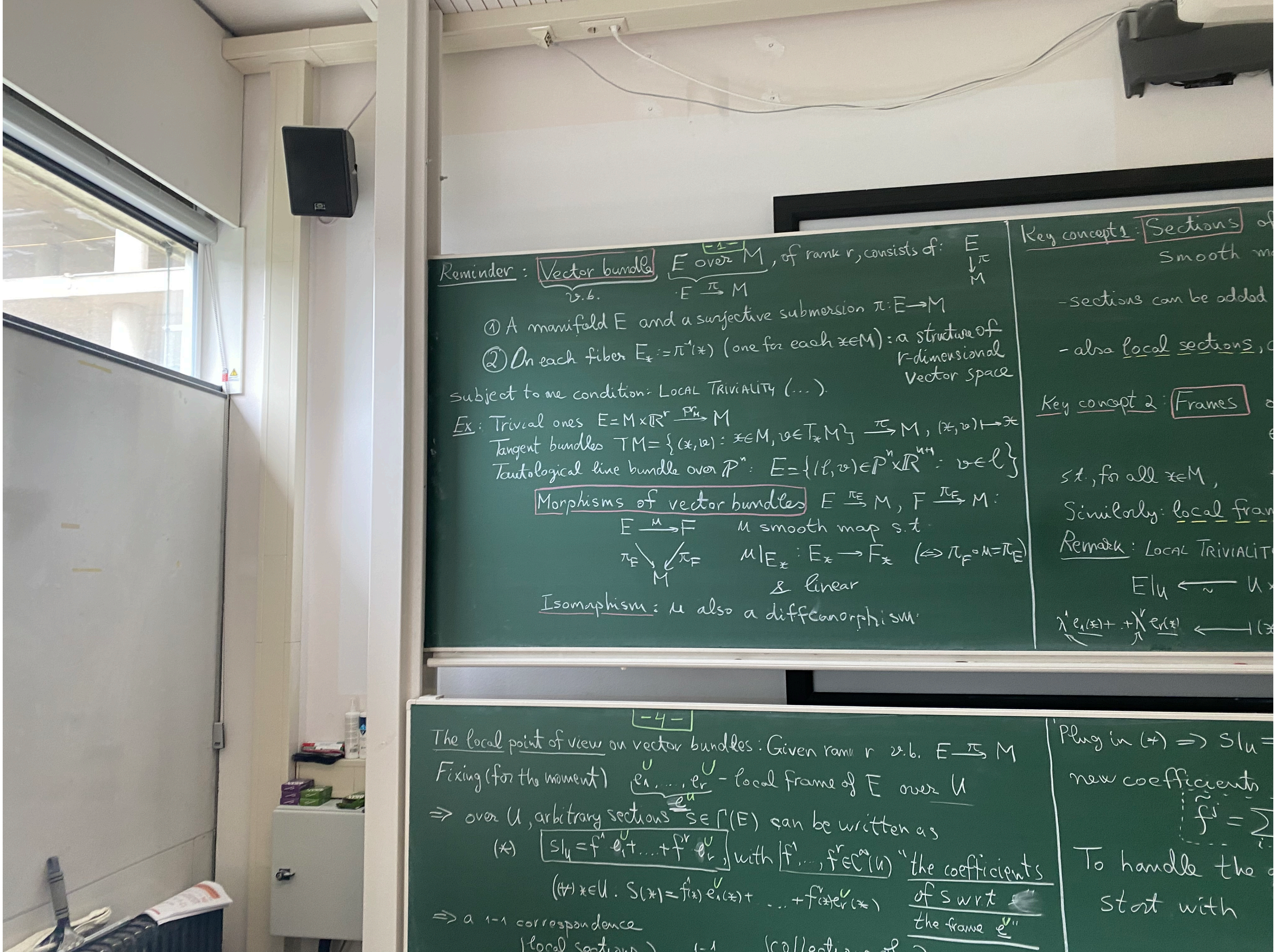

-1-

Reminder : $\underbrace{\boxed{\text{Vector bundle}}}_{v.b.}$ $\underbrace{E \text{ over } M}_{E \xrightarrow{\pi} M}$, of rank r, consists of: $E \xrightarrow{\pi} M$

① A manifold E and a surjective submersion $\pi: E \to M$

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subject to one condition: LOCAL TRIVIALITY (...).

Ex: Trivial ones $E = M \times \mathbb{R}^r \xrightarrow{pr_M} M$

Tangent bundles $TM = \{(x, v) : x \in M, v \in T_xM\} \xrightarrow{\pi} M$, $(x, v) \mapsto x$

Tautological line bundle over $\mathbb{P}^n$: $E = \{(\ell, v) \in \mathbb{P}^n \times \mathbb{R}^{n+1} : v \in \ell\}$

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$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

u smooth map s.t.

$u|_{E_x} : E_x \longrightarrow F_x$ $(\Leftrightarrow \pi_F \circ u = \pi_E)$

& linear

Isomorphism : u also a diffeomorphism.

Key concept 1 : Sections of

Smooth m

- sections can be added

- also local sections, c

Key concept 2 : Frames o

e

s.t., for all $x \in M$,

Similarly: local fram

Remark : LOCAL TRIVIALIT

$E|_U \xleftarrow{\sim} U \times$

$\lambda^1 e_1(x) + \dots + \lambda^r e_r(x) \longleftarrow\!\shortmid (x$

-4-

The local point of view on vector bundles: Given rank r v.b. $E \xrightarrow{\pi} M$

Fixing (for the moment) $\underbrace{e_1^U, \dots, e_r^U}_{e^U}$ - local frame of E over U

⇒ over U, arbitrary sections $s \in \Gamma(E)$ can be written as

(*) $\boxed{s|_U = f^1 e_1^U + \dots + f^r e_r^U}$, with $f^1, \dots, f^r \in C^\infty(U)$ "the coefficients of s w.r.t. the frame e^U"

$(\forall)\ x \in U.\ s(x) = f^1(x) e_1^U(x) + \dots + f^r(x) e_r^U(x)$

⇒ a 1-1 correspondence

(local sections) 1-1 (collection of

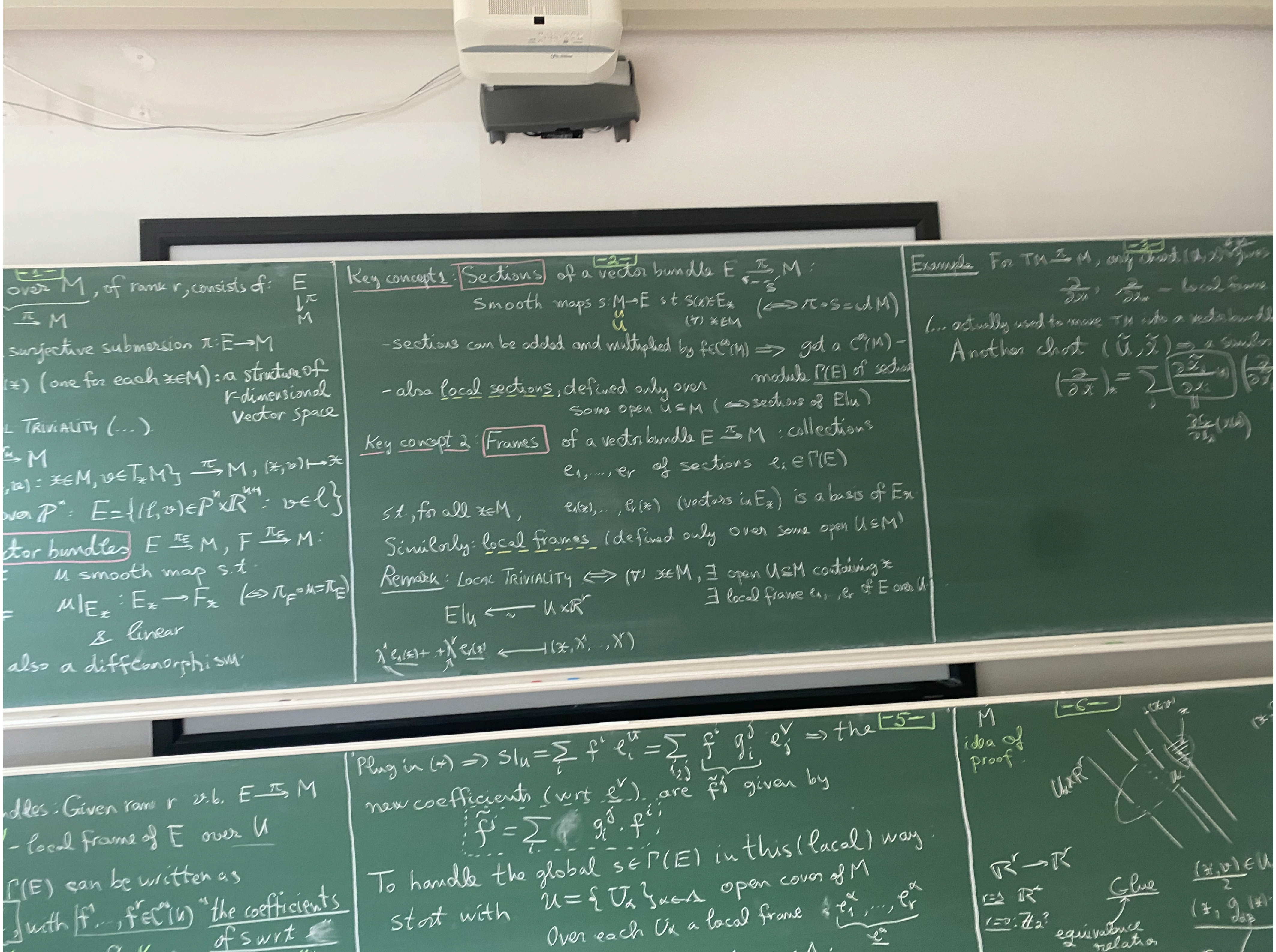
Plug in (*) ⇒ $s|_U =$

new coefficients

$\tilde{f}^j = \sum_i$

To handle the

start with



–1–

over M, of rank r, consists of:

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–2–

Key concept 1: $\boxed{\text{Sections}}$ of a vector bundle $E \underset{s}{\overset{\pi}{\rightleftarrows}} M$:

Smooth maps $s : \overset{\cup}{\underset{U}{M}} \to E$ s.t. $s(x) \in E_x$ $(\forall)\ x \in M$ $(\Leftrightarrow \pi \circ s = \mathrm{id}\, M)$

- sections can be added and multiplied by $f \in C^\infty(M) \Rightarrow$ get a $C^\infty(M)$-module $\Gamma(E)$ of sections
- also local sections, defined only over some open $U \subseteq M$ ($\Leftrightarrow$ sections of $E|_U$)

Key concept 2: $\boxed{\text{Frames}}$ of a vector bundle $E \xrightarrow{\pi} M$: collections $e_1, \dots, e_r$ of sections $e_i \in \Gamma(E)$

s.t., for all $x \in M$, $e_1(x), \dots, e_r(x)$ (vectors in E_x) is a basis of E_x

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Remark: LOCAL TRIVIALITY $\Leftrightarrow$ $(\forall)\ x \in M$, $\exists$ open $U \subseteq M$ containing x $\exists$ local frame $e_1, \dots, e_r$ of E over U.

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–3–

Example For $TM \xrightarrow{\pi} M$, any chart (U, x) gives

$\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}$ – local frame

(... actually used to make TM into a vector bundle

Another chart $(\tilde{U}, \tilde{x}) \Rightarrow$ a similar

$$\left(\frac{\partial}{\partial x}\right)_x = \sum_j \boxed{\frac{\partial \tilde{x}_j}{\partial x_i}(x)} \left(\frac{\partial}{\partial \tilde{x}_j}\right.$$

$\parallel$

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with $\boxed{f^1, \dots, f^r \in C^\infty(U)}$ "the coefficients of s wrt"

–5–

Plug in (*) $\Rightarrow$ $s|_U = \sum_i f^i e_i^U = \sum_{i,j} \underbrace{f^i g_i^j}_{\tilde{f}^j} e_j^V \Rightarrow$ the new coefficients (wrt e^V) are $\tilde{f}^j$ given by

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start with $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ open cover of M

Over each U_α a local frame $\underbrace{e_1^\alpha, \dots, e_r^\alpha}_{e^\alpha}$

–6–

M

idea of proof.

$U_\alpha \times \mathbb{R}^r$

(x,v)

x

U

$\mathbb{R}^r \to \mathbb{R}^r$

r=1: $\mathbb{R}^*$

r=0: $\mathbb{Z}_2$?

Glue

equivalence relation

$\frac{(x,v) \in U_\alpha}{2}$

$(x, g_{\alpha\beta}(x)$

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E s t $s(x) \in E_x$ $(\forall)\ x \in M$ $(\Longleftrightarrow \pi \circ s = \mathrm{id}\, M)$

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$\boxed{-3-}$

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \text{ – local frame of } TM \text{ over } U$$

(... actually used to make TM into a vector bundle !!)

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$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j \boxed{\frac{\partial \tilde{x}_j}{\partial x_i}(x)} \left(\frac{\partial}{\partial \tilde{x}_j}\right)_x$$

$$\parallel$$

$$\frac{\partial c_j}{\partial t_i}(x(x))$$

$c = \tilde{x} \circ x^{-1}$

$\boxed{-5-}$

$\underbrace{f^i\, g_i^j}_{\tilde{f}^j}\, e_j^V \Rightarrow$ the

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open cover of M

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$\boxed{-6-}$

M

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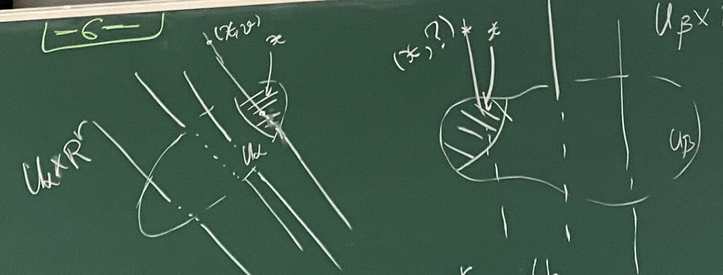

$\mathbb{R}^r \longrightarrow \mathbb{R}^r$

<u>r=1</u> : $\mathbb{R}^*$

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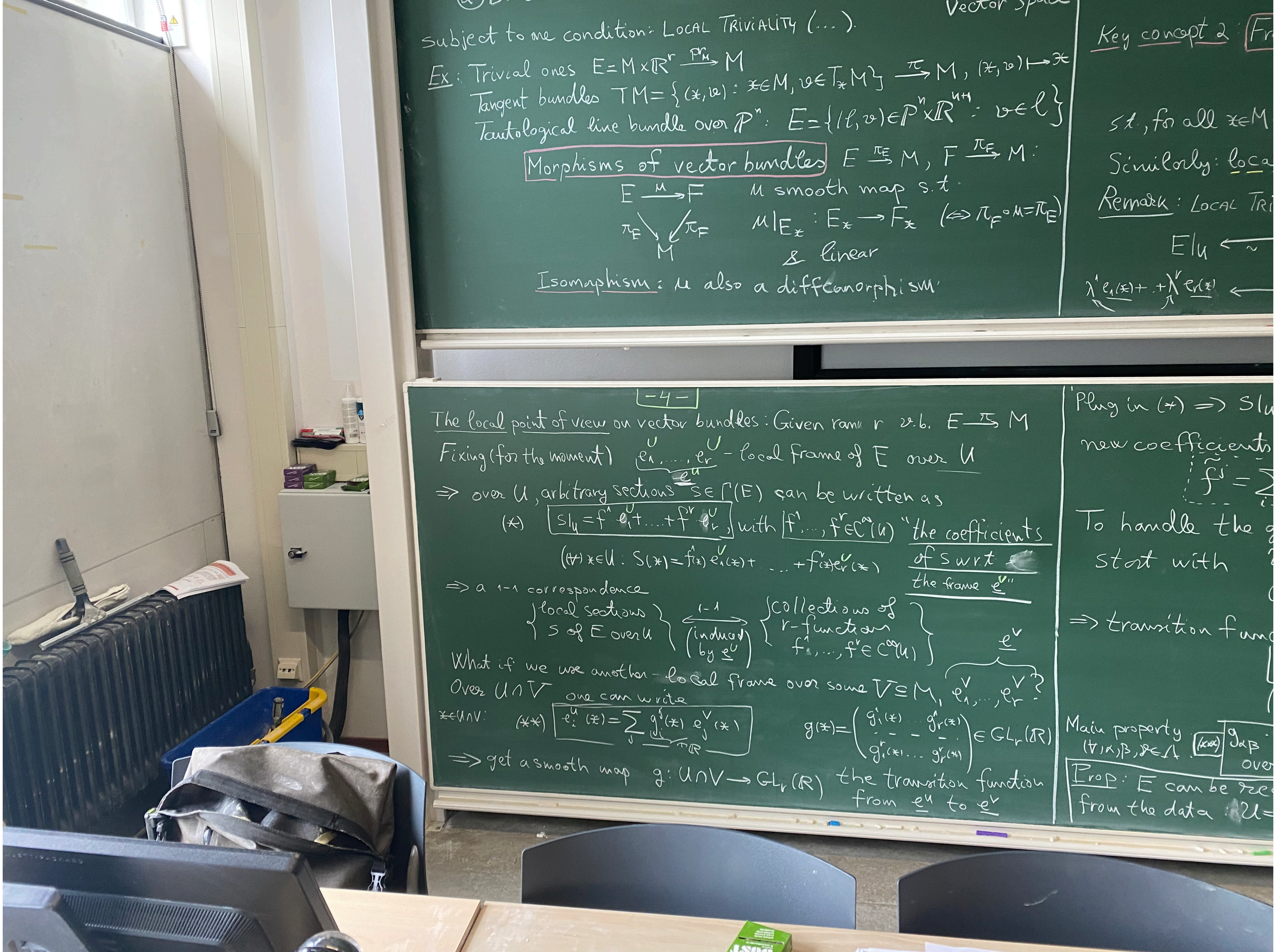
equivalence relation

$\frac{(x, v)}{2} \in U_\alpha \times \mathbb{R}^r$ with

$(x, g_{\alpha\beta}(x) \cdot v) \in U_\beta \times \mathbb{R}^r$

$\xrightarrow{\pi} M$



Vector spa

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Over $U\cap V$ one can write

$x\in U\cap V$: (**) $\boxed{e_i^U(x) = \sum_j \underbrace{g_i^j(x)}_{\in\mathbb{R}}\, e_j^V(x)}$ $\quad g(x) = \begin{pmatrix} g_1^1(x) & \dots & g_r^1(x)\\ \vdots & & \vdots \\ g_1^r(x) & \dots & g_r^r(x)\end{pmatrix}\in GL_r(\mathbb{R})$

⇒ get a smooth map $g: U\cap V\to GL_r(\mathbb{R})$ the transition function from $\underline{e}^U$ to $\underline{e}^V$

Key concept 2: Fr

s.t., for all $x\in M$

Similarly: loca

Remark: LOCAL TRI

$E|_U \xleftarrow{\sim}$

$\lambda^1 e_1(x) + \dots + \lambda^r e_r(x) \longleftarrow$

Plug in (*) ⇒ $s|_U$

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To handle the

start with

⇒ transition func

Main property $(\forall)\ \alpha,\beta,\gamma\in\Lambda$: (***) $g_{\alpha\beta}$

over

Prop: E can be re

from the data: $\mathcal{U}=$

$\} \xrightarrow{\pi} M,\ (x,v) \mapsto x$

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be written as

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$+ f^r(x) e_r^V(x)$

llections of ∞-functions $f^1, \dots, f^r \in C^\infty(U)$ $\}$ $\overbrace{\qquad}^{\underline{e}^V}$

over some $V \subseteq M$, $e_1^V, \dots, e_r^V$?

$$g(x) = \begin{pmatrix} g_1^1(x) & \dots & g_r^1(x) \\ - & - & - \\ g_1^r(x) & \dots & g_r^r(x) \end{pmatrix} \in GL_r(\mathbb{R})$$

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$\boxed{-5-}$

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Over each U_α a local frame $\underbrace{e_1^\alpha, \dots, e_r^\alpha}_{\underline{e}^\alpha}$

$\Rightarrow$ transition functions, one for each $\alpha, \beta \in \Lambda$:

$$\boxed{g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_r(\mathbb{R})} \qquad (A \cdot B)_j^i$$

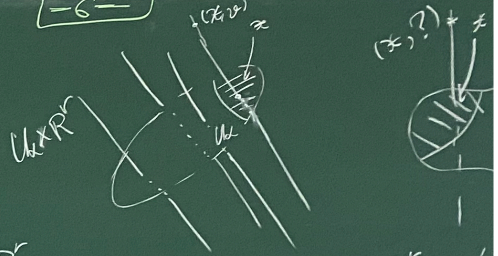
$$\left(e_i^\beta = \sum_j (g_{\alpha\beta})_i^j\, e_j^\alpha \right)$$

Main property $(\forall)\ \alpha, \beta, \gamma \in \Lambda$: $\boxed{(**)}$ $\boxed{g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma} \text{ over } U_\alpha \cap U_\beta \cap U_\gamma}$ $\left(\Rightarrow \begin{array}{l} g_{\alpha\alpha} = Id_r \\ g_{\alpha\beta}^{-1} = g_{\beta\alpha} \end{array} \quad \begin{array}{l} \alpha = \beta = \gamma \\ \gamma = \alpha \end{array} \right)$

Prop: E can be reconstructed from the data $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$, $g_{\alpha\beta}$'s satisfying $(**)$.

$\boxed{-6-}$

idea of proof.



$\mathbb{R}^r \to \mathbb{R}^r$

$\underline{r=1}$: $\mathbb{R}^*$

$\underline{r=0}$: $\mathbb{Z}_2$?

$\underline{\text{Glue}}$

equivalence relation

$(x,v) \in U_\alpha \times \mathbb{R}^r$ wit

$(x, g_{\alpha\beta}(x) \cdot v) \in U_\beta$

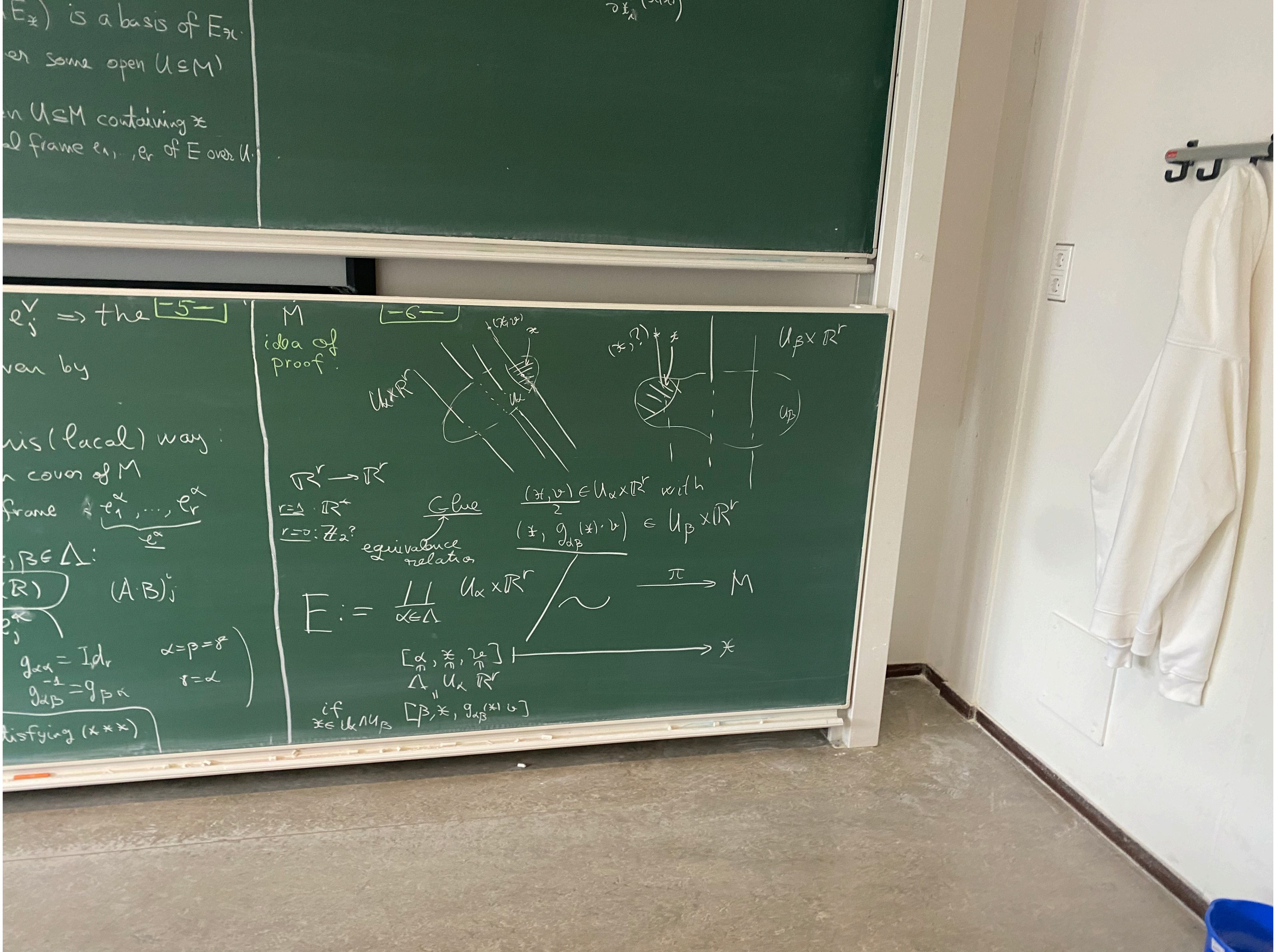
$$E := \coprod_{\alpha \in \Lambda} U_\alpha \times \mathbb{R}^r \Big/ \sim \quad \xrightarrow{\pi}$$

$[\overset{\alpha}{\underset{\Lambda}{}}, \overset{x}{\underset{U_\alpha}{}}, \overset{v}{\underset{\mathbb{R}^r}{}}] \longmapsto$

$[\alpha, x, v]$ with $\alpha \in \Lambda$, $x \in U_\alpha$, $v \in \mathbb{R}^r$

$\shortparallel$

if $x \in U_\alpha \cap U_\beta$ $[\beta, x, g_{\alpha\beta}(x) v]$



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M $\boxed{-6-}$

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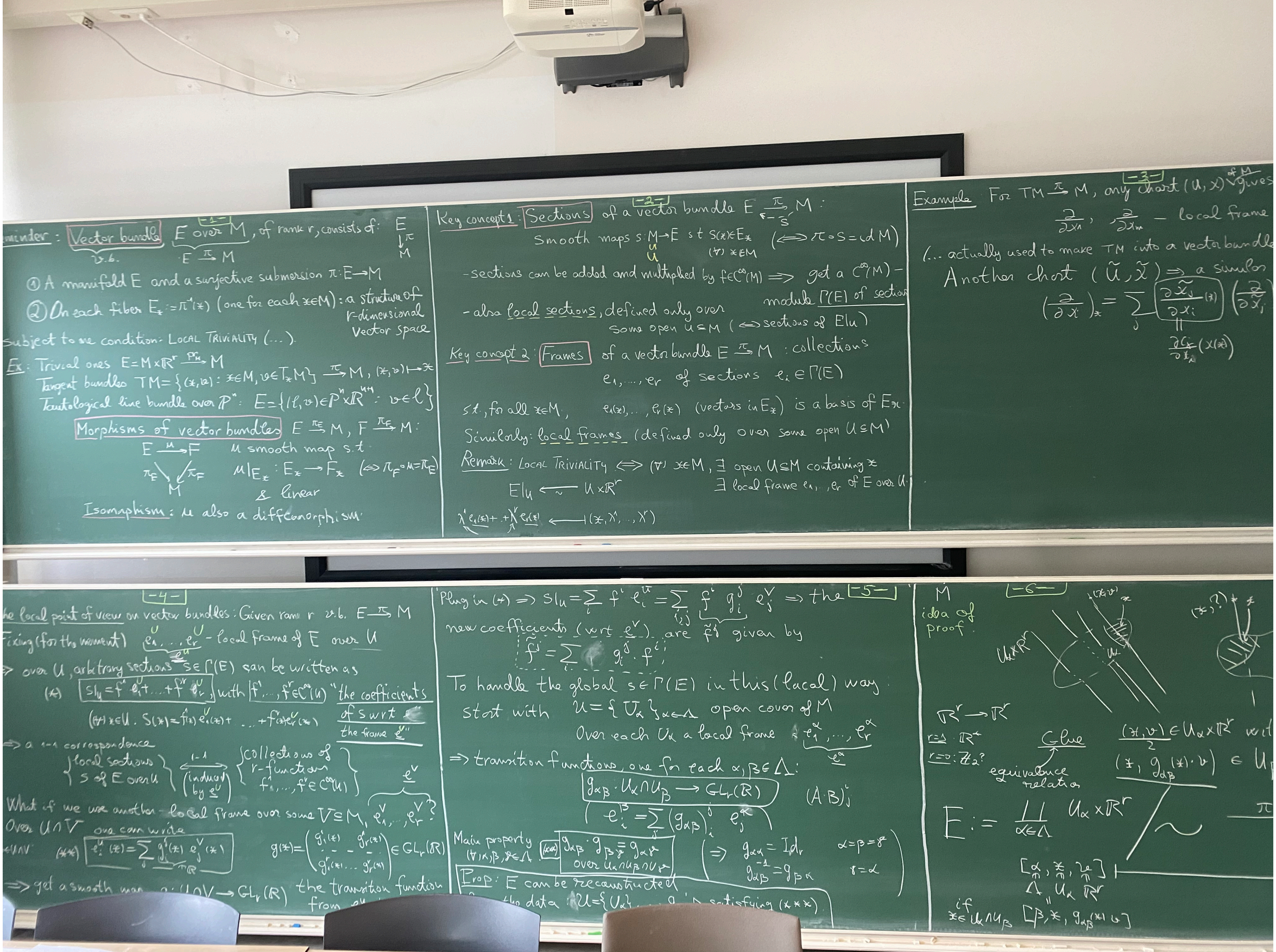

-1-

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-4-

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-5-

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$\Rightarrow$ transition functions, one for each $\alpha,\beta\in\Lambda$:

$$\boxed{g_{\alpha\beta}: U_\alpha\cap U_\beta\longrightarrow GL_r(\mathbb{R})}$$

$$\left(e_i^\beta = \sum_j (g_{\alpha\beta})_i^j\, e_j^\alpha\right)$$

$(A\cdot B)_j^i$

Main property $(\forall)\,\alpha,\beta,\gamma\in\Lambda$: (***) $\boxed{\begin{matrix}g_{\alpha\beta}\cdot g_{\beta\gamma} = g_{\alpha\gamma}\\ \text{over } U_\alpha\cap U_\beta\cap U_\gamma\end{matrix}}$

$\left(\Rightarrow \begin{matrix} g_{\alpha\alpha} = Id_r & \alpha=\beta=\gamma\\ g_{\alpha\beta}^{-1} = g_{\beta\alpha} & \gamma=\alpha\end{matrix}\right)$

Prop: E can be reconstructed … the data: $\mathcal{U}=\{U_\alpha\}$, … g… satisfying (***)

-6-

M

idea of proof.

$U_\alpha\times\mathbb{R}^r$

(x,v)

x

U_α

$(x,?)$

x

x

$\mathbb{R}^r\longrightarrow\mathbb{R}^r$

r=1: $\mathbb{R}^*$

r=0: $\mathbb{Z}_2$?

Glue

equivalence relation

$\frac{(x,v)\in U_\alpha\times\mathbb{R}^r}{2}$ with

$(x, g_{\alpha\beta}(x)\cdot v)\in U_\beta$…

$$E := \coprod_{\alpha\in\Lambda} U_\alpha\times\mathbb{R}^r \Big/ \sim$$

$\xrightarrow{\pi}$

$[\underset{\Lambda}{\underset{\pitchfork}{\alpha}}, \underset{U_\alpha}{\underset{\pitchfork}{x}}, \underset{\mathbb{R}^r}{\underset{\pitchfork}{v}}] \longmapsfrom$

if $x\in U_\alpha\cap U_\beta$ $\;=\; [\beta, x, g_{\alpha\beta}(x)v]$

of rank r, consists of: $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$

ubmersion $\pi : E \to M$

each $x \in M$): a structure of r-dimensional vector space

.).

$_x M\} \xrightarrow{\pi} M$, $(x, v) \mapsto x$

$(\ell, v) \in P^n \times \mathbb{R}^{n+1} : v \in \ell\}$

$E \xrightarrow{\pi_E} M$, $F \xrightarrow{\pi_F} M$:

map s.t.

$\to F_x$ $(\Leftrightarrow \pi_F \circ u = \pi_E)$

ar

rphism.

Looking back at TM, transition functions etc $\Rightarrow$

$\Rightarrow$ one more "remark": general construction of vector bundles)

Start with: $r \in \mathbb{N}$

① our (base) manifold M

② a collection of r-dimensional vector spaces $\{E_x\}_{x \in M}$

... organise them as:

$$E = \{(x, v) : x \in M, v \in E_x\} \xrightarrow{\pi} M$$

③ $\mathcal{U}$ = open cover of M

④ for each $U \in \mathcal{U}$ a "set theoretical" local frame $e_1^U, \dots, e_r^U$ over U (i.e., for each $x \in U$ a basis $e_1^U(x), \dots, e_r^U(x)$ of E_x)

CONDITION: For each $U, V \in \mathcal{U}$, defining $g_i^j : U \cap V \to \mathbb{R}$ by $(**)$ these are all required to be smooth.

CONCLUSION: E can be made into a vector bundle s.t. all $e_1^U, \dots$ are smooth)

<u>Example</u>: For $TM \xrightarrow{\pi} M$,

$\frac{\partial}{\partial x_1}$,

(... actually used to make

Another chart

$\Big(\frac{\partial}{\partial x}$

---

ifold

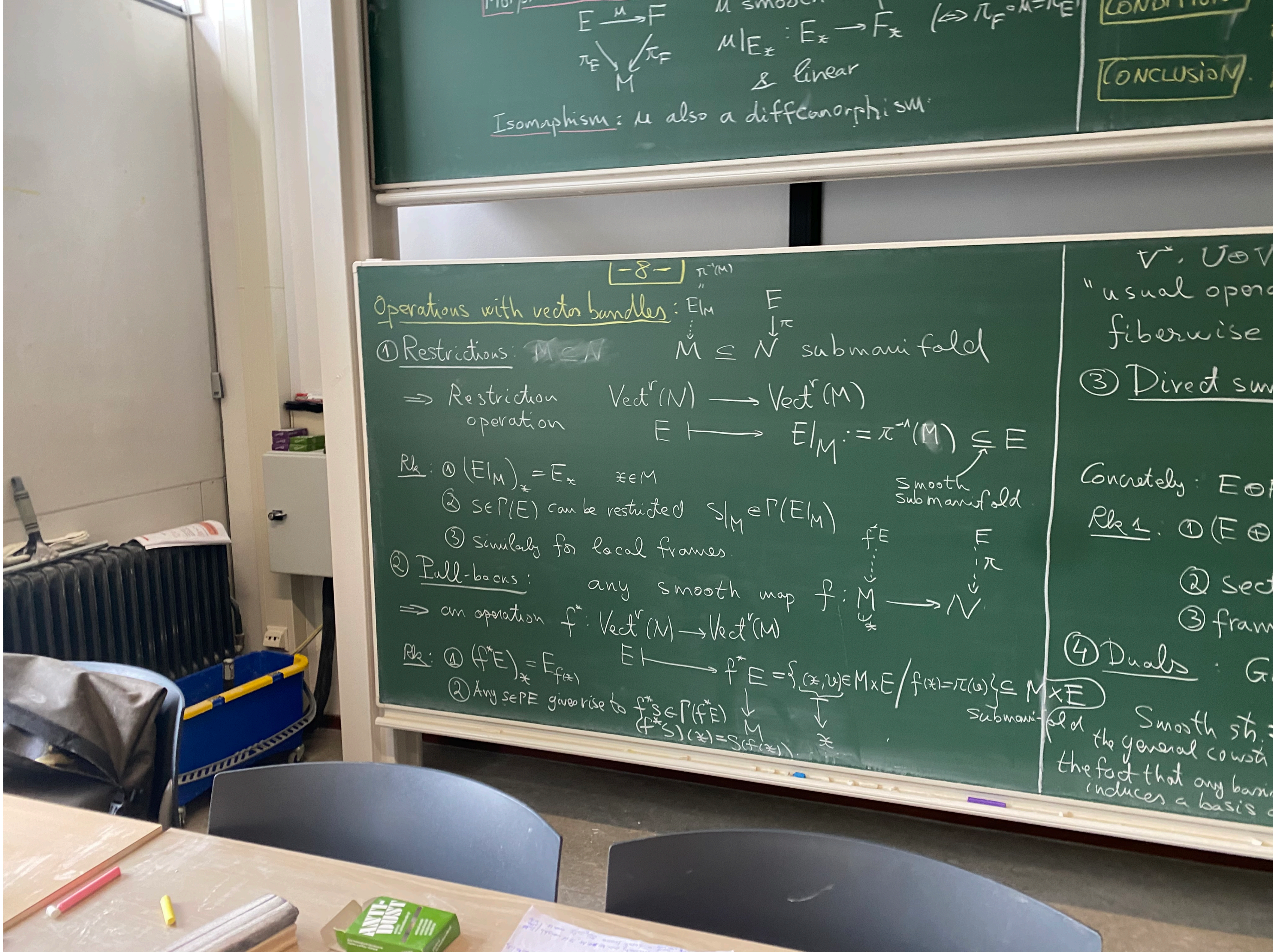
V^*, $U \oplus V$, $\mathrm{Hom}(U, V)$, ...

"usual operations with vector spaces can be applied to fiberwise $\Rightarrow$ an upgrade of those operations to vector bundles"

---

⑤ Tensor prod

⑥ Hom-bun



$$\begin{tikzcd} E \arrow[rr, "\mu"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

μ smoo

$\mu|_{E_x} : E_x \longrightarrow F_x$ ($\Leftrightarrow \pi_F \circ \mu = \pi_E$)

& linear

Isomorphism: μ also a diffeomorphism.

CONCLUSION.

-8-

Operations with vector bundles: $E|_M \overset{\pi^{-1}(M)}{=}$

$$\begin{tikzcd} E|_M \arrow[d, dashed] & E \arrow[d, "\pi"] \\ M \arrow[r, phantom, "\subseteq"] & N \end{tikzcd}$$

① Restrictions: $M \subseteq N$ submanifold

⇒ Restriction operation $\mathrm{Vect}^r(N) \longrightarrow \mathrm{Vect}^r(M)$

$E \longmapsto E|_M := \pi^{-1}(M) \subseteq E$

smooth submanifold

Rk: ① $(E|_M)_x = E_x$ $x \in M$

② $s \in \Gamma(E)$ can be restricted $s|_M \in \Gamma(E|_M)$

③ similarly for local frames.

② Pull-backs: any smooth map $f: M \longrightarrow N$

$$\begin{tikzcd} f^*E \arrow[d, dashed] & E \arrow[d, dashed, "\pi"] \\ M \arrow[r] & N \end{tikzcd}$$

$x \in M$

⇒ an operation $f^*: \mathrm{Vect}^r(N) \to \mathrm{Vect}^r(M)$

$E \longmapsto f^*E = \{\underbrace{(x,v)}_{\downarrow\, x} \in M \times E \,/\, f(x) = \pi(v)\} \subseteq M \times E$

$f^*E \downarrow M$

submanifold

Rk: ① $(f^*E)_x = E_{f(x)}$

② Any $s \in \Gamma E$ gives rise to $f^*s \in \Gamma(f^*E)$

$(f^*s)(x) = s(f(x))$.

V^*, $U \oplus V$

"usual oper

fiberwise

③ Direct su

Concretely: $E \oplus$

Rk 1. ① $(E \oplus$

② Sect

③ fram

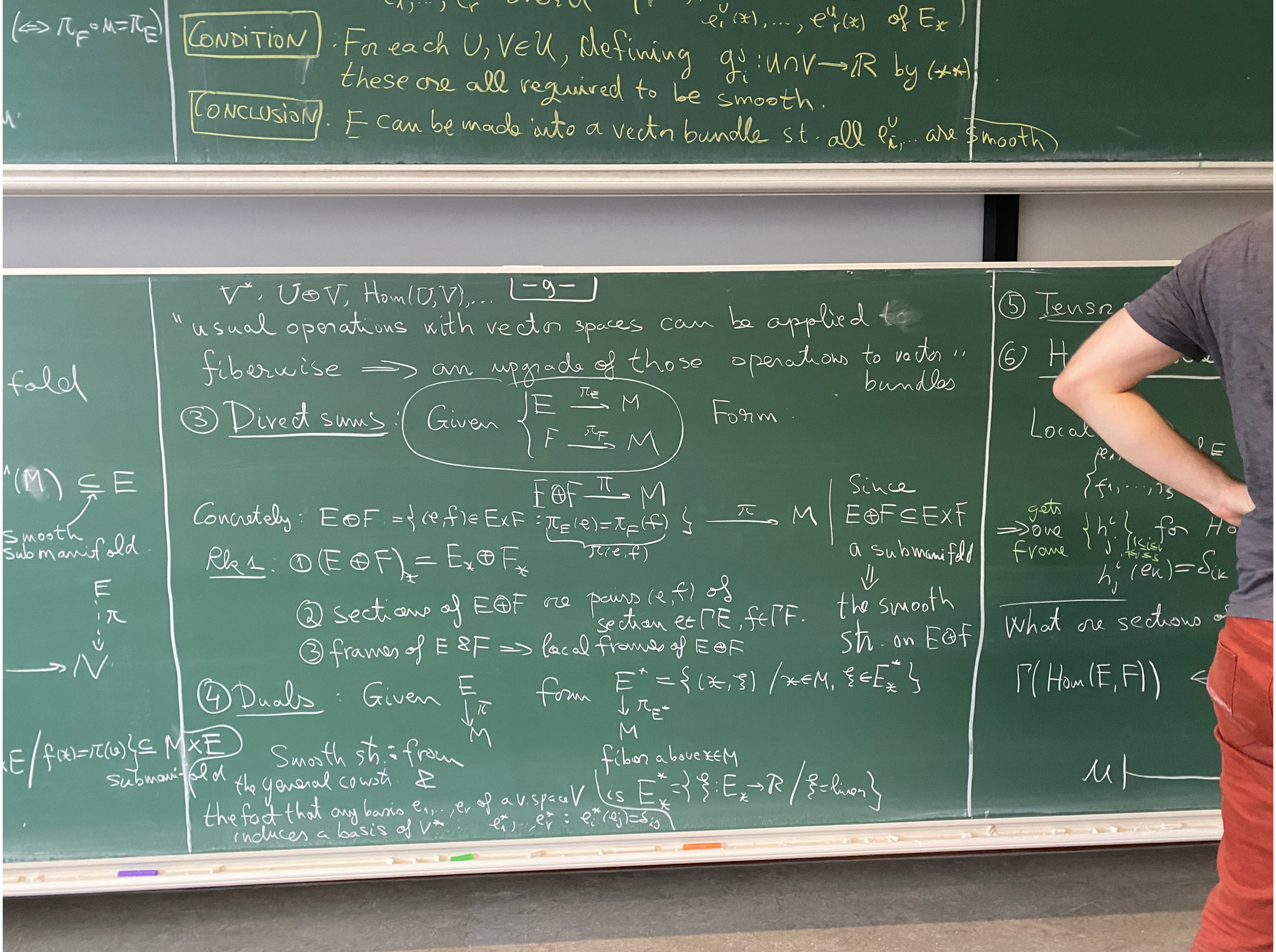
④ Duals: G

Smooth str.

the general constr

the fact that any basi

induces a basis



$(\Leftrightarrow \pi_F \circ M = \pi_E)$

$e_1^U(x), \dots, e_r^U(x)$ of E_x)

$\boxed{\text{CONDITION}}$: For each $U, V \in \mathcal{U}$, defining $g_i^j : U \cap V \to \mathbb{R}$ by $(**)$ these are all required to be smooth.

$\boxed{\text{CONCLUSION}}$: E can be made into a vector bundle s.t. all $e_i^U, \dots$ are smooth)

fold

$(M) \subseteq E$

smooth submanifold.

$$\begin{tikzcd} E \arrow[d, dashed, "\pi"] \\ N \end{tikzcd}$$

$\longrightarrow N$

$E / f(x) = \pi(e) \} \subseteq M \times E$

submanifold

V^*, $U \oplus V$, $\mathrm{Hom}(U,V)$, ... $\boxed{-9-}$

"usual operations with vector spaces can be applied to fiberwise $\Longrightarrow$ an upgrade of those operations to vector bundles"

③ Direct sums: Given $\left\{ \begin{array}{l} E \xrightarrow{\pi_E} M \\ F \xrightarrow{\pi_F} M \end{array} \right.$ Form

$E \oplus F \xrightarrow{\pi} M$

Concretely: $E \oplus F = \{ (e,f) \in E \times F : \underbrace{\pi_E(e) = \pi_F(f)}_{\pi(e,f)} \} \xrightarrow{\pi} M$

Since $E \oplus F \subseteq E \times F$ a submanifold $\Downarrow$ the smooth str. on $E \oplus F$

Rk 1: ① $(E \oplus F)_x = E_x \oplus F_x$

② sections of $E \oplus F$ are pairs (e,f) of section $e \in \Gamma E$, $f \in \Gamma F$.

③ frames of E & F $\Longrightarrow$ local frames of $E \oplus F$

④ Duals: Given $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ form $\begin{array}{c} E^* \\ \downarrow \pi_{E^*} \\ M \end{array} = \{ (x, \xi) \,/\, x \in M, \xi \in E_x^* \}$

fiber above $x \in M$ is $E_x^* = \{ \xi : E_x \to \mathbb{R} \,/\, \xi = \text{linear} \}$

Smooth str.: from the general constr. & the fact that any basis $e_1, \dots, e_r$ of a v.space V induces a basis of V^*: $e_1^*, \dots, e_r^*$: $e_i^*(e_j) = \delta_{ij}$

⑤ Tensor

⑥ H

Local

$e_1,$ E

$\{ f_1, \dots, f_s$

$\Longrightarrow$ gets one frame $\{ h_j^i \}_{\substack{1 \le i \le r \\ 1 \le j \le s}}$ for Ho

$h_j^i (e_k) = \delta_{ik}$

What are sections of

$\Gamma(\mathrm{Hom}(E,F)) \leftarrow$

$\mathcal{U} \longmapsto$

s.t. all $\rho^U_i, \dots$ are smooth

plied to

tions to vector bundles

M | Since $E \oplus F \subseteq E \times F$ a submanifold $\Downarrow$ the smooth str. on $E \oplus F$

ΓF.

$/ x \in M, \xi \in E^*_x \}$

$\to \mathbb{R} / \xi = \text{linear}\}$

—10—

⑤ Tensor products

~~$V \otimes W = \text{Hom}(V^*, W)$~~

⑥ Hom-bundles: $\begin{matrix} E & & F \\ \downarrow & & \downarrow \\ M & & M \end{matrix} \Rightarrow$ new v.bdle $\begin{matrix} \text{Hom}(E,F) \\ \downarrow \\ M \end{matrix}$

whose fiber above $x \in M$

$$\text{Hom}(E,F)_x := \text{Hom}(E_x, F_x)$$

(of vector spaces (linear maps $E_x \to F_x$))

$$\text{Hom}(E,F) = \{(x, T) : \overset{x \in M}{T \in \text{Hom}(E_x, F_x)}\}$$

Local frames

$\left\{ \begin{matrix} e_1, \dots, e_r \text{ of } E \\ f_1, \dots, f_s \text{ of } F \end{matrix} \right.$ over U

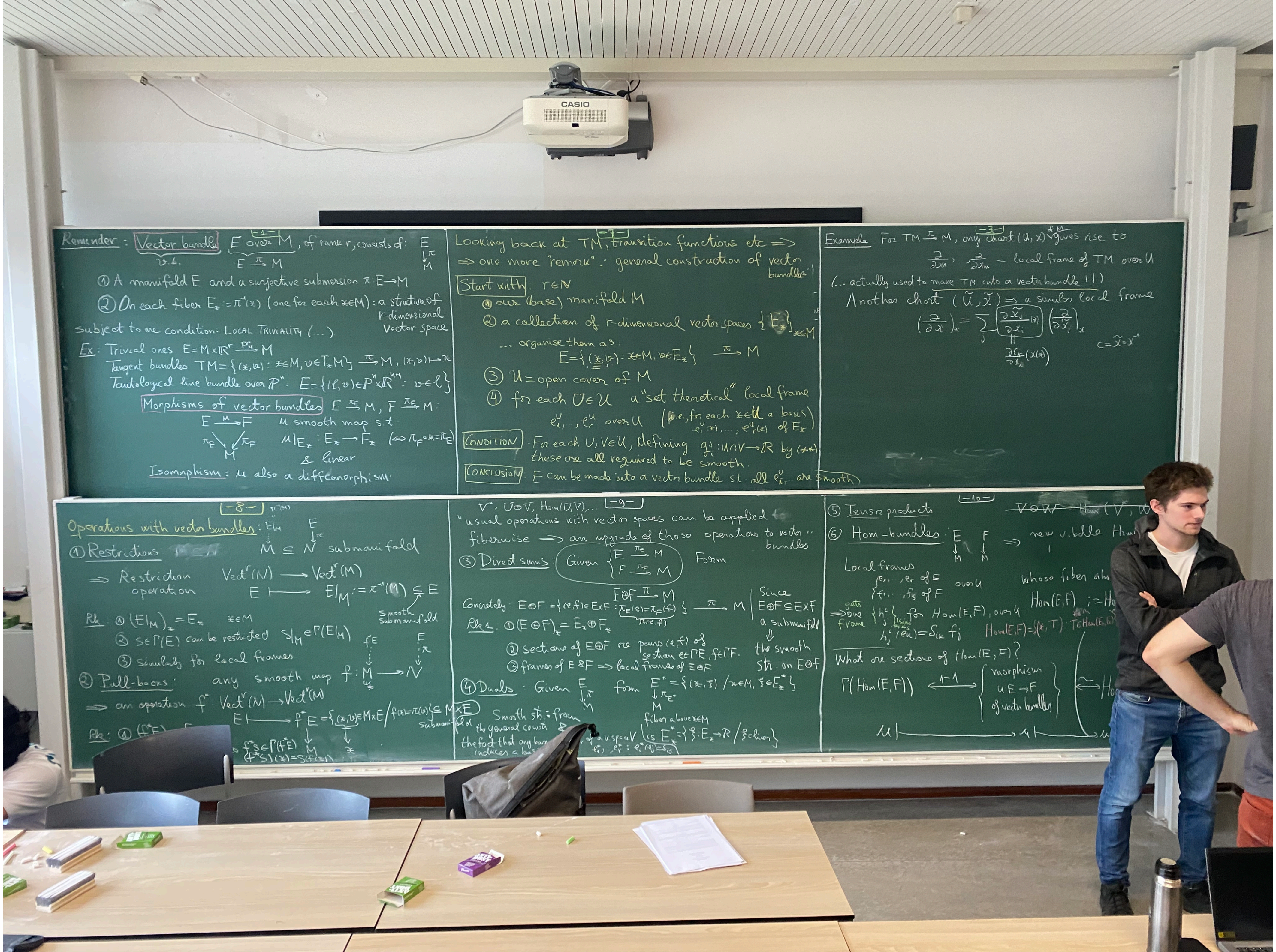
$\Rightarrow$ gets one frame $\{h^i_j\}_{\substack{1 \le i \le r \\ 1 \le j \le s}}$ for $\text{Hom}(E,F)$, over U

$$h^i_j(e_k) = \delta_{ik} f_j$$

What are sections of $\text{Hom}(E,F)$?

$$\Gamma(\text{Hom}(E,F)) \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{morphisms} \\ u : E \to F \\ \text{of vector bundles} \end{matrix} \right\} \xleftrightarrow{\sim} \underset{C^\infty(M)}{\text{Hom}}(\Gamma E, \Gamma F)$$

$$u \longmapsto u \longmapsto u_*$$



$\boxed{-1-}$

Reminder: $\underbrace{\boxed{\text{Vector bundle}}}_{\text{v.b.}}$ $\underbrace{E \text{ over } M}_{E \xrightarrow{\pi} M}$, of rank r, consists of: $E \downarrow^{\pi} M$

① A manifold E and a surjective submersion $\pi : E \to M$

② On each fiber $E_x := \pi^{-1}(x)$ (one for each $x \in M$): a structure of r-dimensional vector space

subject to one condition: LOCAL TRIVIALITY (...)

Ex: Trivial ones $E = M \times \mathbb{R}^r \xrightarrow{pr_M} M$

Tangent bundles $TM = \{(x, v) : x \in M, v \in T_xM\} \xrightarrow{\pi} M$, $(x, v) \mapsto x$

Tautological line bundle over $\mathbb{P}^n$: $E = \{(\ell, v) \in \mathbb{P}^n \times \mathbb{R}^{n+1} : v \in \ell\}$

$\boxed{\text{Morphisms of vector bundles}}$ $E \xrightarrow{\pi_E} M$, $F \xrightarrow{\pi_F} M$:

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

u smooth map s.t. $u|_{E_x} : E_x \to F_x$ ($\Leftrightarrow \pi_F \circ u = \pi_E$) & linear

Isomorphism: u also a diffeomorphism

$\boxed{-7-}$

Looking back at TM, transition functions etc $\Rightarrow$

$\Rightarrow$ one more "remark": general construction of vector bundles !)

$\boxed{\text{Start with}}$: $r \in \mathbb{N}$

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$\boxed{\text{CONCLUSION}}$: E can be made into a vector bundle s.t. all $e_{1}^U, \dots$ are smooth

$\boxed{-3-}$

Example For $TM \xrightarrow{\pi} M$, any chart (U, x) of M gives rise to

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ – local frame of TM over U

(... actually used to make TM into a vector bundle !!)

Another chart $(\tilde{U}, \tilde{x}) \Rightarrow$ a similar local frame

$$\left(\frac{\partial}{\partial x_i}\right)_x = \sum_j \boxed{\frac{\partial \tilde{x}_j}{\partial x_i}(x)} \left(\frac{\partial}{\partial \tilde{x}_j}\right)_x$$

$$\frac{\partial \tilde{x}_j}{\partial x_i}(x) = \frac{\partial c_j}{\partial x_i}(x(x))$$

$c = \tilde{x} \circ x^{-1}$

$\boxed{-8-}$

Operations with vector bundles: $\overset{\pi^{-1}(M)}{\overset{\shortparallel}{E|_M}}$

① Restrictions $\quad M \subseteq N$ submanifold, $E \downarrow^{\pi} N$

$\Rightarrow$ Restriction operation $\quad \mathrm{Vect}^r(N) \longrightarrow \mathrm{Vect}^r(M)$

$E \longmapsto E|_M := \pi^{-1}(M) \subseteq E$

smooth submanifold

Rk: ① $(E|_M)_x = E_x \quad x \in M$

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$\Rightarrow$ an operation $f^* : \mathrm{Vect}^r(N) \to \mathrm{Vect}^r(M)$

$E \longmapsto f^*E = \{(x, v) \in M \times E \,/\, f(x) = \pi(v)\} \subseteq M \times E$ submanifold

$f^*E \downarrow M$, $(x, v) \mapsto x$

Rk: ① $(f^*E) \dots$

$\dots f^*s \in \Gamma(f^*E)$

$(f^*s)(x) = s(f(x))$

$\boxed{-9-}$

V^*, $U \oplus V$, $\mathrm{Hom}(U, V)$, ...

"usual operations with vector spaces can be applied fiberwise $\Rightarrow$ an upgrade of those operations to vector bundles"

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$\boxed{-10-}$

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⑥ Hom-bundles: $E \downarrow M$, $F \downarrow M$ $\Rightarrow$ new v. bdle Hom...

whose fiber abo...

$\mathrm{Hom}(E, F)_x := \mathrm{Ho}\dots$

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$\Rightarrow$ one gets frame $\{h_j^i\}_{\substack{1 \le i \le r \\ 1 \le j \le s}}$ for $\mathrm{Hom}(E, F)$, over U

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