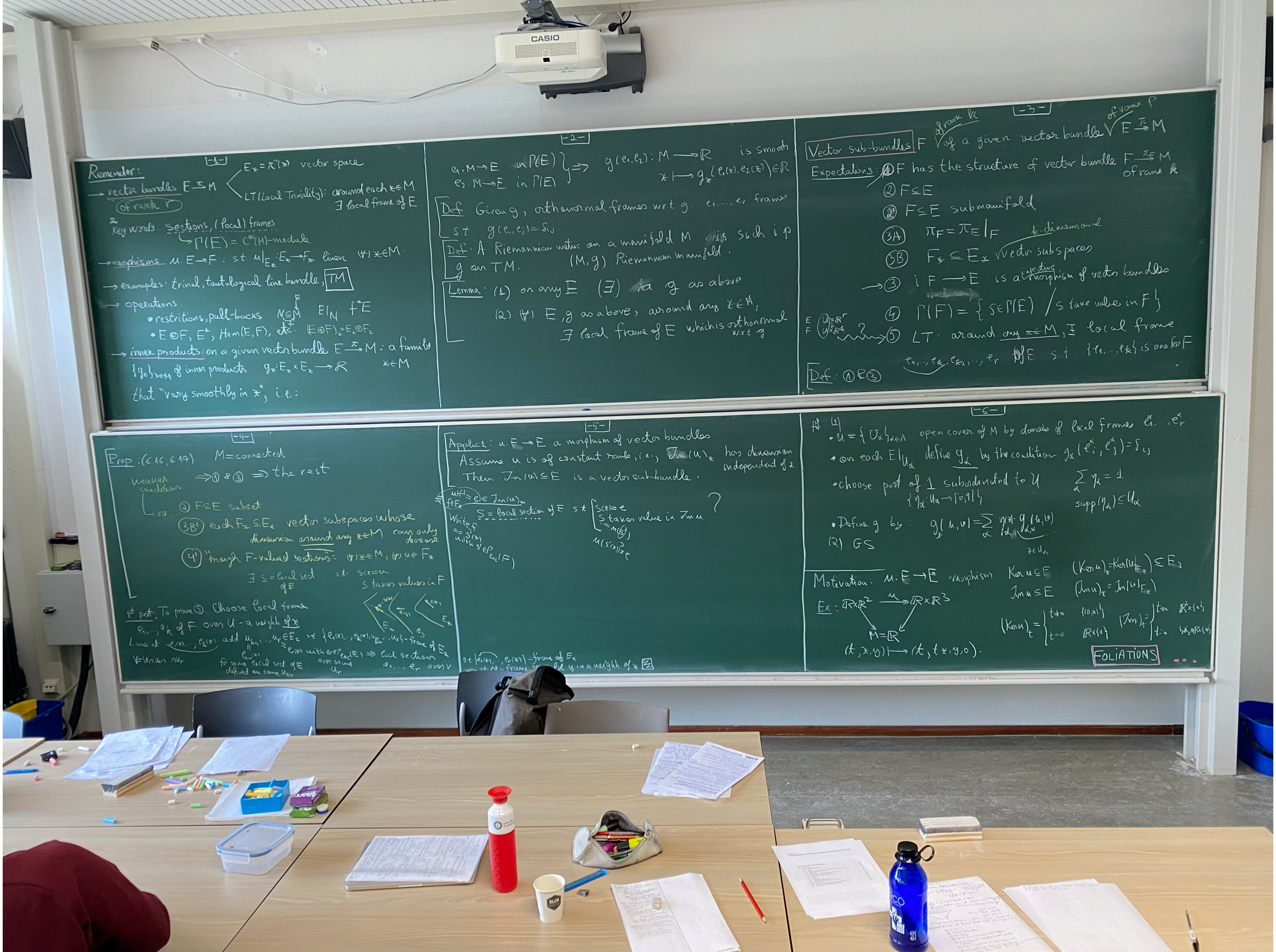

-1-

Reminder:

→ vector bundles $E \xrightarrow{\pi} M$ (of rank r)
- $E_x = \pi^{-1}(x)$ vector space
- LT (Local Triviality): around each $x \in M$ $\exists$ local frame of E

& Key words. Sections, (local) frames
↳ $\Gamma(E) = C^\infty(M)$-module

→ morphisms $u: E \to F$: s.t. $u|_{E_x}: E_x \to F_x$ linear $(\forall)\, x \in M$.

→ examples: trivial, tautological line bundle, $\boxed{TM}$

→ operations.
- restritions, pull-backs $N \subseteq M$ $E|_N$ f^*E
- $E \oplus F$, E^*, $Hom(E,F)$, etc. $(E \oplus F)_x = E_x \oplus F_x$

→ inner products on a given vector bundle $E \xrightarrow{\pi} M$: a family $\{g_x\}_{x \in M}$ of inner products $g_x: E_x \times E_x \to \mathbb{R}$ $x \in M$ that "vary smoothly in x", i.e.:

-2-

$e_1: M \to E$ in $\Gamma(E)$
$e_2: M \to E$ in $\Gamma(E)$
$\} \Rightarrow g(e_1, e_2): M \to \mathbb{R}$ is smooth
$x \mapsto g_x(e_1(x), e_2(x)) \in \mathbb{R}$

Def Given g, orthonormal frames w.r.t g $e_1, \dots, e_r$ frames s.t. $g(e_i, e_j) = \delta_{ij}$

Def. A Riemannian metric on a manifold M is such i.p g on TM. (M, g) Riemannian manifold.

Lemma: (1) on any E $(\exists)$ a g as above
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-3-

$\boxed{\text{Vector sub-bundles}}$ F of rank k of a given vector bundle of rank r $E \xrightarrow{\pi} M$

Expectations:
① F has the structure of vector bundle $F \xrightarrow{\pi_F} M$ of rank k
② $F \subseteq E$
②' $F \subseteq E$ submanifold
(3A) $\pi_F = \pi_E|_F$
(3B) $F_x \subseteq E_x$ k-dimensional vector subspaces
→ ③ $i\ F \to E$ is a injective morphism of vector bundles
④ $\Gamma(F) = \{ s \in \Gamma(E) \,/\, s \text{ take value in } F \}$
E $U \times \mathbb{R}^r$, F $U \times \mathbb{R}^k$? ⇝ ⑤ LT: around any $x \in M$, $\exists$ local frame $\underbrace{e_1, \dots, e_k}, e_{k+1}, \dots, e_r$ of E s.t. $\{e_1, \dots, e_k\}$ is one for F

Def: ① & ③

-4-

Prop: (6.16, 6.17) M = connected
$\Rightarrow$ ① & ③ $\Rightarrow$ the rest

weaker conditions
↳ ② $F \subseteq E$ subset
(3B') each $F_x \subseteq E_x$ vector subspaces whose dimension around any $x \in M$ can only decrease
(4') "enough F-valued sections": $(\forall)\, x \in M$, $(\forall)\, u \in F_x$ $\exists$ s = local sect of E s.t. $s(x) = u$, s takes values in F

$s(x)$, $s(y)$, $s(z)$, E_x, E_y, E_z

1st part. To prove ⑤. Choose local frame $e_1, \dots, e_k$ of F over U - a neighb. of x
Look at $e_1(x), \dots, e_k(x)$ add $u_{k+1}, \dots, u_r \in E_x$ s.t. $\{e_1(x), \dots, e_k(x), u_{k+1}, \dots, u_r\}$ - frame of E_x
$u_{k+1} = e_{k+1}(x)$ for some local sect. of E defined on some U_{k+1}; $u_r = e_r(x)$ with $e_r \in \Gamma_{loc}(E)$ over some U_r ⇒ local sections $e_1, \dots, e_r$ over V
$V = U \cap U_{k+1} \dots \cap U_r$
s.t. $\{e_1(x), \dots, e_r(x)\}$ - frame of E_x
⇒ s.t. $e_1, \dots$ frame [illegible] y in a neighb. of x $\boxed{Z}$

-5-

Applic 1: $u: F \to E$ a morphism of vector bundles
Assume u is of constant rank, i.e., $Im(u)_x$ has dimension independent of x
Then $Im(u) \subseteq E$ is a vector sub-bundle.

$\exists\ f \in F_x$ $u(f) = e \in Im(u)_x$
S = local section of E s.t. $\begin{cases} S(x) = e \\ S \text{ takes value in } Im\, u \\ := u(S') \end{cases}$?
Write f as $S'(x)$ with $s' \in \Gamma_{loc}(F)$
$u(s'(x)) \overset{?}{=} e$

-6-

Pf (1)
- $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ open cover of M by domains of local frames $e_1^\alpha, \dots, e_r^\alpha$
- on each $E|_{U_\alpha}$ define $\underline{g_\alpha}$ by the condition $g_\alpha(e_i^\alpha, e_j^\alpha) = \delta_{ij}$
- choose part of 1 subordinated to $\mathcal{U}$ $\{\eta_\alpha : U_\alpha \to [0,1]\}$ $\sum_\alpha \eta_\alpha = 1$ $supp(\eta_\alpha) \subseteq U_\alpha$
- Define g by: $g_x(u, v) = \sum_\alpha \eta_\alpha(x) \cdot \underbrace{g_{\alpha,x}(u,v)}_{x \in U_\alpha}$

(2) GS

Motivation: $u: F \to E$ morphism
$Ker\, u \subseteq F$ $(Ker\, u)_x = Ker(u|_{F_x}) \subseteq F_x$
$Im\, u \subseteq E$ $(Im\, u)_x = Im(u|_{F_x})$

Ex:
$$\begin{tikzcd} \mathbb{R} \times \mathbb{R}^2 \arrow[rr, "u"] \arrow[dr] & & \mathbb{R} \times \mathbb{R}^3 \arrow[dl] \\ & M = \mathbb{R} & \end{tikzcd}$$
$(t, x, y) \mapsto (t, tx, y, 0)$.

$(Ker\, u)_t = \begin{cases} t \neq 0 & \{(0,0)\} \\ t = 0 & \mathbb{R} \times \{0\} \end{cases}$ $(Im)_t = \begin{cases} t \neq 0 & \mathbb{R}^2 \times \{0\} \\ t = 0 & \{0\} \times \mathbb{R} \times \{0\} \end{cases}$

$\boxed{\text{FOLIATIONS}}$ ...

-1-

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that "vary smoothly in x", i.e.:

$e_1 : M \to$

$e_2 : M \to$

Def: Gi

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Lemma:

-4-

Prop: (6.16, 6.17): M = connected

Applic

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$x \in M$

of E

1.

family

-2-

$$\left.\begin{array}{l} e_1 : M \to E \quad \text{in } \Gamma(E) \\ e_2 : M \to E \quad \text{in } \Gamma(E) \end{array}\right\} \Rightarrow \quad \begin{array}{l} g(e_1, e_2) : M \longrightarrow \mathbb{R} \quad \text{is smooth} \\ \qquad x \longmapsto g_x(e_1(x), e_2(x)) \in \mathbb{R} \end{array}$$

__Def__: Given g, orthonormal frames w.r.t. g: $e_1, \dots, e_r$ frames s.t. $g(e_i, e_j) = \delta_{ij}$.

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Vector sub-

Expectation

E $U \times \mathbb{R}^r$

F $U \times \mathbb{R}^k$

__Def__: ①

-5-

__Applic 1__: $u : F \to E$ a morphism of vector bundles.

has dimension

Pf (1)

is smooth

$, e_2(x)) \in \mathbb{R}$

er frames

such i.p.

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-3-

Vector sub-bundles F of rank k of a given vector bundle of rank p $E \xrightarrow{\pi} M$

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→ ③ $i : F \longrightarrow E$ (inclusion) is a injective morphism of vector bundles

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$E \; (U \times \mathbb{R}^r$
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Def: ① & ③

-6-

Pf (1)

that "vary smoothly in x", i.e.:

-4-

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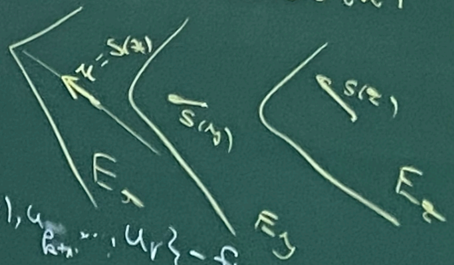

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$u_{k+1} = e_{k+1}(x)$ for some local sect. of E defined on some U_{k+1}

$u_r = e_r(x)$ with $e_r \in \Gamma_{loc}(E)$ over some U_r $\Rightarrow$ local sections $e_1, \dots, e_r$ over V

$V = U \cap U_{k+1} \dots \cap U_r$

Applic

Assu

Then

$\nexists$ $\boxed{u(f) = e}$ $\in$

$f \in F_x$

$S =$

Write $f = s'(x)$ as with $s' \in \Gamma_{loc}$

s.t. $\{e_1(x), \dots, e$

$\Rightarrow$ still a f

Applic1: $u : F \to E$ a morphism of vector bundles.
Assume u is of constant rank, i.e., $\mathrm{Im}(u)_x$ has dimension independent of x
Then $\mathrm{Im}(u) \subseteq E$ is a vector subbundle.

$\exists\, f \in F_x$ $u(f) = e \in \mathrm{Im}(u)_x$

S = local section of E s.t. $\left[\begin{array}{l} S(x) = e \\ S \text{ takes value in } \mathrm{Im}\, u \\ \quad = u(s') \end{array}\right.$?

$u(s'(x)) \overset{?}{=} e$

Write $f =:$ as $s'(x)$ with $s' \in \Gamma_{loc}(F)$

s.t. $\{e_1(x), \dots, e_r(x)\}$ − frame of E_x
$\Rightarrow$ still a frame for all y in a neighb. of x ▨

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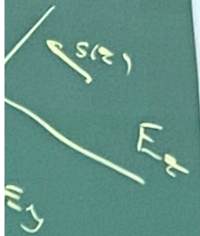

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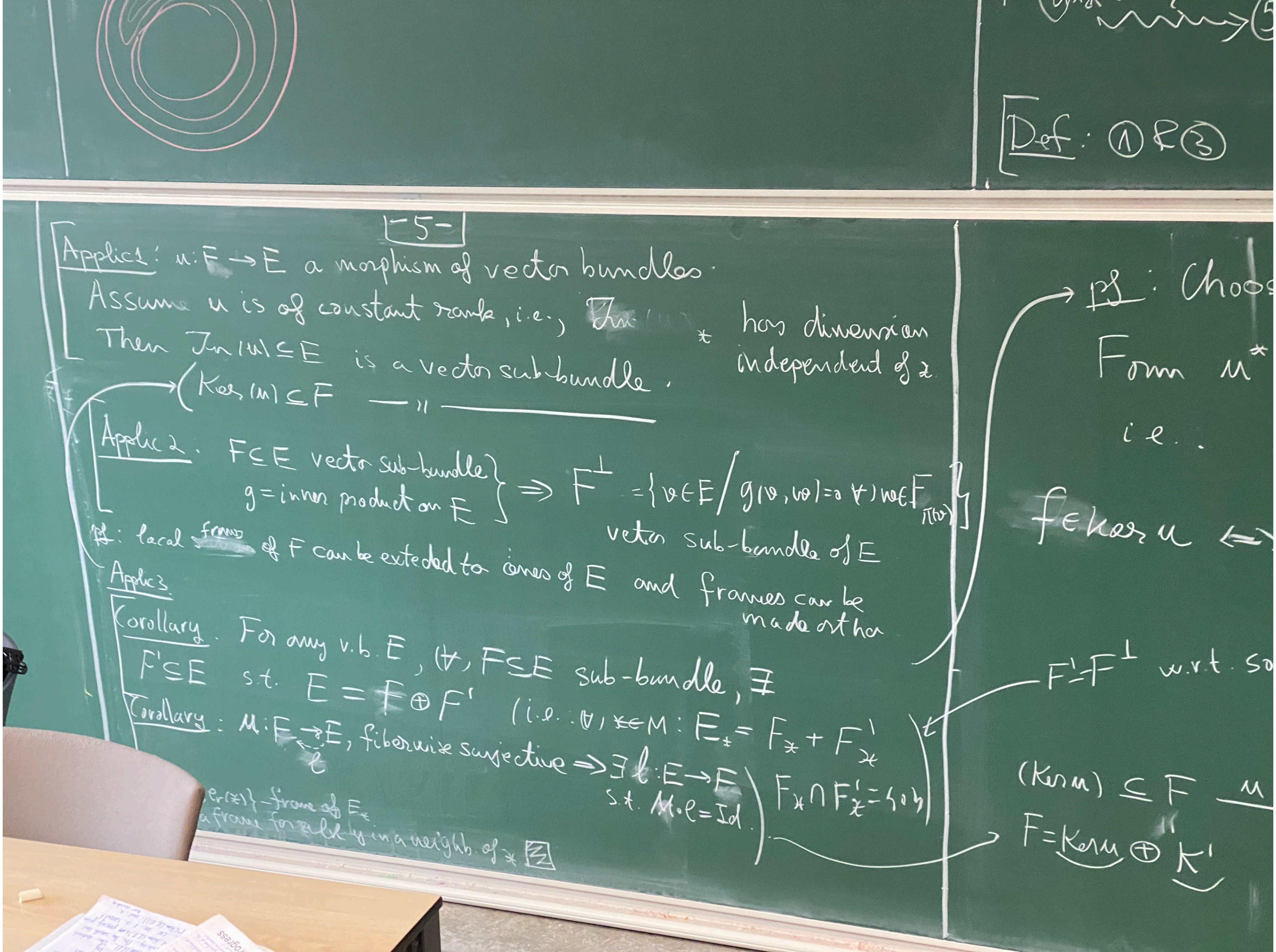

Def: ① & ③

-5-

Applic1: $u: F \to E$ a morphism of vector bundles.
Assume u is of constant rank, i.e., $\operatorname{Im}(u_x)$ has dimension independent of x.
Then $\operatorname{Im}(u) \subseteq E$ is a vector sub-bundle.
$(\operatorname{Ker}(u) \subseteq F$ — " —

Applic2: $\left.\begin{matrix} F \subseteq E \text{ vector sub-bundle} \\ g = \text{inner product on } E \end{matrix}\right\} \Rightarrow F^{\perp} = \{ v \in E \,/\, g(v, w) = 0 \ \forall w \in F_{\pi(v)} \}$
vector sub-bundle of E

Pf: local frames of F can be extended to ones of E and frames can be made ortho

Applic3.

Corollary. For any v.b. E, $\forall$ $F \subseteq E$ sub-bundle, $\exists$ $F' \subseteq E$ s.t. $E = F \oplus F'$ (i.e. $\forall x \in M$: $E_x = F_x + F'_x$, $F_x \cap F'_x = \{0\}$)

Corollary: $u: F \underset{\ell}{\rightleftarrows} E$, fiberwise surjective $\Rightarrow \exists\, \ell: E \to F$ s.t. $u \circ \ell = Id$.

$e_r(x)\}$ - frame of E_x
a frame for all y in a neighb. of x $\boxed{\Sigma}$

Pf: Choos

Form u^*

i.e. .

$f \in \ker u \Leftrightarrow$

$F' = F^{\perp}$ w.r.t. so

$(\operatorname{Ker} u) \subseteq F \xrightarrow{u}$

$F = \operatorname{Ker} u \oplus K'$

$\underbrace{e_1, \dots, e_k}, e_{k+1}, \dots, e_r$ of E s.t. $\{e_1, \dots, e_k\}$ is one for F.

Def: ① & ③

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Pf (1)

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(2) GS

Motivation: $u : F \to E$ morphism $\quad \mathrm{Ker}\, u \subseteq F \quad (\mathrm{Ker}\, u)_x = \mathrm{Ker}(u|_{F_x}) \subseteq F_x$

$\mathrm{Im}\, u \subseteq E \quad (\mathrm{Im}\, u)_x = \mathrm{Im}(u|_{F_x})$

Ex:

$$\begin{tikzcd} \mathbb{R} \times \mathbb{R}^2 \arrow[rr, "u"] \arrow[dr] & & \mathbb{R} \times \mathbb{R}^3 \arrow[dl] \\ & M = \mathbb{R} & \end{tikzcd}$$

$(t, x, y) \longmapsto (t, tx, y, 0)$.

$$(\mathrm{Ker}\, u)_t = \begin{cases} t \neq 0 & \{(0,0)\} \\ t = 0 & \mathbb{R} \times \{0\} \end{cases} \qquad (\mathrm{Im})_t = \begin{cases} t \neq 0 & \mathbb{R}^2 \times \{0\} \\ t = 0 & \{0\} \times \mathbb{R} \times \{0\} \end{cases}$$

FOLIATIONS ...

Def: ① & ③

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end of x.

$\forall w \in F_{\pi(x)}$]

of E

be
ortha

$+ F'_x$

$\cap F'_x = \{0\}$

Pf: Choose g_E on E and g_F on F.

Form $u^*: E \longrightarrow F$ the adjoint of u w.r.t. these inner products

i.e. $g_F(u^*(e), f) = g_E(e, u(f))$ $\forall e, f$

$f \in \ker u \iff f \in (\operatorname{Im} u^*)^\perp$

$\ker u = (\underbrace{\operatorname{Im} u^*})^\perp$ ☒

$F' = F^\perp$ w.r.t. some g $(\exists)$

$(\ker u) \subseteq F \xrightarrow{u} E$

$F = \underbrace{\ker u} \oplus \underbrace{K'}$ $\quad u|_{K'} : K' \longrightarrow E$ isomorphism

Take ℓ to be the inverse

Foliations:

$M =$ m-dimensional manifold

- p-dimensional distributions on M: a vector sub-bundle $\mathcal{F} \subseteq TM$ of rank p
- given $\mathcal{F}$ an 'integral submanifold of $\mathcal{F}$': $N \subseteq M$ submanifold s.t. $T_pN = \mathcal{F}_p$ $(\forall) p \in N$
- given $\mathcal{F}$: called integrable if $(\forall) x \in M$ $\exists N$ as above with $x \in N$

$\Updownarrow$

- given $\mathcal{F}$: called involutive if $(\forall) X, Y \in \Gamma(\mathcal{F}) \Longrightarrow \underbrace{[X,Y]}_{\text{the Lie bracket again in } \Gamma(TM)} \in \Gamma(\mathcal{F})$

Ex1 $M = \mathbb{R}^n$, $\mathcal{F}_{[illegible]}$ spanned by $\frac{\partial}{\partial x_1}$ $\frac{\partial}{\partial x_{[illegible]}}$ [illegible]

Integral: $\mathbb{R}^p \times \{a\}$ $\subseteq \mathbb{R}^n$

$B \cap S \{x\}$ [illegible] $\mathbb{R}^p$ [illegible]

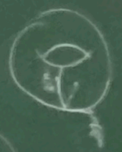
Involutive

Ex2 $X \in \mathfrak{X}(M)$ vector field nowhere vanishing

$\mathcal{F}_x = \mathrm{Span}_{\mathbb{R}}\{X_x\} \subseteq T_xM$ Then integral curves of X give integral submanifolds

Involutive

Ex3. On torus



$\mathcal{F}_1$ spanned by $\frac{\partial}{\partial \theta}$

$\mathcal{F}_2$ — — $\frac{\partial}{\partial \varphi}$

$\mathcal{F}_3$ — — $\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi}$

-5-

Applic1: $u: F \to E$ a morphism of vector bundles

Assume u is of constant rank [illegible]

Then $\mathrm{Im}(u) \subseteq E$ is a vector subbundle

$(\mathrm{Ker}(u) \subseteq F$ — [illegible]

Applic2 $F \subseteq E$ vector subbundle $\Rightarrow F$ [illegible]

generated by [illegible] E

[illegible] of F can be extended to [illegible] of E

local [illegible] of F can be extended to [illegible] of E

[illegible] E [illegible] $F \subseteq E$ [illegible]

Ex1: $M = \mathbb{R}^n$, $\mathcal{F}$ can spanned by $\underbrace{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p} \ \left(\dots \frac{\partial}{\partial x_n} \right)}_{TM}$

Integral: $\mathbb{R}^p \times \{0\}_{n-p} \subseteq \mathbb{R}^n$

$B(0,5) \times \{1\}$ $\qquad$ $\mathbb{R}^p \times \{y\}$ $\quad y \in \mathbb{R}^{n-p}$

Involutive

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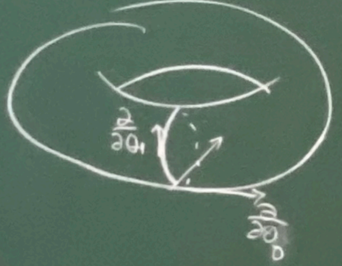
Involutive

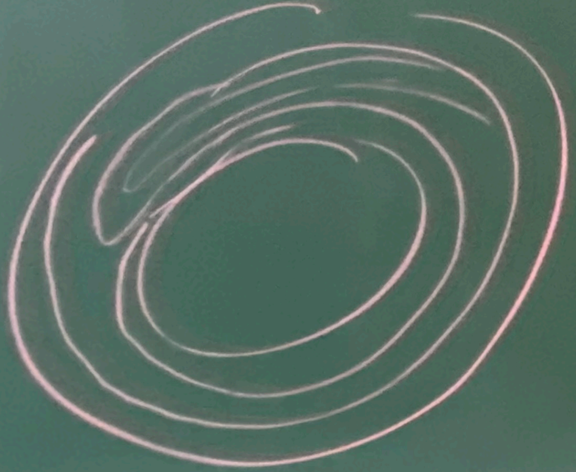
Ex3: On torus



$\mathcal{F}_1$ spanned by $\frac{\partial}{\partial \theta_1}$

$\mathcal{F}_0$ — " — $\frac{\partial}{\partial \theta_0}$

$\mathcal{F}_\lambda$ — " — $\frac{\partial}{\partial \theta_0} + \lambda \frac{\partial}{\partial \theta_1}$



Vector sub-bundles F

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E ($U \times \mathbb{R}^r$
F ($U \times \mathbb{R}^k$? ⇝ ⑤ LT

Def: ① & ③

-5-

of vector bundles

has dimension

independent of x

Choose

From u^*: E