**Exam Differential Geometry (mastermath) 2015/2016**

**Exercise 1.** In the lecture notes, during the discussion on symplectic forms and intrinsic torsion (more precisely, in the proof of Lemma 4.51, with a bit of change in the notation) it is claimed that for any symplectic vector space $(W, \omega)$ the sequence

$$\mathrm{Hom}(W, gl(W, \omega)) \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2 W, W) \xrightarrow{\partial_\omega} \Lambda^3 W^*$$

is exact in the middle, i.e. $\mathrm{Im}(\partial) = \mathrm{Ker}(\partial_\omega)$, where

$$\partial(\xi)(u, v) = \xi(u)(v) - \xi(v)(u),$$

$$\partial_\omega(\phi)(u, v, w) = \omega(\phi(u, v), w) + \omega(\phi(v, w), u) + \omega(\phi(w, u), v).$$

To show this, let $\phi \in \mathrm{Hom}(\Lambda^2 W, W)$ such that $\partial_\omega(\phi) = 0$. To find $\xi \in \mathrm{Hom}(W, gl(W, \omega))$ such that $\phi = \partial(\xi)$:

a. Show that for any $a, b \in \mathbb{R}$ one can find a unique $\xi_{a,b} \in \mathrm{Hom}(V, gl(V))$ such that

$$\omega(\xi_{a,b}(u)(v), w) = a \cdot \omega(\phi(u, v), w) + b \cdot \omega(\phi(u, w), v) \quad \forall\, u, v, w \in W.$$

b. For which $a$ and $b$ does one have

$$\phi(u, v) = \xi_{a,b}(u)(v) - \xi_{a,b}(v)(u) \;\forall\, u, v \in W?$$

c. For which $a$ and $b$ does one have $\xi_{a,b} \in \mathrm{Hom}(W, gl(W, \omega))$?

Finally, conclude that $\phi$ is in the image of $\partial$. (this exercise is worth 1.5 points)

**Exercise 2.** Let $V$ be an $n$-dimensional (real) vector space, $W \subset V$ a linear subspace and $\omega_W : W \times W \to \mathbb{R}$ a linear symplectic form on $W$. We consider

$$GL(V, W, \omega) = \{g \in GL(V) : g(W) \subset W, g|_W \in GL(W, \omega)\}.$$

Show that:

a. $GL(V, W, \omega)$ is a Lie group.

b. the Lie algebra of $GL(V, W, \omega)$, viewed as a subspace of $gl(V)$, is:

$$gl(V, W, \omega) = \{A \in gl(V) : A(W) \subset W, A|_W \in gl(W, \omega)\}.$$

c. We would like to understand the torsion space corresponding to $gl(V, W, \omega)$, $\mathcal{T} := \mathcal{T}(gl(V, W, \omega))$; recall that it is defined as the cokernel of

$$\partial : \mathrm{Hom}(V, gl(V, W, \omega)) \to \mathrm{Hom}(\Lambda^2 V, V),\; \partial(\xi)(u, v) = \xi(u)(v) - \xi(v)(u). \quad (*)$$

Consider the subspace $\mathrm{Hom}_W(\Lambda^2 V, V)$ of $\mathrm{Hom}(\Lambda^2 V, V)$ consisting of bilinear skew-symmetric maps $\phi : V \times V \to V$ with the property that, when restricted to $W \times W$, they take values in $W$. Define

$$\partial_\omega : \mathrm{Hom}_W(\Lambda^2 V, V) \longrightarrow \Lambda^3 W^*,$$

$$\partial_\omega(\phi)(u, v, w) = \omega(\phi(u, v), w) + \omega(\phi(v, w), u) + \omega(\phi(w, u), v).$$

Show that, for $\phi \in \mathrm{Hom}(\Lambda^2 V, V)$, the corresponding element $[\phi] \in \mathcal{T}$ is zero (i.e. $\phi$ is in the image of $(*)$ if and only if

$$\phi \in \mathrm{Hom}_W(\Lambda^2 V, V) \text{ and } \partial_\omega(\phi) = 0.$$

d. Remarking that $\mathrm{Hom}_W(\Lambda^2 V, V)$ is the kernel of the map

$$p_W : \mathrm{Hom}(\Lambda^2 V, V) \longrightarrow \mathrm{Hom}(\Lambda^2 W, V/W),\ p_W(\phi)(w_1, w_2) = \phi(w_1, w_2) \text{ mod } W,$$

deduce that one has a short exact sequence of vector spaces

$$0 \longrightarrow \Lambda^3 W^* \longrightarrow \mathcal{T} \xrightarrow{\overline{p}_W} \mathrm{Hom}(\Lambda^2 W, V/W) \longrightarrow 0.$$

(this exercise is worth 2 points, with 0.5 points per item)

**Exercise 3.** Assume that $\mathcal{F}$ is a vector sub-bundle of $TM$ and we look at "$k$-differential forms along $\mathcal{F}$" (for any $k$) by which we mean sections of the vector bundle $\Gamma(\Lambda^k \mathcal{F}^*)$. These are related to the usual $k$-forms on $M$ by the obvious restriction map

$$\Omega^k(M) \longrightarrow \Gamma(\Lambda^k \mathcal{F}^*), \quad \eta \mapsto \eta|_{\mathcal{F}}$$

which, pointwise, restricts a $k$-multilinear skew-symmetric form on $T_x M$ to one on $\mathcal{F}_x$. We would like talk about "DeRham differential along $\mathcal{F}$",

$$d_{\mathcal{F}} : \Gamma(\Lambda^k \mathcal{F}^*) \to \Gamma(\Lambda^{k+1} \mathcal{F}^*)$$

(defined for all $k$) such that it is compatible with the usual DeRham differential $d$ in the sense that

$$d_{\mathcal{F}}(\eta|_{\mathcal{F}}) = d(\eta)|_{\mathcal{F}} \quad \forall\, \eta \in \Omega^k(M).$$

Note that, since the restriction map is surjective, $d_{\mathcal{F}}$ will be unique if it exists. Show that $d_{\mathcal{F}}$ exists if and only if $\mathcal{F}$ is involutive. (this exercise is worth 1 point)

(Hint: look first in low degrees. And do not forget about the Koszul formula).

**Exercise 4.** An almost symplectic foliation on $M$ is a pair $(\mathcal{F}, \omega_{\mathcal{F}})$ where:

- $\mathcal{F} \subset TM$ is a vector sub-bundle of $M$.
- $\omega_{\mathcal{F}} \in \Gamma(\Lambda^2 \mathcal{F}^*)$ is pointwise a linear symplectic form i.e., at each $x \in M$, $\omega_x$ (a bi-linear form on $\mathcal{F}_x$) is non-degenerate.

We say that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation if:

- $\mathcal{F}$ is involutive (hence a foliation).
- $\omega_{\mathcal{F}}$ is closed along $\mathcal{F}$ i.e., using the $d_{\mathcal{F}}$ from the previous exercise, $d_{\mathcal{F}}(\omega_{\mathcal{F}}) = 0$.

Given an almost symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$, we say that a connection $\nabla$ on $TM$ is compatible with $(\mathcal{F}, \omega_{\mathcal{F}})$ if:

$$\nabla_X(V) \in \Gamma(\mathcal{F}) \quad \forall\, V \in \Gamma(\mathcal{F}),$$

$$L_X(\omega_{\mathcal{F}}(V, W)) = \omega_{\mathcal{F}}(\nabla_X(V), W) + \omega_{\mathcal{F}}(V, \nabla_X(W)) \quad \forall\, X \in \mathcal{X}(M), V, W \in \Gamma(\mathcal{F}).$$

Prove that that the existence of such a connection, which is also torsion free, implies that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation. (this exercise is worth 1 point)

**Exercise 5.** Finally, we look at (almost) symplectic foliations from the point of view of $G$-structures. We fix $n$ and $p$ and we consider almost symplectic foliations $(\mathcal{F}, \omega_{\mathcal{F}})$ of rank $p$ on $n$-dimensional manifolds $M$.

a. Describe a Lie group $G \subset GL_n$ (depending on $p$) such that such rank $p$ almost symplectic foliations correspond to $G$-structures on $M$. Explain the correspondence.

b. Show that linear $G$-structures on an $n$-dimensional vector space $V$ correspond to pairs $(W, \omega)$ where $W \subset V$ is a $p$-dimensional vector subspace and $\omega$ is a linear symplectic form on $W$.

c. Show that a connection $\nabla$ on $TM$ is compatible with $(\mathcal{F}, \omega_{\mathcal{F}})$ in the sense of the previous exercise if and only if it is compatible with the corresponding $G$-structure.

d. Explain how the previous exercise is a consequence of the general properties of $G$-structures.

e. Show that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation if and only if the intrinsic torsion of the corresponding $G$-structure vanishes.

f. Show that the vanishing of the intrinsic torsion implies integrability. Moreover, explain the meaning of integrability in this case (what is the local model?).

(this exercise is worth 5 points: 1.5 points for each of the items c. and e., and 0.5 points for each of the other items)

**Remark:** The points assigned to the various items do not reflect the difficulty of the items but their importance for evaluating you knowledge (for instance, the very last item is by far the most difficult one, but it is worth only 0.5 points).