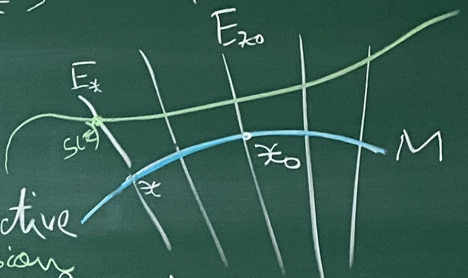
$\boxed{\text{Vector bundles}}$: 1 vector bundle E over $\underbrace{M}_{\text{a manifold}}$, of rank $r \in$

($\underbrace{\text{vector bundle}}_{\text{v.b.}}$)

→ intuitively: "smooth" collection of vector spaces E_x, one for each $x \in M$



→ precisely:

- a manifold E & $\pi : \overset{v}{\underset{\cap}{E}} \to \overset{x}{\underset{\cap}{M}}$ smooth surjective submersion
- each fiber $E_x := \pi^{-1}(x)$ is a real vector space

with local triviality condition:

$(\forall)\, x_0 \in M$, $\exists$ $U \subseteq M$ neighborhood of x_0, $\exists$ diffeomorphism

$h|_{E_x} : E_x \to \{x\} \times \mathbb{R}^r$ linear isom.

$$\begin{tikzcd} \pi^{-1}(U) =: E|_U \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

$x \in U$

s.t. $(\forall)\, x \in U$: h takes E_x to $\{x\} \times \mathbb{R}^r$ by linear isomorphisms

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M, of rank $r\in\mathbb{N}$)

manifold

$x\in M$

M

feomorphism

r

ms

②

$\boxed{\text{Morphisms}}$: Given vector bundles $\pi_E : E \to M$, $\pi_F : F \to M$
a morphism $u : E \to F$ vector bundle: any smooth map s.t. $u|_{E_x} : E_x \longrightarrow F_x$ $(\forall)\, x\in M$
and this is linear.

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

$\pi_F \circ u = \pi_E$

Call u an isomorphism: if furthermore, u = diffeomorphism

Ex: trivial vector bundle of rank r (r any $\in\mathbb{N}$) over M:

$$E = M\times\mathbb{R}^r, \quad \pi = pr_M : M\times\mathbb{R}^r \longrightarrow M$$
$$E_x = \{x\}\times\mathbb{R}^r$$

Ex: Any vector bundle $E \xrightarrow{\pi} M$, $U\subseteq M$ open $\Longrightarrow$
$\Longrightarrow$ the restriction of E to U: $E|_U := \pi^{-1}(U)$, vector bundle over U

Rk: local triviality: $(\forall)\, x\in M$, $\exists\, U$ neighborhood s.t. $E|_U$ is isom. to trivial $U\times\mathbb{R}^r$.

Examp

Exam

3

Sections

Given a v.b. E over M, with projection $\pi : E \to M$

a section of E: a smooth map

$$s : M \to E \text{ s.t. } \underbrace{s(x) \in E_x \ (\forall)\, x \in M}_{(\Leftrightarrow\ \pi \circ s = id_M)}$$

Ex: The zero section $0 : M \to E$, $x \longmapsto 0_x$ = the zero/origin of the vector space E_x.

Notations: $\Gamma(M,E) = \Gamma(E)$:= the collection of all such sections

For $U \subseteq M$ open. $\Gamma(U,E) = \Gamma(E|_U)$ local sections of E (over U).

Rk: $(s_1+s_2)(x) := s_1(x) + s_2(x) \in E_x$

$(f \cdot s)(x) := \underset{\mathbb{R}}{\underset{\cap}{f(x)}} \cdot \underset{E_x}{\underset{\cap}{s(x)}} \in E_x$ $\Big\}$ $\Rightarrow \Gamma(E)$ is a $C^\infty(M)$-module.

$C^\infty(M) = \{f : M \to \mathbb{R} \,/\, f = \text{smooth}\}$

Prop: Given v.b. $(E, \pi_E), (F, \pi_F)$ over M

$\Rightarrow$ a bijection $\left\{\begin{array}{c}\text{morphisms}\\ E \to F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{morphisms}\\ \Gamma(E) \to \Gamma(F)\\ \text{of } C^\infty(M)\text{-modules}\end{array}\right\}$

$u \longmapsto u_*$ where $u_*(s) = u \circ s$

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[dr] & F \arrow[d] \\ & M \arrow[ul, dotted, bend left, "s"] \end{tikzcd}$$

4

Examples: Trivial (product) bundles. $E = M \times \mathbb{R}^r$, $\pi = pr_M$.

Case $r=1$: called the trivial <u>line</u> bundle.

$x \longmapsto (x, f(x))$

Example 2: The tautological line bundle:

$$\begin{cases} M := \boxed{\mathbb{P}^n} = \mathbb{R}\mathbb{P}^n := \{ \ell \subseteq \mathbb{R}^{n+1} \,/\, \ell = \text{line through the origin} \} & \text{real projective space} \\ E := \{ \underset{\mathbb{P}^n \times \mathbb{R}^{n+1}}{\underset{\cap}{(\ell, v)}} \,/\, \ell \in \mathbb{P}^n, v \in \ell \} \subseteq \mathbb{P}^n \times \mathbb{R}^{n+1} \\ \pi : E \to \mathbb{P}^n, \ \pi(\ell, v) = \ell . \end{cases}$$

(ℓ, v)

Particular case $n=1$:

Exercise: prove local triviality.

5

The smooth str. on $\mathbb{P}^n$: charts?

$\boxed{\mathbb{P}^n \ni [x_0 : x_1 : \dots : x_n]}$:= the lines in $\mathbb{R}^{n+1}$ through $0 \in \mathbb{R}^{n+1}$ and $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$

NB : $[x_0 : x_1 : \dots : x_n] = [x_0' : x_1' : \dots : x_n'] \iff \exists \lambda \in \mathbb{R}$ st. $x_i' = \lambda x_i$ $\forall$, $1 \le i \le n$.

$\Rightarrow [x_0 : x_1 : \dots : x_n] = [1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0}]$ for $x_0 \neq 0$.

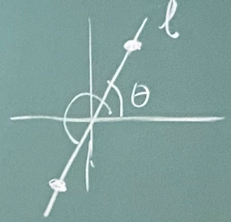
$U_0 := \{[x_0 : \dots : x_n] \in \mathbb{P}^n \,/\, x_0 \neq 0\}$, $\chi_0 : U_0 \longrightarrow \mathbb{R}^n$

$[x_0 : x_1 : \dots : x_n] \longmapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$

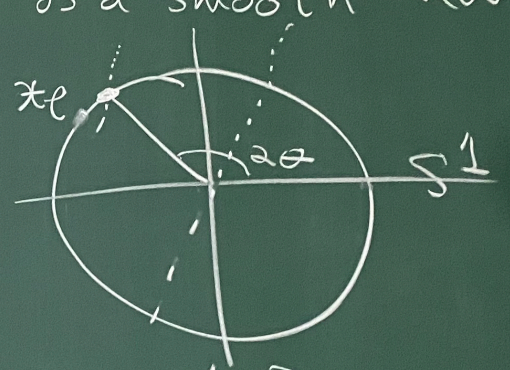
$\Rightarrow (U_0, \chi_0)$ chart of $\mathbb{P}^n$ $\Rightarrow$ $\mathbb{P}^n$ is a smooth manifold.

Similarly $(U_1, \chi_1), \dots, (U_n, \chi_n)$

$\mathbb{P}^1$



S^1



E over $\mathbb{P}^1$

Over S^1 ?

open Moebius band

6

Example 3: tangent bundle of a manifold M $\boxed{m = \dim M}$

- $\{\cancel{T_x M}\}_{x\in M}$ collection of vector spaces
  (E_x)

$$TM = \{(x, v) \,/\, x\in M,\ v\in T_x M\}$$

- chart (U, x) of M, $x : \underset{\cap}{U} \xrightarrow{\sim} \underset{\cap}{x(U)} \Rightarrow$ a frame
  ($U \subset M$, $x(U) \subset \mathbb{R}^m$)

$$\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_m}\right)_x \quad \text{frame (basis) of } T_x M \text{ for each } x\in U$$

Then $\Rightarrow$ can make TM into a smooth manifold. Charts?

Idea: any (U, x) of M induces a chart $(\tilde{U}, \tilde{x})$ of TM

$$TM \supseteq \tilde{U} = \{(x, v)\in TM \,/\, x\in U\} \xrightarrow{\tilde{x}} \underbrace{(\underset{\in\mathbb{R}^m}{x(x)}, \underset{\in\mathbb{R}^m}{\lambda_1^v, \dots, \lambda_m^v})}_{\in\mathbb{R}^{2m}}$$

$v\in T_x M$

$$v = \lambda_1^v \cdot \left(\frac{\partial}{\partial x_1}\right)_x + \dots + \lambda_m^v \left(\frac{\partial}{\partial x_m}\right)_x$$

charts $(\tilde{U}, \tilde{x})$ for TM. Now: $\pi : TM \to M$, $\pi(x, v) = x$

7

Frames In a vector space V, a frame is an ordered basis

$$e = (e_1, \dots, e_r) \qquad e_i \in V \qquad (r = \dim V$$

One has:

$$\left\{ \begin{array}{c} \text{frames} \\ e=(e_1,\dots,e_r) \\ \text{of } V \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{linear isomorphisms} \\ \hat{e} : \mathbb{R}^r \to V \end{array} \right\}$$

<u>Def</u>: E = v.b. over M. A frame of E is a collection

$$e = (e_1, \dots, e_r) \quad \text{with } e_i \in \Gamma E$$

s.t. $\forall x \in M$: $(e_1(x), \dots, e_r(x))$ is a frame of the vector space E_x.

<u>Rk</u>:

$$\left\{ \begin{array}{c} \text{frames} \\ (e_1,\dots,e_r) \\ \text{of } E \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{vector bundle isomorphism} \\ M \times \mathbb{R}^r \xrightarrow{\sim} E \end{array} \right\}$$

Can talk about local frames of E (defined over some $U \subseteq M$ open)

$\Rightarrow$ the local triviality of $E \iff \exists$ of local frames

<u>E.g.</u>: for vector bundles E of rank 1: (local) frame of E $\equiv$ nowhere vanishing (local) section of E.

Cho
$(\tilde{U},$

linear isom

h takes E_x to $\{x\}\times\mathbb{R}^r$ by linear isomorphisms

8

Next time: Lemma 6.10
look at it already!

AARON
a.w.gootjes-dreesbach
@uu.nl

$M = S^1$

$E = S^1\times\mathbb{R}^5$

$x \longmapsto (x, \underset{\mathbb{R}^5}{z(x)})$

$e \in \Gamma E$

$e(\underset{(x,y)}{p}) = (p, x^2, xe^y, \frac{x-y}{5}, 0)$

$e_1, e_2 \in \Gamma E$, $e_1(x) = (x,1,0,0,0,0)$

$e_2(x) = (x,0,1,0,0,0)$

$\{x\}\times\mathbb{R}^5$

Frames

One has:

Def: E

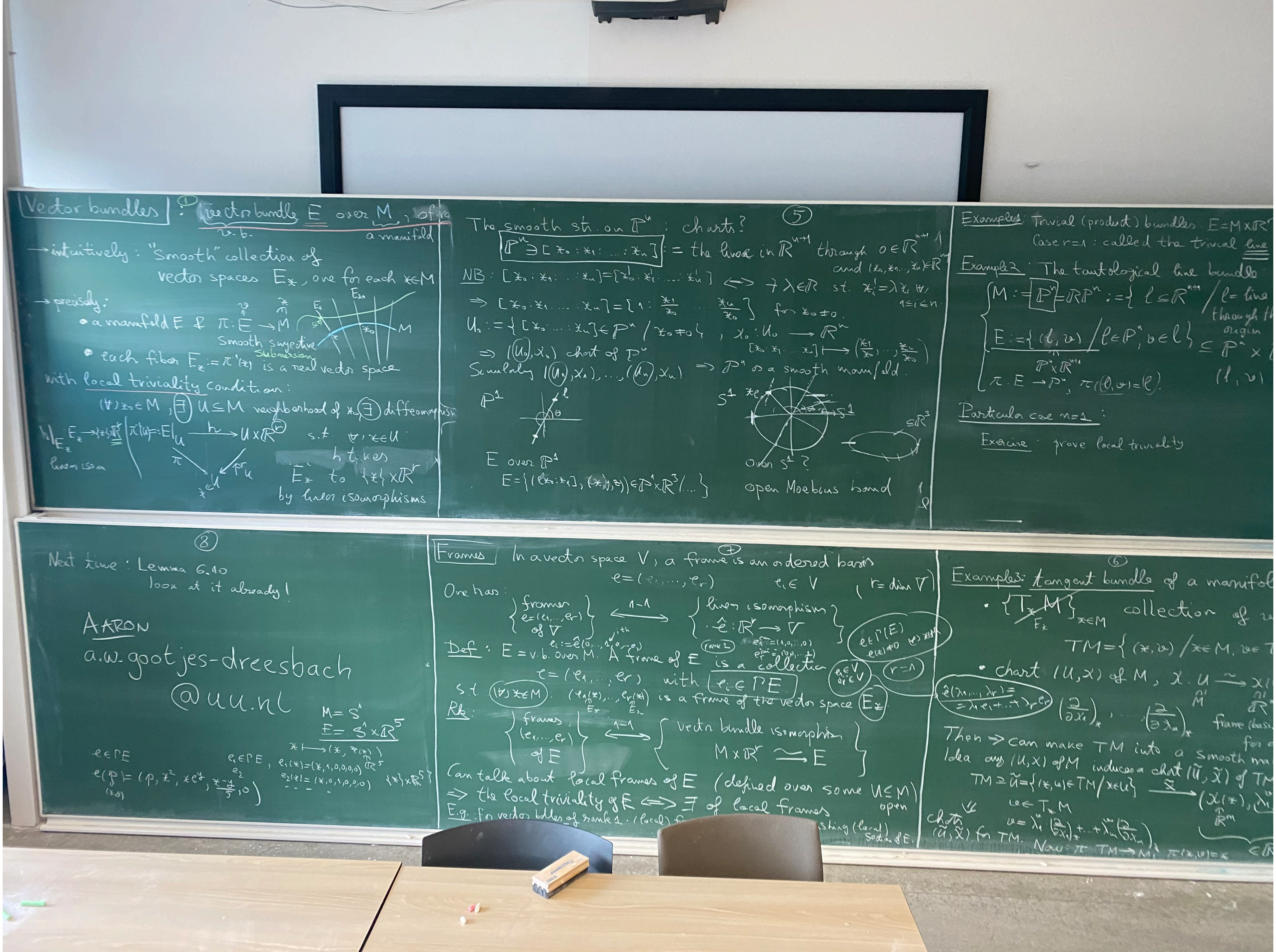
s.t.

Rk:

Can talk abo

$\Longrightarrow$ the local

E.g.: for vector bd


Vector bundles
① vector bundle E over M, of rank r
v.b.
a manifold
→ intuitively: "smooth" collection of vector spaces E_x, one for each x∈M
→ precisely:
• a manifold E & π: E → M
smooth surjective submersion
• each fiber E_x := π⁻¹(x) is a real vector space
with local triviality condition:
(∀) x₀∈M, ∃ U⊆M neighborhood of x₀, ∃ diffeomorphism
π⁻¹(U) =: E|_U → U×ℝ^r
s.t. ∀ x∈U
h takes E_x to {x}×ℝ^r
by linear isomorphisms
⑤
The smooth str. on ℙ^n: charts?
ℙ^n ∋ [x₀ : x₁ : … : x_n] = the line in ℝ^{n+1} through 0∈ℝ^{n+1} and (x₀, x₁, …, x_n)∈ℝ^{n+1}
NB: [x₀ : x₁ : … : x_n] = [x'₀ : x'₁ : … : x'_n] ⟺ ∃ λ∈ℝ st. x'_i = λx_i ∀ 1≤i≤n
⇒ [x₀ : x₁ : … : x_n] = [1 : x₁/x₀ : … : x_n/x₀] for x₀≠0.
U₀ := {[x₀ : … : x_n] ∈ ℙ^n / x₀≠0}, χ₀ : U₀ → ℝ^n
[x₀ : x₁ : … : x_n] ↦ (x₁/x₀, …, x_n/x₀)
⇒ (U₀, χ₀) chart of ℙ^n
Similarly (U₁, χ₁), …, (U_n, χ_n) ⇒ ℙ^n is a smooth manifold.
ℙ¹
S¹
⊆ℝ³
E over ℙ¹
E = {([x₀ : x₁], (x, y, z)) ∈ ℙ¹×ℝ³ / …}
Over S¹?
open Moebius band
Examples: Trivial (product) bundles E = M×ℝ^r
Case r=1: called the trivial line
Example 2: The tautological line bundle
M := ℙ^n = ℝℙ^n := {ℓ ⊆ ℝ^{n+1} / ℓ = line through the origin}
E := {(ℓ, v) / ℓ∈ℙ^n, v∈ℓ} ⊆ ℙ^n × ℝ^{n+1}
π: E → ℙ^n, π(ℓ, v) = ℓ.
Particular case n=1:
Exercise: prove local triviality
⑧
Next time: Lemma 6.10
look at it already!
AARON
a.w.gootjes-dreesbach
@uu.nl
M = S¹
E = S¹×ℝ⁵
Frames
⑦
In a vector space V, a frame is an ordered basis
e = (e₁, …, e_r)   e_i ∈ V   (r = dim V)
One has:
{frames e=(e₁,…,e_r) of V} ←1-1→ {linear isomorphisms ê: ℝ^r → V}
Def: E = v.b. over M. A frame of E is a collection
e = (e₁, …, e_r) with e_i ∈ ΓE
s.t. (∀) x∈M: (e₁(x), …, e_r(x)) is a frame of the vector space E_x
Rk:
{frames (e₁, …, e_r) of E} ←1-1→ {vector bundle isomorphisms M×ℝ^r ≃ E}
Can talk about local frames of E (defined over some U⊆M open)
⇒ the local triviality of E ⟺ ∃ of local frames
⑥
Example 3: tangent bundle of a manifold
• {T_xM}_{x∈M} collection of
TM = {(x, v) / x∈M, v∈T_xM}
• chart (U, χ) of M
Then ⇒ can make TM into a smooth manifold
Idea: any (U, χ) of M induces a chart (Ũ, χ̃) of TM
TM ⊇ Ũ = {(x, v)∈TM / x∈U}
(Ũ, χ̃) for TM. Now π: TM → M, π(x, v) = x