1$	$t - 1$	t
	$p = 2, t = 1$	2	2
$(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$	$p \neq 2$ and $n \neq 2$ or $p = 2, 3$	$t/s + 2$	$t/s + 2$
	$n = 2$	$t + 2$	$t + 3$
$P(2, p^t)$	$\{p^t \neq 5\}$	3	3
A_4, S_4		3	3
A_5	$p \neq 3$	3	3
	$p = 3$	3	4

at least three fixed points on $\mathbf{P}^1$, the set of which should also remain stable under the action of the normalizer of N, which hence is finite.

A cyclic subgroup N of order prime to p has a diagonalizable generator, and by a direct computation, this is seen to be exactly stabilized by the one-dimensional group D generated by the center of $\mathrm{PGL}(2, K)$ and the involution $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

A p-group N can be put into upper diagonal form by conjugation, and a little computation shows that the stabilizer of such a group consists precisely of the two-dimensional group of upper trigonal matrices. □

THEOREM 8.4

Let N be as in Section 7, and suppose a Bass-Serre representation of N is given as in Theorem 7.5. Then

$$\dim H^{\sim}_{N,\phi} = 3c(T_N) - 3 + \sum_{v \in V(T_N)} h(N_v) - \sum_{e \in E(T_N)} h(N_e)$$

and

$$\dim_K D^{\sim}_{N,\phi}\big(K[\epsilon]\big) = 3c(T_N) - 3 + \sum_{v \in V(T_N)} t(N_v) - \sum_{e \in E(T_N)} t(N_e),$$

where for a finite group $G \subset \mathrm{PGL}(2, K)$, the numbers $h(G)$ and $t(G)$ are given in Table 2.

Table 3

G	(p, t, n)	$h^{\mathrm{alg}}(G)$	$t^{\mathrm{alg}}(G)$	$3 - \nu(G)$
$\mathbf{Z}/n\mathbf{Z}$	$(n; p) = 1$	0	0	2
D_n	$p \neq 2$	0	0	3
	$p = 2$	1	1	3
$(\mathbf{Z}/p\mathbf{Z})^t$	$p \neq 2, 3$	$t - 1$	t	1
	$p = 3$	$t - 1$	$t - 1$	1
	$p = 2, t > 1$	$t - 2$	$t - 1$	1
	$p = 2, t = 1$	1	1	1
$(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/n\mathbf{Z}$	$p \neq 2$ and $n \neq 2$ or $p = 2, 3$	$t/s - 1$	$t/s - 1$	3
	$n = 2$	$t - 1$	t	3
$P(2, p^t)$	$\{p^t \neq 5\}^*$	0	0	3
A_4, S_4		0	0	3
A_5	$p \neq 3$	0	0	3
	$p = 3$	0	1	3

Proof

Since N is infinite, the action of $\mathrm{PGL}(2)^\wedge$ is of dimension 3. By Proposition 8.1 and Section 8.2, we are reduced to computing $D_{N,\phi}$ for finite N occuring in T_N. We know which different G can occur on the edges and vertices of T_N by Dickson's theorem (Theorem 8.3.3). For each of these, using (8.3.1), we are reduced to the computation of the algebraic data, for which we appeal to Theorem 5.1, and to the computation of $\nu(G)$, which is in Lemma 8.3.4. If we let $h^{\mathrm{alg}}(G)$ and $t^{\mathrm{alg}}(G)$ denote the Krull dimension of the prorepresentable hull and vector-space dimension of the tangent space to $D_{\mathbf{P}^1,\phi|G}$, respectively, we have Table 3. In this computation, note at * that $\mathrm{PSL}(2, 5) = A_5$ does not occur in Dickson's list if $p = 5$. □

Remark 8.4.1

The main analytic theorem stated in the introduction follows from Theorem 8.4 by excluding the cases $g \leq 2$, $p = 2, 3$.

9. Compatibility between algebraic and analytic deformation

9.1. Deformation of Mumford uniformization

We have already seen how to compare analytic and algebraic deformation functors for finite groups acting on $\mathbf{P}^1$. We now want to compare these functors in general, in particular to achieve equality between the apparently different results from the main algebraic and analytic theorems (Theorems 5.1 and 8.4).

Let X be a Mumford curve over K uniformized by a Schottky group Γ, and let $\phi : N \hookrightarrow \mathrm{PGL}(2, k)$ be a discrete group between Γ and its normalizer in $\mathrm{PGL}(2, K)$. Let $\rho : N/\Gamma \hookrightarrow \mathrm{Aut}(X)$. To be able to compare the functors $D^{\sim}_{N,\phi}$ and $D_{X,\rho}$, it is necessary to develop a theory of algebraic deformation of Mumford uniformization; this just means to translate the formalism of Mumford [16] from fields K to elements in the category $\mathscr{C}_K$. For lack of a reference, we sketch it here; the reader who wants to follow the details is encouraged to take a copy of [16] to hand.

9.2. Analytic objects in $\mathscr{C}_K$

The maximal ideal of an object A in $\mathscr{C}_K$ is denoted by $\mathfrak{m}_A$. Each object A in $\mathscr{C}_K$ can be made into a K-affinoid algebra in a unique way by a suitable surjective homomorphism $K\langle X_1, \ldots, X_n\rangle \to A$ over K (see [2, (6.1)]). We denote by $\mathscr{O}_A$ (resp., $\mathfrak{m}_{\mathscr{O}_A}$) the subring (resp., the ideal in $\mathscr{O}_A$) consisting of power-bounded (resp., topologically nilpotent) elements in A (see [2, (6.2.3)]). Since every element in $\mathfrak{m}_A$ is nilpotent, we have $\mathscr{O}_A \cap \mathfrak{m}_A \subseteq \mathfrak{m}_{\mathscr{O}_A}$. By this, it is easily seen that $\mathscr{O}_A$ is a local ring with the maximal ideal $\mathfrak{m}_{\mathscr{O}_A}$, that $\mathscr{O}_A/\mathfrak{m}_{\mathscr{O}_A} \cong k$, and that $\pi_A^{-1}(\mathscr{O}_K) = \mathscr{O}_A$ and $\pi_A^{-1}(\mathfrak{m}_{\mathscr{O}_K}) = \mathfrak{m}_{\mathscr{O}_A}$, where $\pi_A \colon A \to K$ is the reduction map.

Example

In the ring of dual numbers $K[\epsilon]$, the ring of power-bounded elements is $\mathscr{O}_K + K\epsilon$, whereas the ideal of topologically nilpotent elements is $\mathfrak{m}_{\mathscr{O}_K} + K\epsilon$.

9.3. Lattices

Let A be an object in $\mathscr{C}_K$. By a lattice in A^2 we mean an $\mathscr{O}_A$-submodule M in A^2 that is free of rank 2. By elementary commutative algebra, this is equivalent to $M \subset A^2$ being an $\mathscr{O}_A$-submodule such that the image $\overline{M}$ in K^2 by the reduction map $A^2 \to K^2$ is a lattice in the usual sense. We consider the set $\Delta_A^{(0)}$ of similarity classes of lattices up to multiplication by A^*. Then $\Delta_A^{(0)}$ can be naturally identified with the set of equivalence classes of couples $(\mathbf{P}, \phi)$, where $\mathbf{P}$ is an $\mathscr{O}_A$-scheme isomorphic to $\mathbf{P}^1_{\mathscr{O}_A}$ and ϕ is an isomorphism between $\mathbf{P} \otimes A$ and $\mathbf{P}^1_A$, and two couples $(\mathbf{P}, \phi)$ and $(\mathbf{P}', \phi')$ are equivalent if there exists an $\mathscr{O}_A$-isomorphism $\psi \colon \mathbf{P} \to \mathbf{P}'$ such that $\phi' \circ \psi = \phi$. The identification between $\Delta_A^{(0)}$ and the space of such couples is given by

$$M \mapsto \mathbf{P}(M) = \mathrm{Proj}(\mathrm{Sym}_{\mathscr{O}_A} M),$$

where ϕ is induced from $M \otimes A \cong A^2$.

9.4. Trees

We take a subgroup $N \subset \mathrm{PGL}(2, A)$ such that its image $\overline{N}$ in $\mathrm{PGL}(2, K)$ by the reduction map is finitely generated, discrete, and isomorphic to N. Such a subgroup

N contains a normal free subgroup Γ of finite index since $\overline{N}$ does and N and $\overline{N}$ are isomorphic as groups. This Γ satisfies a flatness condition analogous to [16, (1.4)] (or, equivalently, property $*$ in [16, p. 139]) in the following sense: If Σ is the set of all sections $\operatorname{Spec} A \to \mathbf{P}^1_A$ fixed by nontrivial elements $\gamma \in \Gamma$, then for any $P_1, P_2, P_3, P_4 \in \Sigma$, the cross-ratio $R := R(P_1, P_2; P_3, P_4)$ or its inverse R^{-1} lies in $\mathscr{O}_A$. (Note that Σ does not depend on the choice of Γ in N.) The proof follows easily from the fact that $\pi_A^{-1}(\mathscr{O}_K) = \mathscr{O}_A$. Given P_1, P_2, P_3 with homogeneous coordinates w_1, w_2, w_3, respectively, let $M = \mathscr{O}_A a_1 w_1 + \mathscr{O}_A a_2 w_2 + \mathscr{O}_A a_3 w_3$, where the a_i satisfy a nontrivial linear relation $a_1 w_1 + a_2 w_2 + a_3 w_3 = 0$. The class $v(P_1, P_2, P_3)$ of M in $\Delta_A^{(0)}$ depends only on P_1, P_2, P_3. We let $\Delta_\Gamma^{(0)}$ be the set of all such $v(P_1, P_2, P_3)$. The set $\Delta_\Gamma^{(0)}$ is linked in the sense of [16, (1.11)], and the tree thus obtained is obviously the usual tree with respect to $\overline{\Gamma}$.

9.5

The construction of the formal scheme also parallels the original one. For M_1 and M_2 in $\Delta_\Gamma^{(0)}$, one defines the join $\mathbf{P}(M_1) \vee \mathbf{P}(M_2)$ to be the closure of the graph of the birational map $\mathbf{P}(M_1) \to \cdots \to \mathbf{P}(M_2)$ induced from $\phi_2^{-1} \circ \phi_1$, where $(\mathbf{P}(M_i), \phi_i)$ corresponds to M_i $(i = 1, 2)$ by the correspondence from Section 9.3. The formal scheme $\mathscr{P}_\Gamma$ over $\operatorname{Spf} \mathscr{O}_A$ is then constructed as in [16, p. 156] using these joins. Obviously, its fiber over $\operatorname{Spf} \mathscr{O}_K$ is isomorphic to the usual formal scheme; in particular, their underlying topological spaces are isomorphic. It is clear that the associated rigid space Ω_Γ of $\mathscr{P}_\Gamma$ in the sense of [3, Section 5] is the complement in $\mathbf{P}_A^{1,\mathrm{an}}$ of the closure of the set of fixed rig-points (corresponding to the fixed sections). The quotient and the algebraization can equally well be taken by reasoning as in the usual case. What finally comes out is a scheme X_Γ over A with special fiber over K the Mumford curve corresponding to $\overline{\Gamma}$, and hence one can further take a finite quotient by N/Γ.

9.6

The above construction of infinitesimal deformation of Mumford uniformization induces a morphism of functors

$$\Phi : D^{\sim}_{N,\phi} \longrightarrow D_{X,\rho}$$

by associating to a deformation of $N \hookrightarrow \mathrm{PGL}(2, K)$ to $\bar{N} \hookrightarrow \mathrm{PGL}(2, A)$ the corresponding “Mumford” curve over Spec A. By an argument parallel to that of [16, Section 4], it is not difficult to see the following.

PROPOSITION 9.7
The morphism Φ is an isomorphism.

Remark 9.7.1
If X is a Mumford curve uniformized by a Schottky group Γ and N is between Γ and its normalizer in $\mathrm{PGL}(2, K)$, let $\rho\colon N\backslash\Gamma \to \mathrm{Aut}(X)$ be the corresponding representation. Then the (a priori very different looking) results from the algebraic computation in Theorem 5.1 for $D_{X,\rho}$ and the analytic computation in Theorem 8.4 for $D^{\sim}_{N,\phi}$ agree. There may be a more direct combinatorial proof of this equality.

Example 9.7.2 (*Artin-Schreier-Mumford curves*)
If, for the Artin-Schreier curves of Example 5.1.2 over a nonarchimedean field K, the value of c satisfies $|c| < 1$, then $X_{t,c}$ is a Mumford curve, and the corresponding normalizer of its Schottky group is

$$N_t = \big((\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(p^t-1)\mathbf{Z}\big) *_{\mathbf{Z}/(p^t-1)\mathbf{Z}} D_{p^t-1}.$$

We compute from this that the analytic infinitesimal deformation space is 1-dimensional, in concordance with the algebraic result.

Example 9.7.3 (*Drinfeld modular curves*)
The Drinfeld modular curve $X(\mathfrak{n})$ is known to be a Mumford curve (cf. [8]), and the normalizer of its Schottky group is isomorphic to an amalgam (cf. [5])

$$N(\mathfrak{n}) = \mathrm{PGL}(2, p^t) *_{(\mathbf{Z}/p\mathbf{Z})^t \rtimes \mathbf{Z}/(p^t-1)\mathbf{Z}} (\mathbf{Z}/p\mathbf{Z})^{td} \rtimes \mathbf{Z}/(p^t-1)\mathbf{Z}$$

(at least if $p \neq 2, q \neq 3$). The above formula gives a $(d-1)$-dimensional infinitesimal analytic deformation space, and this agrees with the algebraic result.

Acknowledgments. G. Cornelissen is an honorary postdoctoral fellow of the Fund for Scientific Research–Flanders (FWO-Vlaanderen). This work was done while Cornelissen was visiting Kyoto University and the Max-Planck-Institut für Mathematik (MPIM) in Bonn and while F. Kato was visiting the University of Paris-VI. Thanks to Frans Oort for pointing us to [1].

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