

On the K -Theory of Graph C^* -Algebras

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Abstract We classify graph C^* -algebras, namely, Cuntz-Krieger algebras associated to the Bass-Hashimoto edge incidence operator of a finite graph, up to strict isomorphism. This is done by a purely graph theoretical calculation of the K -theory of the C^* -algebras and the method also provides an independent proof of the classification up to Morita equivalence and stable equivalence of such algebras, without using the boundary operator algebra. A direct relation is given between the K_1 -group of the algebra and the cycle space of the graph.

Keywords Graph C^* -algebras · K -theory · Edge incidence operator · Morita equivalence · Strict isomorphism

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1 Introduction

The purpose of this paper is to give a very simple proof of the classification of graph C^* -algebras, which makes no use of operator algebra techniques but only uses combinatorial and homotopical properties of graphs. Theorem 1 in this paper is well known in the

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operator algebra literature, but we provide a very simple and purely graph-theoretical proof. We then use this proof as the basis to obtain Theorem 2, which completes the classification by identifying the image of the unit in K -theory. We can sometimes reverse our methods and deduce graph theoretical results from known operator algebraic calculations (cf. 2.12).

To set up notation: let \mathcal{U} denote a finite (multi-)graph with m (geometrical) edges. Note that \mathcal{U} might have loops, multiple edges and sinks. Consider the (oriented) graph \mathcal{U}^+ , whose vertices equal those of \mathcal{U} , and whose edges consist of all edges of \mathcal{U} with both possible orientations. We denote the two edges of \mathcal{U}^+ corresponding to an edge e of \mathcal{U} by e and \bar{e} . Let o and t denote the origin, respectively, terminal vertex of an oriented edge e of \mathcal{U}^+ . Let $E\mathcal{U}^+$ denote the set of edges of the oriented graph \mathcal{U}^+ , and let $\mathbf{Z}^{E\mathcal{U}^+}$ denote the free \mathbf{Z} -module spanned by these edges. On this space, we consider the linear operator $T : \mathbf{Z}^{E\mathcal{U}^+} \rightarrow \mathbf{Z}^{E\mathcal{U}^+}$ defined on the basis elements $e \in E\mathcal{U}^+$ by

$$T(e) := \left(\sum_{t(e)=o(e')} e' \right) - \bar{e},$$

the sum extending over all edges $e' \in E\mathcal{U}^+$ satisfying the condition given under the summation sign. We will consider the operator T as a linear map of \mathbf{Z} -modules, and not just of the corresponding \mathbf{Q} - or \mathbf{R} -vector spaces; especially when we write $\ker(1 - T)$ or $\text{im}(1 - T)$, we mean these as submodules of $\mathbf{Z}^{E\mathcal{U}^+}$.

This operator is represented by a $2m \times 2m$ -matrix A , whose entries are in $\{0, 1\}$. The operator T was considered by Hashimoto [13] and Bass [2] in connection with their study of the Ihara zeta function of a graph (see also: Stark and Terras [21]).

Let $\mathcal{O}_{\mathcal{U}}$ denote the Cuntz-Krieger algebra [11] associated to the matrix A . This is the universal C^* -algebra generated by $2m$ partial isometries $\{S_i\}_{i=1}^{2m}$ with orthogonal range projections, subject to the relations $S_i^* S_i = \sum A_{ij} S_j S_j^*$.

We want to classify Cuntz-Krieger algebras that arise in such a way, up to strict isomorphism (denoted \cong in this paper). Although it seems at first that quite a few algebras are possible, we will see that this turns out not to be the case.

First consider the weaker equivalence relation on such algebras given by (strong) Morita equivalence (denoted \sim) and stable isomorphism (also denoted \sim in this paper, since the notions are equivalent in our setting). Then Kumjian and Pask [14] have proven that $\mathcal{O}_{\mathcal{U}}$ is Morita equivalent to a boundary operator algebra. More specifically, let \mathcal{T} denote the universal covering tree of \mathcal{U} . Then $\mathcal{U} = \mathcal{T}/\Gamma$ for Γ a free group of rank $g =$ the first Betti number of \mathcal{U} , and $\mathcal{O}_{\mathcal{U}} \sim C^*(\partial\mathcal{T}) \rtimes \Gamma$, where the right hand side algebra, which we call a *boundary operator algebra*, only depends on g . As a matter of fact, since a result of Rørdam [20] implies that K_0 is a full invariant for the stable isomorphism class of Cuntz-Krieger algebras and stable isomorphism and Morita equivalence are the same for such algebras, this result can be used to compute the K -theory of $\mathcal{O}_{\mathcal{U}}$ by calculating it for one example graph \mathcal{U} , and this is done by Robertson in [18] in an operator algebraic way.

Note that Rørdam has also proven that $K_0(\mathcal{O}_{\mathcal{U}})$ together with the position of the unit in $K_0(\mathcal{O}_{\mathcal{U}})$ is a full invariant for the strict isomorphism type of $\mathcal{O}_{\mathcal{U}}$. We do not see how one can follow the position of the unit through the proof of Kumjian and Pask. Our aim in this paper is twofold. First, we want to recompute the K -theory of \mathcal{U} directly in a graph theoretical way, thus avoiding the boundary operator algebra. Our proof will involve relating the Smith Normal Form reduction of $1 - A$ to contraction of non-loops in the graph. The proof of our second result will rely heavily on this technique. Although knowing K_0 for a Cuntz-Krieger algebra allows one to compute K_1 , we also exhibit a *direct* relation between K_1 and graph homology. The result is

Theorem 1 *Let \mathcal{U} denote a finite graph with first Betti number $g \geq 1$. Then*

$$K_0(\mathcal{O}_{\mathcal{U}}) \cong \mathbf{Z}^g \oplus \mathbf{Z}/(g - 1)\mathbf{Z}.$$

Furthermore, for $g \geq 2$, $K_1(\mathcal{O}_{\mathcal{U}})$ is naturally isomorphic to Z_1 , the space of cycles on \mathcal{U} , so $K_1(\mathcal{O}_{\mathcal{U}}) \cong \mathbf{Z}^s$. As a matter of fact, $K_1(\mathcal{O}_{\mathcal{U}})$ is isomorphic to $\ker(1 - T)$, and we have an isomorphism

$$\begin{aligned} \varphi: Z_1 &\rightarrow \ker(1 - T), \\ \left[\sum e \right] &\mapsto \sum e - \sum \bar{e}. \end{aligned}$$

Hence for $g \geq 2$, the Morita equivalence and stable equivalence type of $\mathcal{O}_{\mathcal{U}}$ only depends on g . On the other hand, if $g = 1$, then $\ker(1 - T)$ is spanned by $\varphi([c])$ for c a representative for a fundamental cycle, and $c + \sum l$, where the sum is over all edges l outside c that point away from c .

The result can now be used, conversely, to compute the K -theory of boundary operator algebras. One can also deduce some results about the operator T from this theorem. In particular, $1 - T$ has kernel $K_1(\mathcal{O}_{\mathcal{U}})$, $\mathbf{Z}^{|E_{\mathcal{U}}|+1}/\text{im}(1 - T) \cong K_0(\mathcal{O}_{\mathcal{U}})$, and one recovers the computation of the rank of $1 - T$ from [2] for both $g \geq 2$ and $g = 1$.

Secondly, we want to study the strict isomorphism class of $\mathcal{O}_{\mathcal{U}}$ for a general graph \mathcal{U} . This problem doesn't seem to have been dealt with in the literature. By Rørdam's classification results, it is known to depend upon the position of the unit in K_0 . Again from the boundary operator algebra point of view, Robertson showed in [18] that the unit of $C^*(\partial\mathcal{T}) \rtimes \Gamma$ is of exact order $g - 1$. This is analogous to Connes's calculation in [5], Corollary 6.7, that the class of the identity $\mathbb{1}$ in $K_0(A)$ has order $g - 1$, where $A = C(\mathbf{P}^1(\mathbf{R})) \rtimes \Gamma$, with Γ a torsion-free cocompact lattice in $PGL(2, \mathbf{R})$, and g the genus of the Riemann surface uniformized by Γ .

Here, we use purely graph theoretical considerations to calculate the exact position of the unit in $K_0(\mathcal{O}_{\mathcal{U}})$ for a general graph \mathcal{U} . Let us denote by (a, b) the greatest common divisor of two integers a and b .

Theorem 2 *Let \mathcal{U} denote a graph with first Betti number $g \geq 2$. Then the image of the unit of $\mathcal{O}_{\mathcal{U}}$ in $K_0(\mathcal{O}_{\mathcal{U}}) \cong \mathbf{Z}^g \oplus \mathbf{Z}/(g - 1)\mathbf{Z}$ has (finite) order*

$$\frac{g - 1}{(g - 1, |V_{\mathcal{U}}|)},$$

where $|V_{\mathcal{U}}|$ is the number of vertices of \mathcal{U} . In particular, the class of the unit in K_0 is annihilated by the Euler characteristic $g - 1$ of \mathcal{U} . Furthermore, every possible strict isomorphism type of an operator algebra with K_0 -group of the given type occurs as graph C^ -algebra for a stable graph; strict isomorphism is not a homotopy invariant for graphs; and $\mathcal{O}_{\mathcal{U}}$ is only strictly isomorphic to a boundary operator algebra of genus g if the number of vertices in \mathcal{U} is coprime to the Euler characteristic $g - 1$ of \mathcal{U} .*

We hope that the purely combinatorial considerations leading to these theorems will help in understanding the higher dimensional analogues of these results (for group actions on buildings) as considered by Robertson in [17, 18]. For some further results, see Vdovina [23].

Since these Cuntz-Krieger algebras are related to topological Markov chains [11], the results can also be read in dynamical terms as saying something about the Bowen-Franks invariants [3] of certain subshifts.

Our results arose from trying to answer the following question in algebraic geometry: one can associate a spectral triple to a Mumford curve X (cf. [7]), in which the operator algebra is the boundary operator algebra for the Schottky group. In this construction, one can replace that algebra by the graph C^* -algebra $\mathcal{O}_{\mathcal{U}}$ for \mathcal{U} the stable reduction of X . Does this new algebra capture more geometrical information about X ? See Remark 2.18 below for some more details. Also, strict isomorphism is important when a C^* -algebra is extended to a spectral triple (Connes [6], compare [22]).

We only consider finite graphs. It might be interesting to extend the methods to row-finite or locally finite graph C^* -algebras.

2 Stable Isomorphism Type

2.1 (Set-up) Let \mathcal{U} denote a graph (possibly with loops, multiple edges, and sinks) with m (geometrical) edges and of first Betti number (“cyclomatic number”) g , viz., g equals the number of independent loops, which equals the number of edges outside a spanning tree. We will mostly be considering the case where $g \geq 2$, but we will comment upon what happens for $g = 1$ at the appropriate place. In this section, we will prove Theorem 1 from Sect. 1.

2.2 (The operator T) We make a new (oriented) graph \mathcal{U}^+ , whose vertices equal those of \mathcal{U} , and whose edges consist of all edges of \mathcal{U} with both possible orientations. We denote the two edges of \mathcal{U}^+ corresponding to an edge e of \mathcal{U} by e and \bar{e} . Let $o(e)$ and $t(e)$ denote the origin, respectively, terminal vertex of an oriented edge e of \mathcal{U}^+ . Let $E^{\mathcal{U}^+}$ denote the set of edges of the oriented graph \mathcal{U}^+ , and let $\mathbf{Z}^{E^{\mathcal{U}^+}}$ denote the free \mathbf{Z} -module spanned by these edges. On this space, we consider the linear operator $T : \mathbf{Z}^{E^{\mathcal{U}^+}} \rightarrow \mathbf{Z}^{E^{\mathcal{U}^+}}$ defined on the basis elements $e \in E^{\mathcal{U}^+}$ by

$$T(e) := \left(\sum_{t(e)=o(e')} e' \right) - \bar{e},$$

the sum extending over all edges $e' \in E^{\mathcal{U}^+}$ satisfying the condition given under the summation sign. We will consider the operator T as a linear map of \mathbf{Z} -modules, and not just of the corresponding \mathbf{Q} - or \mathbf{R} -vector spaces; especially when we write $\ker(1 - T)$ or $\text{im}(1 - T)$, we mean these as submodules of $\mathbf{Z}^{E^{\mathcal{U}^+}}$.

This operator is represented by a $2m \times 2m$ -matrix A , whose entries are in $\{0, 1\}$. The operator T was considered by Hashimoto [13] and Bass [2] in connection with their study of the Ihara zeta function of a graph (see also: Stark and Terras [21]). Although not strictly necessary for the main argument of this paper, we will comment upon this relation in 2.12 below.

2.3 (The Cuntz-Krieger algebra) Let $\mathcal{O}_{\mathcal{U}}$ denote the Cuntz-Krieger algebra associated to the matrix A . This is the universal C^* -algebra generated by $2m$ partial isometries $\{S_i\}_{i=1}^{2m}$ with orthogonal range projections, subject to the relations

$$S_i^* S_i = \sum A_{ij} S_j S_j^*.$$

We want to classify Cuntz-Krieger algebras that arise in such a way. Although it seems at first that quite a few algebras are possible, we will see that this turns out not to be the case.

First some preliminary observations. Since the reduction graph has cyclomatic number $g \geq 2$, it has a vertex of valency ≥ 2 , so A is not a permutation matrix (actually, the only exception would be a graph consisting of a single vertex and a single loop, which corresponds to a Tate elliptic curve). The matrix A is also irreducible, i.e., all entries of A^m for m sufficiently large are positive. This is clear from the interpretation of $A^m_{x,y}$ as the number of paths of length m between the end point of x and the origin of y , since \mathcal{U} is a finite and connected graph. Therefore, $\mathcal{O}_{\mathcal{U}}$ is a simple algebra [11, Theorem 2.14]. By a result of Rørdam [20], two different such algebras for graphs \mathcal{U} and \mathcal{U}' are stably isomorphic (i.e., isomorphic after tensoring with compact operators) if and only if $K_0(\mathcal{O}_{\mathcal{U}}) \cong K_0(\mathcal{O}_{\mathcal{U}'})$. Furthermore, they are isomorphic if and only if there is a group isomorphism between the K_0 -groups that maps the class of the unit of $\mathcal{O}_{\mathcal{U}}$ to that of $\mathcal{O}_{\mathcal{U}'}$. The image of the unit in the abstract group $K_0(\mathcal{O}_{\mathcal{U}})$ up to abstract group automorphisms is called “the position of the unit” in the literature.

2.4 Remark The notions of stable isomorphism and (strong) Morita equivalence are equivalent for algebras of the form $\mathcal{O}_{\mathcal{U}}$, since they are separable, see Brown-Green-Rieffel [4].

2.5 (K -theory computations) We hence start by computing the K -theory of $\mathcal{O}_{\mathcal{U}}$. Theorem 5.3 in [11] says that $K_0(\mathcal{O}_{\mathcal{U}}) = \mathbf{Z}^n / (1 - A')\mathbf{Z}^n$, where n is the dimension of the matrix A . Equivalently, in terms of the operator T , $K_0(\mathcal{O}_{\mathcal{U}}) = \mathbf{Z}^{E^{\mathcal{U}^+}} / \text{im}(1 - T)$. Note that it is irrelevant whether one works with A' or A in this formula. We now claim the following:

2.6 Proposition *If \mathcal{U} is a graph with cyclomatic number $g \geq 1$, then*

$$K_0(\mathcal{O}_{\mathcal{U}}) \cong \mathbf{Z}^g \oplus \mathbf{Z}/(g - 1)\mathbf{Z};$$

in particular, the stable isomorphism type of $\mathcal{O}_{\mathcal{U}}$ only depends on g if $g \geq 2$.

2.7 Remark As follows from [20, 4.3], any abelian group can be the K -group of a Cuntz-Krieger algebra associated to the vertex adjacency matrix of a graph. In light of this, the above result might come as a surprise.

Proof The proof has two parts. We use the following notation: for two square matrices M, N of size $n \times n$, we write $M \sim N$ if there exist matrices $X, Y \in \text{GL}_n(\mathbf{Z})$ such that $XYM = N$. We also write 1_n for the identity matrix of size $n \times n$.

2.6.1 We start by proving this for the case where \mathcal{U} has only one vertex, so $m = g$, see Fig. 1.

The matrix $A - 1$ is of the form $A - 1 = \begin{pmatrix} B & B \\ & B \end{pmatrix}$, where B is a $g \times g$ matrix with zeros along the diagonal and 1 everywhere else.

Obviously, $A - 1 \sim \begin{pmatrix} B & 0 \\ & 0 \end{pmatrix}$. By subtracting the first column from all other in B , and then adding all rows to the last, then adding all columns to the first, we find $B \sim \text{diag}(g - 1, 1, \dots, 1)$. It follows that

$$\text{im}(1 - A) \cong \text{im}(\text{diag}(1, \dots, 1, g - 1, \underbrace{0, \dots, 0}_g)) = \mathbf{Z}^{2m-g-1} \oplus (g - 1)\mathbf{Z},$$

and the result follows.

2.6.2 We now consider what happens to the operator when we contract an edge.

Fig. 1 The genus g graph with one vertex

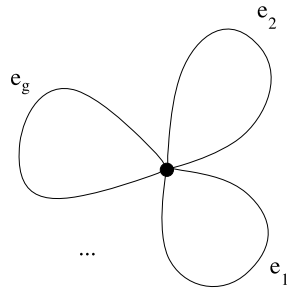
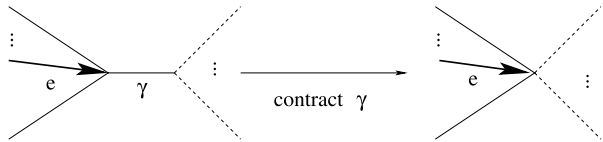


Fig. 2 Contracting the edge γ



Claim If \mathcal{U}' is the graph obtained from contracting a single non-loop edge in \mathcal{U} , and A' and A are the matrices of the respective T -operators on these graphs, then $A - 1 \sim \begin{pmatrix} 1_2 & 0 \\ 0 & A' - 1 \end{pmatrix}$.

Suppose that γ is the edge that is contracted. Suppose γ and $\bar{\gamma}$ correspond to the first and second row of $A - 1$ respectively. Perform the following elementary row operations on $A - 1$: add the first row to every row corresponding to an edge e whose terminal point is the origin of γ ; and add the second row to every row corresponding to an edge e whose terminal point is the origin of $\bar{\gamma}$.

Observe the following facts about the resulting matrix:

- (1) Since γ is not a loop, the left upper corner is -1_2 .
- (2) The cofactor of the first two rows is equal to $A' - 1$; because of the transformation that we did, see Fig. 2: the outgoing (dashed) edges of γ become added to the outgoing edges of e .
- (3) All entries in the first two columns from the third row on are zero; because in the original matrix, there was a 1 at location (e, γ) exactly if e flows into γ , but then our operation has added the -1 from location (γ, γ) to it — and similarly for the second row of $(e, \bar{\gamma})$.

Now any possible non-zero entries in the first two rows from the third column on can be removed by adding the first or second column to the corresponding column, and this will not affect any other entry. In the end, we find a matrix as in the Claim above.

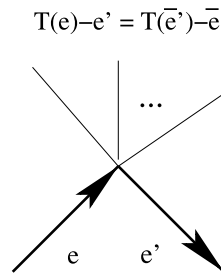
Given a general graph \mathcal{U} , we contract all non-loops one after the other. By the claim, we are left with a matrix of the form $1 - A \sim \begin{pmatrix} -1_{2m-g} & 0 \\ 0 & A' - 1 \end{pmatrix}$, where A' corresponds to a graph with only loops, i.e. with one vertex. But then 2.6.1 can be applied to $A' - 1$, and we find in the end that

$$1 - A \sim \text{diag}(g - 1, 1, \dots, 1, \underbrace{0, \dots, 0}_g).$$

As we remarked above, the algebra $\mathcal{O}_{\mathcal{U}}$ is simple for $g \geq 2$, so K_0 classifies it up to stable isomorphism. This finishes the proof of Proposition 2.6. □

2.8 (K_1 and graph homology) Since K_1 of a Cuntz-Krieger algebra is isomorphic to the torsion free part of K_0 (see [20]), the same proof shows that $K_1(\mathcal{O}_{\mathcal{U}}) \cong \ker(1 - T) \cong \mathbf{Z}^g$

Fig. 3 A basic equation



for $g \geq 2$; and that $K_1(\mathcal{O}_{\mathcal{U}}) \cong \ker(1 - T) \cong \mathbf{Z}^2$ for $g = 1$. We now make a digression to interpret this result in terms of graph homology.

Let $Z_1 = H_1(\mathcal{U}, \mathbf{Z})$ denote the space of (integral) cycles on \mathcal{U} , i.e., the kernel of the boundary map $\mathbf{Z}^{E_{\mathcal{U}}} \rightarrow \mathbf{Z}^{V_{\mathcal{U}}}$. We observe that one can decompose the space $\mathbf{Z}^{E_{\mathcal{U}^+}}$ as follows:

$$\mathbf{Z}^{E_{\mathcal{U}^+}} = \mathbf{Z}^{S_{\mathcal{U}}} \oplus Z_1,$$

where $S_{\mathcal{U}} = E_{\mathcal{U}^+} - \{\bar{\gamma}_1, \dots, \bar{\gamma}_g\}$ for a collection of edges γ_i outside a fixed spanning tree (so $S_{\mathcal{U}}$ consists of the edges of a spanning tree with both orientations, and one oriented edge for each geometrical edge outside that spanning tree).

As was observed by Jakub Byszewski, there is a natural injective group homomorphism $\varphi : Z_1 \hookrightarrow \ker(1 - T)$ given as follows: if a cycle $[c]$ is represented as $\sum_{e \in I} e$, we define

$$\varphi([c]) = \sum_{e \in I} e - \sum_{e' \in I} \bar{e}'. \tag{1}$$

This does not depend upon the choice of a representative for the cycle c . Let us check that, indeed, $(T - 1)\varphi([c]) = 0$. Fix $e \in I$, and suppose $e' \in I$ is the (unique) edge in I such that $t(e) = o(e')$. Observe the basic equation $T(e) - e' = T(\bar{e}') - \bar{e}$, that is illustrated in Fig. 3.

We can therefore calculate

$$\begin{aligned} (T - 1)\varphi([c]) &= (T - 1)\left(\sum e - \sum \bar{e}\right) \\ &= \sum T(e) - \sum e - \sum T(\bar{e}) + \sum \bar{e} \\ &= \sum T(\bar{e}') - \sum \bar{e} + \sum e' - \sum T(\bar{e}) - \sum e + \sum \bar{e} \\ &= 0. \end{aligned}$$

The map φ is injective: choose a basis for the space of loops that doesn't contain any edges in sinks, then if $e \in I$, $\bar{e} \notin I$, so the independence of e and \bar{e} in the image implies injectiveness.

Below is the promised geometrical interpretation of the kernel of T :

2.9 Lemma *For $g \geq 2$, the kernel of $1 - T$ is isomorphic to the cycle space Z_1 via the isomorphism φ defined in (1):*

$$K_1(\mathcal{O}_{\mathcal{U}}) = \ker(1 - T) = Z_1 \cong \mathbf{Z}^g.$$

Proof From 2.6, we know that the kernel of $1 - T$ has the same rank ($= g$) as Z_1 . Alternatively, this follows independently from computations with the graph zeta function as in 2.12

below. Since φ is also injective, the image is of the form $\bigoplus_{i=1}^g a_i \mathbf{Z}$ for some $a_i \in \mathbf{Z}_{>0}$. We should prove that $a_i = 1$ for all i .

An end of \mathcal{U} is a connected contractible subgraph of \mathcal{U} that shares exactly one vertex with its complement in \mathcal{U} .

Given any edge not belonging to an end, choose a loop γ in which e occurs with multiplicity one, and in which \bar{e} doesn't occur. Then e occurs in $\varphi(\gamma)$ with multiplicity one.

Now suppose e belongs to an end. Since φ is bijective after tensoring with \mathbf{Q} , for any $\sum a_e e \in \ker(1 - T)$ we can find $a, b \in \mathbf{Z}$ and a loop γ such that $a(\sum a_e e) = b\varphi(\gamma) = b(\sum e - \sum \bar{e})$. Since e is in an end, it is not in the support of this last sum, so it cannot occur in $\sum a_e e$ either.

The conclusion is that edges in ends don't occur in $\ker(1 - T)$, and any other edge occurs with multiplicity one in some element of $\text{im}(\varphi)$. This would not be the case if some $a_i > 1$. □

2.10 Remark For graphs without ends, this result is also found in [19], where it is applied to prove the following: let \mathcal{T} denote the universal covering of \mathcal{U} and Γ its fundamental group. The group of Γ -invariant integral valued measures on clopen sets of the boundary $\partial\mathcal{T}$ is isomorphic to Z_1 .

2.11 Remark The map φ gives a natural isomorphism $Z_1 \rightarrow \ker(1 - T)$. On the other hand, Cuntz has constructed a natural isomorphism $\ker(1 - T) \rightarrow K_1(\mathcal{O}_{\mathcal{U}})$ (cf. [20, p. 33] and [10, 3.1]). It would be interesting to give a direct formula for the map $Z_1 \rightarrow K_1(\mathcal{O}_{\mathcal{U}})$ that relates graph homology and operator K_1 . From the proof of 3.1 in Cuntz [10], it is not hard to obtain an explicit form of the map $Z_1 \rightarrow K_1(\bar{\mathcal{O}}_{\mathcal{U}})$, where $\bar{\mathcal{O}}_{\mathcal{U}}$ denotes the stabilization of $\mathcal{O}_{\mathcal{U}}$.

2.12 (Relation to the Ihara graph zeta function) We can (independently of the above) compute the rank of the operator $1 - T$ acting on the vector space $\mathbf{Q}^{E_{\mathcal{U}^+}}$. For that, we use the relation between the characteristic polynomial of T and the Ihara zeta function of the graph. Theorem 4.3 from [2] (applied to the trivial representation) implies that

$$\det(1 - uT) = \zeta_{\mathcal{U}}^{-1}(s) = (1 - u^2)^{g-1} \det(\Delta(u))$$

for Δ the graph Laplace operator. Since $g \geq 2$, the universal covering tree of \mathcal{U} is not a linear tree, and *loc. cit.*, Theorem 5.10.b(i) says that $\Delta(u) = (1 - u)D^+(u)$ with $D^+(1) \neq 0$, so

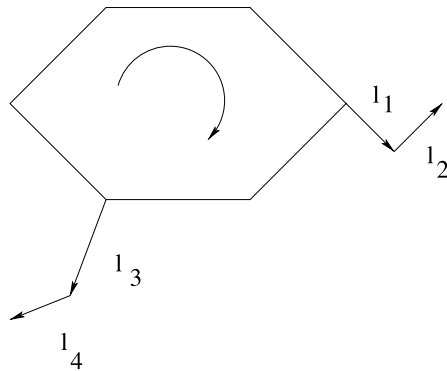
$$\text{ord}_{u=1} \det(1 - Tu) = g.$$

Now $x \in \ker(1 - T)$ if and only if x belongs to the eigenspace of T for the eigenvalue $+1$, and we have just seen that this space is g -dimensional. Therefore, the rank of $1 - T$ is $2m - g$. From $Z_1 \hookrightarrow \ker(1 - T)$, and this result, one finds back Lemma 2.9.

If $g = 1$, the universal covering of the graph is a linear tree with a cyclic action of \mathbf{Z} . This corresponds to Tate's uniformization of an elliptic curve with totally split multiplicative reduction. Then [2, 5.11.10] implies that $1 - T$ has rank $2m - 2$, which is compatible with 2.6.

We can now reverse the logic and use the above (independent) calculation of the Smith Normal Form and kernel of $1 - T$ to deduce facts about the Bass-Hashimoto T -operator and the graph zeta function:

Fig. 4 Generators for $\ker(1 - T)$ are $\varphi(c)$ and $c + \sum_{i=1}^4 l_i$



2.13 Corollary *The multiplicity of the eigenvalue $+1$ for the operator T , which equals the pole order at $+1$ of the Ihara zeta function of \mathfrak{U} , is g if $g \geq 2$ and 2 if $g = 1$. The value $+1$ is a simple zero of the graph Laplacian $\Delta(u)$ if $g \geq 2$ and a double zero if $g = 1$.*

The following is also very easy to see:

2.14 Proposition *Suppose that \mathfrak{U} has Betti number $g = 1$. Let c denote a (fundamental) cycle in \mathfrak{U} . Then two independent elements of $\ker(1 - T)$ are given by $\varphi([c])$ and $c + \sum l$, where the sum is over all edges l outside c that point away from c , see Fig. 4 for an example.*

2.15 Remark As noticed in the introduction, we can revert the logic of this section to apply operator algebra K -theory to graph theory as follows: given a graph \mathfrak{U} of cyclomatic number $g \geq 2$, write $\mathfrak{U} = \Gamma \setminus \mathcal{T}$, where \mathcal{T} is the universal covering tree of \mathfrak{U} and Γ is a free group on g generators. Then Kumjian and Pask [14] have shown that $\mathcal{O}_{\mathfrak{U}} \cong C(\Lambda_{\Gamma}) \rtimes \Gamma$ in the sense of strong Morita equivalence. By the Brown-Green-Rieffel result from 2.4, we find that the stable isomorphism class of $\mathcal{O}_{\mathfrak{U}}$ only depends on g .

In [18], Robertson showed that $C(\Lambda_{\Gamma}) \rtimes \Gamma$ is strictly isomorphic to the Cuntz-Krieger algebra associated to the stable graph in Fig. 1. We can therefore compute $K_0(\mathcal{O}_{\mathfrak{U}})$ for any chosen \mathfrak{U} , and doing this for the “flower” implies that $K_0(\mathcal{O}_{\mathfrak{U}})$ is as expected. This, in its turn, implies the results about the image and rank of $1 - T$, independently of graph theoretical considerations.

2.16 Remark The proof in [18] includes the calculation of the K -theory of the Cuntz-Krieger algebra for Fig. 1, but directly “on the boundary”, whereas our proof takes place on the reduction graph. This can be seen as another manifestation of a “holography principle”, by which information on the boundary can be equivalently expressed on the tree itself, cf. [15]. It would be very interesting to extend this kind of calculation to higher dimensional buildings, cf. [8, 17].

2.17 (Analogue for Riemann surfaces) The analogue of this result in the classical theory of global uniformization of Riemann surfaces is as follows. Let $A = C(\mathbf{P}^1(\mathbf{R})) \rtimes \Gamma$, with Γ a torsion-free cocompact lattice in $PGL(2, \mathbf{R})$, and g the genus of the Riemann surface uniformized by Γ . Then Anantharaman-Delaroche [1] proved that the K -theory of A is given by $K_0(A) = \mathbf{Z}^{2g+1} \oplus \mathbf{Z}/(2g - 2)\mathbf{Z}$. The proof is topological (via a Thom isomorphism), and although it follows that A is isomorphic to a Cuntz-Krieger algebra, there is no apparent direct link between the matrix of that algebra and some combinatorial structure on the Riemann surface, as is the case for the non-Archimedean theory.

2.18 (Analogue for Mumford curves) Let k be a non-Archimedean complete discretely valued field of mixed characteristic with absolute value $|\cdot|$. A projective curve X over k is called a *Mumford curve* if it is uniformized over K by a *Schottky group*. This means that there exists a free subgroup Γ in $PGL(2, K)$ of rank g , acting on \mathbf{P}_K^1 with *limit set* Λ_Γ such that X satisfies $X^{\text{an}} \cong \Gamma \backslash (\mathbf{P}_K^{1,\text{an}} - \Lambda_\Gamma)$ as rigid analytic spaces. Mumford [16] has shown that these conditions are equivalent to the existence of a stable model of X over the ring of integers \mathcal{O}_k of k whose special fiber consists only of rational components with double points over the residue field.

Suppose the ground field k is large enough so that the group Γ acts on the Bruhat-Tits tree of $PGL(2, k)$ without inversions. This is always possible by a finite extension of k if necessary. Let \mathcal{T}_Γ denote the subtree of the Bruhat-Tits tree spanned by geodesics connecting fixed points of hyperbolic elements in Γ . Then $\mathcal{U}_X := \mathcal{T}_\Gamma / \Gamma$ is a finite graph that is intersection dual to the stable reduction of the curve X . In particular, the cyclomatic number of \mathcal{U}_X equal the rank of Γ , which equals the genus g of X (cf. [16, Theorem 3.3]). Note that \mathcal{U}_X is allowed to have loops and multiple edges.

As is observed in [12, p. 124], any graph \mathcal{U} can occur as the stable reduction graph of a Mumford curve, as soon as \mathcal{U} is finite, connected, and every vertex which is not connected to itself is the origin of at least three edges. We call such a graph a *stable graph*.

In [7], a spectral triple was associated to X in which the operator algebra is the boundary operator algebra of Γ . For this algebra, the Mumford curve plays the rôle of the Riemann surface in the results of 2.17. The current work is inspired by the question whether finer invariants of X are detected by replacing this algebra with $\mathcal{O}_\mathcal{U}$ for $\mathcal{U} = \mathcal{U}_X$ the stable reduction of X . For some results for Schottky uniformization of Riemann surfaces, see [9].

3 Strict Isomorphism Type

3.1 The (not only stable) isomorphism type of $\mathcal{O}_\mathcal{U}$ is determined by the image of the unit of that algebra in its K_0 -group. Suppose that \mathcal{U} and \mathcal{U}' are two stable graphs with the same cyclomatic number $g \geq 2$. To ease notation, add a prime to any symbol pertaining to \mathcal{U}' . Since we have already shown that $\mathcal{O} := \mathcal{O}_\mathcal{U}$ and $\mathcal{O}' := \mathcal{O}_{\mathcal{U}'}$ are stably isomorphic, to prove that they are actually isomorphic, by Rørdam [20] it suffices to decide whether or not there exists an isomorphism of abelian groups $K_0(\mathcal{O}) \cong K_0(\mathcal{O}')$ that maps the class of the unit 1 of \mathcal{O} to that of the unit $1'$ of \mathcal{O}' . We can now recast this in terms of linear algebra as follows: in our setting, an isomorphism of K_0 -groups is given by an automorphism of $\mathbf{Z}^g \oplus \mathbf{Z} / (g - 1)\mathbf{Z}$. Now Cuntz's isomorphism $\varphi : K_0(\mathcal{O}) \rightarrow \mathbf{Z}^n / (1 - A')\mathbf{Z}^n$ is explicitly given by

$$\varphi([1]) = \varphi\left(\left[\sum e_i e_i^*\right]\right) = \sum \varphi([e_i e_i^*]) = (1, \dots, 1).$$

Hence it suffices to check whether there is an isomorphism $\mathbf{Z}^n / (1 - A') \cong \mathbf{Z}^{n'} / (1 - (A')')$ that fixes the class of $1 = (1, \dots, 1)$.

We now look at two examples to show that the isomorphism type can indeed vary.

3.2 Example For a flower as in Fig. 1, any preimage of 1 by $1 - T$ has $g - 1$ in the denominator; this is easily seen, since the matrix B from 2.6.1 is invertible. Hence the image of 1 in $K_0(\mathcal{O}_\mathcal{U})$ is an element of exact order $g - 1$.

3.3 Example For a graph consisting of two vertices that are connected by $g + 1$ edges, the matrix of $1 - T$ is a block matrix of the form $\begin{pmatrix} -1_{g+1} & B \\ B & -1_{g+1} \end{pmatrix}$. It is easy to check that 1 is the

image of

$$\left(-1, \underbrace{\frac{2}{g-1}, \dots, \frac{2}{g-1}}_g, \frac{g+1}{g-1}, \underbrace{0, \dots, 0}_g\right).$$

In particular, if g is odd, $(g - 1)/2 \cdot \mathbb{1} \in \text{im}(1 - T)$, so the order of $\mathbb{1}$ in $K_0(\mathcal{O}_{\mathcal{U}})$ divides $(g - 1)/2$. One can check that if g is odd, $\mathbb{1}$ has order exactly equal to $(g - 1)/2$ in $K_0(\mathcal{O}_{\mathcal{U}})$, whereas if g is even, the order is exactly equal to $g - 1$.

3.4 Remark Connes showed in [5, Corollary 6.7], that in the setting of Riemann surfaces, and sticking to the notations of Remark 2.17, the class of the identity $\mathbb{1}$ in $K_0(A)$ has order $2g - 2$, the Euler characteristic of the Riemann surface. This fits with the results of [1] cited in 2.17, in the sense that the class of $\mathbb{1}$ generates the subgroup $\mathbf{Z}/(2g - 2)\mathbf{Z}$ of $K_0(A)$.

3.5 (Proof of Theorem 2) Let λ denote the minimal positive integer such that the equation

$$(1 - A) \cdot x = \lambda \mathbb{1}$$

has a solution in an integral vector $x \in \mathbf{Z}^{2m}$; then, if it exists, λ is the exact order of $\mathbb{1}$ in $K_0(\mathcal{O}_{\mathcal{U}})$. We know from 2.6.2 that there exist $X, Y \in GL_{2m}(\mathbf{Z})$ such that

$$X(1 - A)Y = \text{diag}(1, \dots, 1, g - 1, \underbrace{0, \dots, 0}_g),$$

where X (resp. Y) is given by the row (resp. column) operations performed in the course of proving 2.6. In particular, if $(1 - A) \cdot x = \lambda \mathbb{1}$, then

$$\text{diag}(1, \dots, 1, g - 1, \underbrace{0, \dots, 0}_g) \cdot y = \lambda X \cdot \mathbb{1} \tag{2}$$

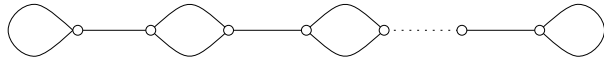
has an integral solution $y \in \mathbf{Z}^{2m}$. The equations corresponding to the first $2m - g - 1$ rows of (2) obviously have a solution in integers for any integral λ . The only question that remains is whether the remaining $g + 1$ equations have an integral solution.

We calculate these equations by finding the entries of $X \cdot \mathbb{1}$ by performing the row operations specified by X on $\mathbb{1}$. At the start, $\mathbb{1}$ has entry 1 at every place. The general Smith Normal Form reduction process in 2.6 starts in 2.6.2. The row operations that were performed in 2.6.2 correspond to the contraction of all non-loop edges, so that in the end, only one vertex remains. This operation can be seen as collapsing all vertices to a given one, i.e., to reach the final form of the matrix, we have to collapse $|V\mathcal{U}| - 1$ vertices. To each of the contractions of an edge corresponds adding that row to any ingoing edge of the source of that edge. Hence each of the edges (then loops) that remain after the complete contraction process in 2.6.2 has $|V\mathcal{U}| - 1$ rows added to it. The result of this operation on $\mathbb{1}$ is that it has been transformed into

$$\mathbb{1} \rightsquigarrow (\dots, \underbrace{|V\mathcal{U}|, \dots, |V\mathcal{U}|}_{2g}).$$

Then the row operations in reducing the matrix from 2.6.1 correspond to subtracting from the last g rows the corresponding of the first g rows, and then adding all rows to the

Fig. 5 The stable graph \mathcal{U}_g with $2g - 2$ vertices



$(2m - g)$ -th one. After the reduction in 2.6.1, one finally arrives at

$$\mathbb{1} \rightsquigarrow X \cdot \mathbb{1} = (\dots, g \cdot |V\mathcal{U}|, \underbrace{0, \dots, 0}_g).$$

The last g equations in (2) therefore also admit a solution. Now the final equation to consider is that on the $(g + 1)$ -to-last row:

$$? \exists z \in \mathbf{Z}: (g - 1)z = \lambda \cdot g \cdot |V\mathcal{U}|,$$

and one sees that the minimal integral λ for which a solution exists is

$$\lambda = \frac{g - 1}{(g - 1, |V\mathcal{U}|)}.$$

This proves the first part of the theorem, and proves in particular that $g - 1$ annihilates the image of $\mathbb{1}$, that hence has finite order.

There is an automorphism of the group $\mathbf{Z}^g \oplus \mathbf{Z}/(g - 1)\mathbf{Z}$ that carries an element of $\mathbf{Z}/(g - 1)\mathbf{Z}$ to another exactly if these elements have the same order in $\mathbf{Z}/(g - 1)\mathbf{Z}$.

To see that any isomorphism type occurs for a stable graph \mathcal{U} of genus $g \geq 2$, we will show that one can realise any number of vertices m with $1 \leq m \leq g - 1$ for such a stable graph. We will show that one can find a stable graph \mathcal{U}_g of genus g with $2g - 2$ vertices. Collapsing these vertices one after the other keeps the graph stable of the same genus, decreasing the number of vertices by one every time. The graph \mathcal{U}_g is as in Fig. 5. It has $2g - 2$ vertices. The two terminal vertices have a loop attached to it, and then consecutive vertices are connected alternatingly by a simple edge and by two edges. The genus of \mathcal{U}_g is easily seen to be g , and \mathcal{U}_g is stable.

Since for fixed g , all contractions of \mathcal{U}_g are homotopic but the isomorphism type of the corresponding operator algebra varies, strict isomorphism is not a homotopy invariant.

Since the boundary operator algebra of genus g is strictly isomorphic to $\mathcal{O}_{\mathcal{U}}$ with \mathcal{U} a “flower” as in Fig. 1 (see Robertson [18]) and in K_0 of the latter algebra, $\mathbb{1}$ has order $g - 1$, it follows that only algebras $\mathcal{O}_{\mathcal{U}}$ for graphs with

$$\frac{g - 1}{(g - 1, |V\mathcal{U}|)} = g - 1,$$

i.e., graphs for which $(g - 1, |V\mathcal{U}|) = 1$, are strictly isomorphic to boundary operator algebras. This finishes the proof of Theorem 2.

3.6 Remark Note that by Euler’s formula,

$$\text{ord } \mathbb{1} = \frac{g - 1}{(g - 1, |V\mathcal{U}|)} = \frac{g - 1}{(g - 1, |E\mathcal{U}|)}.$$

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