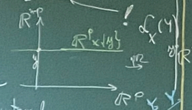
-1-

Reminder: M = manifold (m-dimensional), $p \in \mathbb{N}$

- p-dimensional distribution on M: rank p sub-bundle $\mathcal{F} \subseteq TM$
- integral of $\mathcal{F}$: any submanifold $N \subseteq M$ s.t. $T_pN = \mathcal{F}_p$ $(\forall)\, p \in N$
- $\mathcal{F}$ = integrable if $(\forall)\, x_0 \in M$ $\exists$ integral N passing through x_0 ($x_0 \in N$).
- $\mathcal{F}$ = involutive if $X, Y \in \Gamma(\mathcal{F}) \Longrightarrow [X,Y] \in \Gamma(\mathcal{F})$

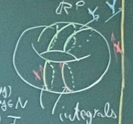
! $\mathcal{L}_X(Y)$

Ex1. $M = \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p} \xrightarrow{pr_2} \mathbb{R}^{m-p} \ni y$; $\mathcal{F}$ = spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}$ } integrals: $\mathbb{R}^p \times \{y\}$



Ex2. $\mathcal{F}$ = 1-dimensional, spanned by some $X \in \mathfrak{X}(M)$ } then the integral curves of X become integrals of $\mathcal{F}$

Ex3. $\forall$ Submersion $\pi : M \to N$ induces $\mathcal{F}_\pi := \operatorname{Ker} d\pi = \{v \in TM : d\pi(v) = 0\}$ involutive!! ✓ } then the fibers $\pi^{-1}(y)$ with $y \in N$ are integrals of $\mathcal{F}_\pi$



-2-

Theorem: Given $\mathcal{F}$ = p-dim distribution, t.f.a.e.:

1. $\mathcal{F}$ = integrable
2. around each $x_0 \in M$ $\exists$ chart (U, x) s.t. $\boxed{\mathcal{F}_x = \operatorname{Span}\left\{\left(\frac{\partial}{\partial x_1}\right)_x, \dots, \left(\frac{\partial}{\partial x_p}\right)_x\right\} \quad (\forall)\, x \in U}$
3. around each $x_0 \in M$ $\exists$ chart (U, x) s.t. all the slices $N_y = \chi^{-1}(\mathbb{R}^p \times \{y\})$ (with $y \in \mathbb{R}^q$) are integrals of $\mathcal{F}$
4. $\mathcal{F}$ = involutive

NB when this happens: call $\mathcal{F}$ a foliation on M.

charts as in 2 & 3: foliated charts for $\mathcal{F}$

It follows that $(\forall)\, x_0 \in M$ $\exists!$ maximal connected integral of $\mathcal{F}$, denoted $L = L_{x_0}$, called leaf of the foliation $\mathcal{F}$. Form a partition of M

also induced by $\sim$ on M: $x \sim y$ iff $\exists$ path $\gamma : [0,1] \to M$ s.t. $\dot\gamma(t) \in \mathcal{F}_{\gamma(t)}$ $\forall t$, $\gamma(0) = x$, $\gamma(1) = y$

-3-

proof: $2 \Rightarrow 3 \Rightarrow 1 \Rightarrow 4$ ✓

Now: $4 \Rightarrow 2$. Fix $x_0 \in M$

Start with any chart (U, x) of M around x_0. May assume

$$T_{x_0}M = \tilde{\mathcal{F}}_{x_0} \oplus \operatorname{Span}_{\mathbb{R}}\left\{\left(\frac{\partial}{\partial x_{p+1}}\right)_{x_0}, \dots, \left(\frac{\partial}{\partial x_m}\right)_{x_0}\right\}$$

(eventually after re-arranging some x_i's)

It follows that $\exists\, W \subseteq M$ open, $x_0 \in W$ s.t.

$$\boxed{(*) \quad T_xM = \mathcal{F}_x \oplus \operatorname{Span}_{\mathbb{R}}\left\{\left(\frac{\partial}{\partial x_{p+1}}\right)_x, \dots, \left(\frac{\partial}{\partial x_m}\right)_p\right\}} \quad (\forall)\, x \in W$$

May assume $M = W$ ball in $\mathbb{R}^m$

Use projection on 1st p coord. $\pi : W \longrightarrow \mathbb{R}^p$; look at $(d\pi)_x : T_xM \to T_{\pi(x)}\mathbb{R}^p$

and notice that $(*) \Longrightarrow (d\pi)_x\big|_{\mathcal{F}_x} : \mathcal{F}_x \xrightarrow{\sim} T_{\pi(x)}\mathbb{R}^p$ linear isom.

$\Rightarrow V^1_x, \dots, V^p_x \in \mathcal{F}_x$ which $\xmapsto{(d\pi)_x} \left(\frac{\partial}{\partial x_1}\right)_{\pi(x)}, \dots, \left(\frac{\partial}{\partial x_p}\right)_{\pi(x)}$ $\quad x \in W = M$

$\Rightarrow$ find $V^1, \dots, V^p \in \Gamma(\mathcal{F})$ which are π-projectable to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p} \in \mathfrak{X}(\mathbb{R}^p)$

-4-

Look at $[V^i, V^j]$ $\xrightarrow{\text{invol}}$ $[V^i, V^j] \in \Gamma\mathcal{F}$

$\xrightarrow{\pi\text{-proj}}$ $[V^i, V^j]$ is π-proj. to $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, i.e., they land in $\operatorname{Span}\left\{\frac{\partial}{\partial x_{p+1}}, \dots, \frac{\partial}{\partial x_m}\right\}$

$\xRightarrow{(*)} [V^i, V^j] = 0.$

To do: find new coord $\tilde{x}$ s.t. $V^i = \frac{\partial}{\partial \tilde{x}_i}$ $\quad 1 \le i \le p$

Use the flows $\varphi^t_{V^i}$ of the vector fields V^i. Define

$h : \mathbb{R}^p \longrightarrow M$, $h(t_1, \dots, t_p) = \varphi^{t_1}_{V^1} \cdots \varphi^{t_p}_{V^p}(x_0)$

$(dh)_0 : T_0\mathbb{R}^p \longrightarrow T_{x_0}M$, $\left(\frac{\partial}{\partial t_1}\right)_0 \longmapsto \frac{d}{dt_1}\Big|_{t_1=0} \varphi^{t_1}_{V^1}\big(\underbrace{\varphi^0_{V^2} \cdots \varphi^0_{V^p}(x_0)}_{x_0}\big) = V^1_{x_0}$

sends the basis $\frac{\partial}{\partial t_i}$ to $V^1_{x_0}, \dots, V^p_{x_0}$ $\Longrightarrow (dh)_0$ = an isom. $\Longrightarrow$

$\Rightarrow h$ = diffeom around 0. Take $\tilde{x} = h^{-1}$ ▨

$V^1_{h(t_1,\dots,t_p)}$ $\quad t \neq 0$

-5-

Complex Vector Bundles: same as Real but work $/\mathbb{C}$, with complex vector spaces.

... but keep in mind:

- duals F^* will be complex duals
- analogue of inner products g: hermitian products ($\mathbb{C}$-valued & $h(v,u) = \overline{h(u,v)}$)
- basic examples: tautological L over $\mathbb{CP}^n$, tangent bundles of complex manifolds ← to be discussed

New operation: Conjugation:

-6-

Complexification:

$\boxed{-1-}$

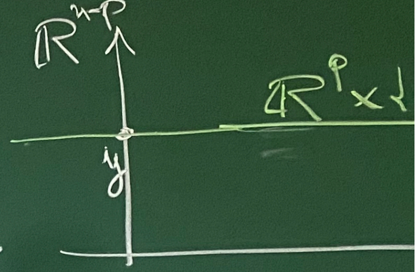
Reminder: M = manifold (n-dimensional), $p \in \mathbb{N}$

- p-dimensional distribution on M: rank p sub-bundle $\mathcal{F}$ ...
- integral of $\mathcal{F}$: any submanifold $N \subseteq M$ s.t. $T_pN = \mathcal{F}_p$ $(\forall$ ...
- $\mathcal{F}$ = integrable: if $(\forall)\, x_0 \in M$ $\exists$ integral N passing th...
- $\mathcal{F}$ = involutive: if $X, Y \in \Gamma(\mathcal{F}) \Longrightarrow [X,Y] \in \Gamma(\mathcal{F}$ ...

Ex1: $M = \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p} \xrightarrow{pr_2} \mathbb{R}^{n-p} \ni y$ ($\mathbb{R}^{n-p} = N$)

$\mathcal{F}$ = spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}$

integrals: $\mathbb{R}^p \times \{y\}$



Ex2: $\mathcal{F}$ = 1-dimensional spanned by some $X \in \mathfrak{X}(M)$

then the integral curves of X become integrals of $\mathcal{F}$

Ex3: $\forall$ Submersion $\pi : M \to N$ ($\pi^{-1}(y) \subseteq M$, $y \in N$)

induces $\mathcal{F}_\pi := \operatorname{Ker} d\pi = \{ v \in TM : d\pi(v) = 0 \}$

involutive!!

then the fibers $\pi^{-1}(y)$ with $y \in N$ are integrals of $\mathcal{F}_\pi$

$\boxed{-2-}$

Theorem: Given $\mathcal{F}$ = p-d...

[-1-]

<u>Reminder</u>: M = manifold (m-dimensional), $p \in \mathbb{N}$

- p-dimensional <u>distribution on M</u>: rank p sub-bundle $\mathcal{F} \subseteq TM$
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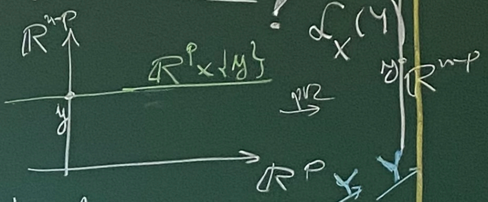

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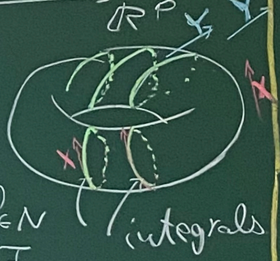

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involutive !! ✓

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[-2-]

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form a partition of M

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-5-

Complex Vector Bundles: same as Real but work /$\mathbb{C}$, with complex vector spaces.

... but keep in mind:

- duals F^* will be complex duals V^* $V \to \mathbb{C}$
- analogue of inner product g: hermitian products ($\mathbb{C}$-valued & $h(v,u) = \overline{h(u,v)}$) $h(u,u) = \overline{h(u,u)}$
- basic examples: $\boxed{\text{tautological } L \text{ over } \mathbb{CP}^n}$, tangent bundles of complex manifolds (to be discussed)

$\boxed{\text{New operation: Conjugation}}$: $F \in Vect_{\mathbb{C}}(M) \Longrightarrow$ a new one $\overline{F}$ which is F but with new structure of complex vector space on the fibers: $v +_{\overline{F}} w = v + w$ $\quad \lambda \cdot_{\overline{F}} v := \overline{\lambda} \cdot_{F} v$

Rk: F being iso with $\overline{F} \iff \exists$ $u: F \to F$ diffeom $\mathbb{C}$-antilinear. THIS MAY NOT EXIST

Rk: For $E \in Vect_{\mathbb{R}}(M) \Longrightarrow E^*$ is isomorphic to E
$g(\cdot, v) \longleftarrow v$

Similarly, for $F \in Vect_{\mathbb{C}}(M)$:

$$F^* \simeq \overline{F}$$

-6-

$\boxed{\text{Complexification}}$: $Vect^r_{\mathbb{R}}(M) \longrightarrow Vect^r_{\mathbb{C}}(M)$, $E \longmapsto E_{\mathbb{C}}$

Def 1: $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} = \{ \underbrace{u \otimes 1 + v \otimes i}_{u + i \cdot v} : u, v \in E \text{ same fiber} \}$

Def 1' $E_{\mathbb{C}} = E \oplus E = \{ (u,v) : u, v \in \text{same fiber of } E \}$

with complex mult def. by $\underset{\mathbb{C}}{(a+bi)} \cdot (u,v) := (au - bv, av + bu)$

$(u,v) + (u',v') = (u+u', v+v')$

$a \cdot (u,v) = (au, av)$

Rk: Unlike general $F \in Vect_{\mathbb{C}}(M)$, those of type $E_{\mathbb{C}}$ come with a conjugation map

$$E_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}, \quad \underset{u+iv}{w} \longmapsto \underset{u-iv}{\overline{w}}$$

Note: $E_{\mathbb{C}} \simeq \overline{E_{\mathbb{C}}}$

-7-

$\boxed{\text{From real to complex}}$: $Vect^s_{\mathbb{C}}(M) \longrightarrow Vect^{2s}_{\mathbb{R}}(M)$

$F \longmapsto F_{\mathbb{R}} := F$ but forget mult. by i

Def: A complex str. on a real vector bundle E is any morphism $J: E \to E$ s.t. $J \circ J = -Id_E$.

Def: An almost complex str. on a manifold M is a complex str. J on the tangent bundle.

? integrability, involutivity ?

-8-

Given $E \in Vect_{\mathbb{R}}(M)$ when can it be made complex?

Missing "multiplication by i"

$(a+bi) \cdot e := ae + b J(e)$ makes E into a complex v. bd

-5-

COMPLEX VECTOR BUNDLES : Same as REAL but work $/\mathbb{C}$, with complex vector spaces.

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From real to complex : -7- $Vect^s_{\mathbb{C}}(M) \longrightarrow Vect^{2s}_{\mathbb{R}}(M)$

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-6-

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-8-

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? integrability, involutivity ?

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$(a+$

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y i

Given $E \in \mathrm{Vect}_{\mathbb{R}}(M)$ wh[illegible] - 8 -

can it be made complex?

Missing "multiplication by i"

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-7-

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