— 1 —

Reminder:

- operations that work over $K \in \{\mathbb{R}, \mathbb{C}\}$: sums $E_1 \oplus E_2$, duals, etc
- operations relating $/\mathbb{R}$ with $/\mathbb{C}$
  - → Complexification $\mathrm{Vect}^r_{\mathbb{R}}(M) \longrightarrow \mathrm{Vect}^r_{\mathbb{C}}(M)$, $E \longmapsto E_{\mathbb{C}}$
  - → Forgetting $\mathbb{C}$ $\quad \mathrm{Vect}^r_{\mathbb{C}}(M) \longrightarrow \mathrm{Vect}^{2r}_{\mathbb{R}}(M)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\Psi}{E}$ $\boxed{\text{when does it come from a complex v.b. ?}}$

Def: A complex structure on a real vector bundle E is any v.b morphism $J: E \to E$ s.t $J^2 = -\mathrm{Id}$.
We denote by E_J the resulting complex v.b. structure on E:
$$z \cdot e = ae + bJ(e) \quad (\forall)\, z = a + bi \in \mathbb{C},\ e \in E$$

— 2 —

Geometric structures: use $E = TM$. Eg

- Riemannian structures on M: inner product on $E = TM$
- foliations: – distributions on M: vector sub-bundles $\mathcal{F} \subseteq TM$
  – integrability of $\mathcal{F}$: through each $p \in M$ $\exists$ integral N of $\mathcal{F}$
  $\Leftrightarrow$ —— " —————— chart (U, χ) foliated
  – Frobenius thm: $\mathcal{F}$ is integrable iff $[X, Y] \in \Gamma\mathcal{F}$, $(\forall)\, X, Y \in \Gamma\mathcal{F}$

Def: An almost complex structure on M is any $J: TM \to TM$, complex structure on its tangent bundle.

— 3 —

Reminder: smooth structure $\mathcal{A}$ on a space M

- consider (topological) charts $\chi: \underset{M}{\overset{\text{open}}{U}} \longrightarrow \underset{\mathbb{R}^m}{\overset{\text{open}}{\Omega}}$
- two of them, χ & χ' called smoothly compatible if
  $c = \chi' \circ \chi^{-1}: \underset{\mathbb{R}^m}{\text{open}} \longrightarrow \underset{\mathbb{R}^m}{\text{open}}$ is smooth, with
  —"— inverse
- Smooth atlas $\mathcal{A}$ on M: collection of charts, each two smoothly compatible
- smooth structure on M = maximal smooth atlas $\mathcal{A}$

Call $(M, \mathcal{A})$ = (smooth) manifold.

Reminder: smooth structure $\mathcal{A}$ on a space M

- consider (topological) charts $\chi : \underset{\underset{M}{\cap\,\text{open}}}{U} \longrightarrow \underset{\underset{\cancel{\mathbb{R}^m}\ \mathbb{C}^m = \mathbb{R}^{2m}}{\cap\ \text{open}}}{\Omega}$
- two of them, χ & χ' called ~~smoothly~~ holom. compatible if

$$c = \chi' \circ \chi^{-1} : \underset{\underset{\cancel{\mathbb{R}^m}\ \mathbb{C}^m}{\cap}}{\text{open}} \longrightarrow \underset{\underset{\cancel{\mathbb{R}^m}\ \mathbb{C}^m}{\cap}}{\text{open}}$$ is ~~smooth~~ holom, with —"— inverse

- ~~smooth~~ holomorphic atlas $\mathcal{A}$ on M: collection of charts with U's covering M, each two ~~smoothly~~ holom. compatible
- ~~smooth~~ holomorphic structure on M := maximal ~~smooth~~ holom atlas $\mathcal{A}$

Call $(M, \mathcal{A})$ = (~~smooth~~ complex) manifold.

Holomorphic structures $\mathcal{C}$: Same, but replace $\mathbb{R}$ by $\mathbb{C}$ and smooth by holomorphic.

$\boxed{\text{Rk}}$: $\mathcal{C}$ induces a smooth structure $\mathcal{A} =: Re(\mathcal{C})$.

Q: $\mathcal{C}$ = a real $\mathcal{A}$ plus "what else"? → ~~an almost complex structure J s.t. (*)~~

$= \mathbb{R}^{2m}$

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M

<u>Partial answer 1</u>: While TM is determined by $\mathcal{A} = \mathrm{Re}(\mathcal{C})$

$\mathcal{C}$ induces an almost complex structure $J_{\mathcal{C}} : TM \to TM$.

<u>How?</u> $p \in M$ Choose $x \in \mathcal{C}$, $x . U \to \Omega \subseteq \mathbb{C}^m$ $(x_1, \dots, x_m, y_1, \dots, y_m)$

Take differential $(dx)_p : T_pM \longrightarrow \mathbb{C}^m$ ← complex v.space

$J_{\mathcal{C}}$ acting on T_pM, i acting on $\mathbb{C}^m$

$J_{\mathcal{C}}(v) = (dx)_p^{-1}\left(i (dx)_p(v)\right)$

Equivalently $x = (x_1, \dots, x_m, y_1, \dots, y_m) : U \to \mathbb{R}^{2m}$

$$\begin{array}{ccccccccc}
\text{bases} & \left(\frac{\partial}{\partial x_1}\right)_p, & \dots, & \left(\frac{\partial}{\partial x_m}\right)_p, & \left(\frac{\partial}{\partial y_1}\right)_p, & \dots, & \left(\frac{\partial}{\partial y_m}\right)_p & \in T_pM \\
& \downarrow & & \downarrow & \downarrow & & \downarrow & \\
& \left(\frac{\partial}{\partial y_1}\right)_p & \dots & \left(\frac{\partial}{\partial y_m}\right)_p & -\left(\frac{\partial}{\partial x_1}\right)_p & \dots & -\left(\frac{\partial}{\partial x_m}\right)_p & \in T
\end{array}$$

x

(*)

-5-

Partial answer 2: If $\mathcal{C}_1, \mathcal{C}_2$ are two holom structures on M inducing the same smooth str. then $\mathcal{C}_1 = \mathcal{C}_2 \iff J_{\mathcal{C}_1} = J_{\mathcal{C}_2}$

Pf: "$\Leftarrow$" $\chi_1 \in \mathcal{C}_1$, $\chi_2 \in \mathcal{C}_2$ $\Rightarrow$ is $c = \chi_2^{-1} \circ \chi_1$: open $O_1 \subset \mathbb{C}^m$ $\to$ open $O_2 \subset \mathbb{C}^m$

is it holomorphic ??

$(f_1, \dots, f_m, g_1, \dots, g_m) : O_1 \to O_2$

We know: $(dc)_p : \mathbb{C}^m \to \mathbb{C}^n$ is $\mathbb{C}$-linear. Write out

$$\Rightarrow \frac{\partial f_k}{\partial x_j} = \frac{\partial g_k}{\partial y_j}, \quad \frac{\partial f_k}{\partial y_j} = -\frac{\partial g_k}{\partial x_j}$$

Precisely the C-R charactarization of holomorphicity !

smooth structure, $\mathcal{A} =: Re(\mathcal{C})$.

Q: $\mathcal{C}$ = a real $\mathcal{A}$ plus "what else"? ~~an almost complex structure J~~ s.t. (*)

TM

N of J.

(U, x)

ated

, $(\forall) X, Y \in \Gamma \mathcal{F}$

$TM \to TM$,

n. structure $\mathcal{C}$

be unique!)

Thm (Newlander-Nirenberg): J is integrable iff —6—

(*) $\underbrace{[X,Y] + J\big([JX,Y]+[X,JY]\big) - [JX,JY]}_{N_J(X,Y)} = 0 \quad (\forall) X, Y \in \mathfrak{X}(M)$.

$N_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

- skew-symmetric
- $C^\infty(M)$ linear in X and Y: $\begin{cases} N_J(fX,Y) = f N_J(X,Y) & f \in C^\infty(M) \\ \dots \end{cases}$

i.e. N_J is a tensor $\in \Gamma(\Lambda^2 T^*M \otimes TM)$

Pf: "$\Rightarrow$" In a holom. chart $\chi = (x_1, \dots, x_n, y_1, \dots, y_n)$:

(*) applied to vectors of type $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ is obviously 0.

For arbitrary X, Y: are $C^\infty(M)$-linear combinations of the previous ones

$N_J\left(\sum f_j \frac{\partial}{\partial x_j} + \dots, \sum g \frac{\partial}{\partial x_n}\right) = \sum f \cdot g \, \underbrace{N_J\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}_{} = 0$

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Pf:

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is $\mathbb{C}$-

$\Rightarrow$

Preci

$z \cdot e = ae + b J(e)$ $\quad (\forall) z = a+bi \in \mathbb{C},\ e \in E$
$z = a+bi \in \mathbb{C}$, $e \in E$

Holomorphic structures

Rk: $\mathcal{C}$ induces a smooth structur

Q: $\mathcal{C}$ = a real $\mathcal{A}$ plus

Geometric structures: now $E = TM$. E.g. $\boxed{2}$

- Riemannian structures on M: inner product on $E = TM$
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    $\Leftrightarrow$ — " — chart (U, χ) foliated
  - Frobenius thm: $\mathcal{F}$ is integrable iff $[X,Y] \in \Gamma\mathcal{F}$, $(\forall) X, Y \in \Gamma\mathcal{F}$

Def: An almost complex structure on M is any $J: TM \to TM$, complex structure on its tangent bundle TM.
We say that J is integrable if $J = J_{\mathcal{C}}$ for some holom. structure $\mathcal{C}$ (will be unique!)

Thm (Newlander-Nirenberg): J

(*) $\underbrace{[X,Y] + J([JX, \ldots}_{N_J(X,Y)}$

$N_J: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow$

- skew-symmetric
- $C^\infty(M)$ linear in X and Y.

i.e. N_J is

Pf: "$\Rightarrow$" In a holom. chart $\chi = (x_1, \ldots$
(*) applied to vectors of type $\frac{\partial}{\partial x}$
For arbitrary X, Y: are $C^\infty(M)$-linea
$N_J(\sum f_j \frac{\partial}{\partial x_j} + \ldots \sum$

-1-

Reminder

- operations that work over $K \in \{\mathbb{R}, \mathbb{C}\}$: sums $E_1 \oplus E_2$, duals, etc, pull-backs
- operations relating $/\mathbb{R}$ with $/\mathbb{C}$
  - $\to$ Complexification $\mathrm{Vect}^r_{\mathbb{R}}(M) \to \mathrm{Vect}^r_{\mathbb{C}}(M)$, $E \mapsto E_{\mathbb{C}}$
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-2-

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-3-

Reminder: smooth structure $\mathcal{A}$ on a space M

- consider (topological) charts $\chi: U \longrightarrow \Omega$, $U \subset M$ open, Ω open $\subset$ ~~$\mathbb{R}^m$~~ $\mathbb{C}^m = \mathbb{R}^{2m}$
- two of them, χ & χ' called ~~smoothly~~ holom. compatible if $c = \chi' \circ \chi^{-1}$: open $\longrightarrow$ open (in ~~$\mathbb{R}^m$~~ $\mathbb{C}^m$) is ~~smooth~~ holom, with — " — inverse
- ~~smooth~~ holomorphic atlas $\mathcal{A}$ on M: collection of charts, with U's covering M, each two ~~smoothly~~ holom compatible
- ~~smooth~~ holomorphic structure on M = maximal ~~smooth~~ holom atlas $\mathcal{A}$

Call $(M, \mathcal{A})$ = (~~smooth~~ complex) manifold

Holomorphic structures $\mathcal{C}$: same, but replace $\mathbb{R}$ by $\mathbb{C}$

Rk: $\mathcal{C}$ induces a smooth structure, $\mathcal{A} =: \mathrm{Re}(\mathcal{C})$ and smooth by holomorphic

Q: $\mathcal{C}$ = a real $\mathcal{A}$ plus "what else"? an ~~almost complex structure~~ J s.t. $(*)$

-4-

Partial answer 1 While TM is determined by $\mathcal{A} = \mathrm{Re}(\mathcal{C})$, $\mathcal{C}$ induces an almost complex structure $J_{\mathcal{C}}: TM \to TM$

How? $p \in M$. Choose $\chi \in \mathcal{C}$, $\chi: U \to \Omega \subset \mathbb{C}^m$ $(x_1, \dots, x_m, y_1, \dots, y_m)$

Take differential $(d\chi)_p: T_pM \longrightarrow \mathbb{C}^m$ $\leftarrow$ complex v.space

$J_{\mathcal{C}}(v) = (d\chi)_p^{-1}\left(i\,(d\chi)_p(v)\right)$

Equivalently $\chi = (x_1, \dots, x_m, y_1, \dots, y_m): U \to \mathbb{R}^{2m}$

basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_m}\right)_p \in T_pM$

$\downarrow$

$\left(\frac{\partial}{\partial y_1}\right)_p, \dots, \left(\frac{\partial}{\partial y_m}\right)_p, -\left(\frac{\partial}{\partial x_1}\right)_p, \dots, -\left(\frac{\partial}{\partial x_m}\right)_p \in T_pM$

-5-

Partial answer 2 If $\mathcal{C}_1, \mathcal{C}_2$ are two holom structures on M inducing the same smooth str then $\mathcal{C}_1 = \mathcal{C}_2 \Longleftrightarrow J_{\mathcal{C}_1} = J_{\mathcal{C}_2}$

Pf: "$\Leftarrow$" $\chi_1 \in \mathcal{C}_1$, $\chi_2 \in \mathcal{C}_2$ $\Rightarrow$ is $c = \chi_2^{-1} \circ \chi_1$: open $O_1 \subset \mathbb{C}^m$ $\to$ open $O_2 \subset \mathbb{C}^m$ is it holomorphic??

We know $(dc)_p: \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is $\mathbb{C}$-linear. Write out $(f_1, \dots, f_m, g_1, \dots, g_m): O_1 \to O_2$

$\Rightarrow \frac{\partial f_k}{\partial x_j} = \frac{\partial g_k}{\partial y_j}, \frac{\partial f_k}{\partial y_j} = -\frac{\partial g_k}{\partial x_j}$

Precisely the C-R characterization of holomorphicity!

-6-

Thm (Newlander-Nirenberg): J is integrable iff

$(*)$ $\underbrace{[X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]}_{N_J(X,Y)} = 0$ $(\forall) X, Y \in \mathfrak{X}(M)$

$N_J: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$
- skew-symmetric
- $C^\infty(M)$ linear in X and Y: $N_J(fX, Y) = f N_J(X, Y)$, $f \in C^\infty(M)$ ...

i.e. N_J is a tensor $\in \Gamma(\Lambda^2 T^*M \otimes TM)$

Pf: "$\Rightarrow$" In a holom chart $\chi = (x_1, \dots, x_m, y_1, \dots, y_m)$:
$(*)$ applied to vectors of type $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ is obviously 0
[illegible] X, Y are $C^\infty(M)$-linear combinations of the previ[illegible]
[illegible] $\dots, \sum g \frac{\partial}{\partial x_k}) = \sum f \cdot g \, N_J(\frac{\partial}{\partial x}, [illegible])$

— 6 —

Thm (Newlander-Nirenberg): J is integrable iff

(*) $\underbrace{[X,Y] + J\left([JX,Y]+[X,JY]\right) - [JX,JY]}_{N_J(X,Y)} = 0 \quad (\forall)\, X,Y \in \mathfrak{X}(M)$.

$N_J : \mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

- skew-symmetric
- $C^\infty(M)$ linear in X and Y. $\begin{cases} N_J(fX,Y) = f N_J(X,Y) \\ \dots \end{cases}$ $f \in C^\infty(M)$

i.e. N_J is a tensor $\in \Gamma(\Lambda^2 T^*M \otimes TM)$

Cor: Complex manifolds as a real manifold together with J satisfying (*)

pf: "$\Rightarrow$" In a holom. chart $\chi = (x_1,\dots,x_n,y_1,\dots,y_n)$:

(*) applied to vectors of type $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ is obviously 0.

For arbitrary X,Y: are $C^\infty(M)$-linear combinations of the previous ones

$N_J\left(\sum f_j \frac{\partial}{\partial x_j} + \dots, \sum g \frac{\partial}{\partial x_n}\right) = \sum f \cdot g\, \underbrace{N_J\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}_{0} = 0$

"$\Leftarrow$" Way too difficult to discuss it

Partial ans...

inducing ...

pf: "$\Leftarrow$" ...

We know: ...

is $\mathbb{C}$-linear ...

$\Rightarrow \frac{\partial f_k}{\partial x_j} = \dots$

Precisely ...

...ms $E_1 \oplus E_2$, duals, etc
pull-backs
quotients/sub E
same fibers

(M), $E \mapsto E_{\mathbb{C}}$

$^{2r}(M)$

hen does it come
complex v.b. ?

ructure on E

$e \in E$

$\boxed{-7-}$

A slightly different flavor of $(*)$: Via re-interpretation of complex structure $J : E \to E$ on a $E \in Vect^r_{\mathbb{R}}(M)$ in terms of sub-bundle

① complexify E to $E_{\mathbb{C}}$

② complexify J to $J_{\mathbb{C}} : E_{\mathbb{C}} \to E_{\mathbb{C}}$, $u + iv \mapsto J(u) + iJ(v)$ "$J(u+iv)$"

③ $J_{\mathbb{C}}^2 = -Id$, now over $\mathbb{C}$ $\Longrightarrow$ $E_{\mathbb{C}} = E^{1,0} \oplus E^{0,1}$

the $+i$ eigenspaces
$= \{ e \in E_{\mathbb{C}} : J_{\mathbb{C}}(e) = ie \}$
$= \{ u - iJ(u) : u \in E \}$

$-i$ eigenspace
$= \overline{E^{1,0}}$

$e = u + iv$
$Ju + iJv = iu - v$

④ Denoting $\mathcal{C} = E^{1,0}$. $\boxed{\mathcal{C} \subseteq E_{\mathbb{C}}}$ complex subspace with $\mathcal{C} \oplus \overline{\mathcal{C}} = E_{\mathbb{C}}$

Rk: J can be recovered from $\mathcal{C}$!

$$\mathcal{C} \xrightarrow[\simeq]{Re} E \overset{J}{\dashrightarrow} E, \qquad \mathcal{C} \xrightarrow[\simeq]{-Im} E$$

Back to J ...

$\mathcal{C} = \{ u - iJ$ ...

$\Gamma\mathcal{C} = \{ X - iJ$ ...

$[X - iJX, Y - i$ ...

$= \underbrace{([X,Y] - [JX,JY]}_{Z}$ ...

Conclusion: $(*)$ ...

Thm (Newlander-Nirenberg): J is integrable iff $\boxed{-6-}$

$(*)$ $\underbrace{[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]}_{N_J(X,Y)} = 0 \quad \forall X, Y \in \mathfrak{X}(M)$.

Partial answer?

-8-

Back to $J: TM \to TM$

$\mathcal{C} = \{ u - iJ(u) : u \in TM \} \subseteq TM \otimes \mathbb{C}$

$\Gamma\mathcal{C} = \{ X - iJ(X) : X \in \mathfrak{X}(M) \} \subseteq \underbrace{\mathfrak{X}(M)}_{[\cdot,\cdot]} \otimes \mathbb{C}$

extend $[\cdot,\cdot]$ by $\mathbb{C}$-linearity

$$[X - iJX, Y - iJY] =$$

$$= \underbrace{([X,Y] - [JX,JY])}_{Z} - i\,\underbrace{([JX,Y] + [X,JY])}_{?\,J(Z)\,?} \overset{?}{\in} \Gamma\mathcal{C}$$

Conclusion: $(*) \iff [\tilde{X}, \tilde{Y}] \in \Gamma\mathcal{C} \quad \forall \tilde{X}, \tilde{Y} \in \Gamma\mathcal{C}$.

-5-

Partial answer 2: If $\mathcal{C}_1, \mathcal{C}_2$ are two holom structures on M inducing the same smooth st. then $\mathcal{C}_1 = \mathcal{C}_2 \iff J_{\mathcal{C}_1} = J_{\mathcal{C}_2}$

$\boxed{-g-}$

$g(u,u)$

$\omega(u,v) = 0 \ \forall v \Rightarrow u = 0$

Geom. structures: variations on inner products:

- inner product g on $E \in Vect^r_{\mathbb{R}}(M)$:

  fiberwise bilinear, symmetric, positive definite $g_x : E_x \times E_x \longrightarrow \mathbb{R}$, "smooth in x"

- symplectic form ω on $E \in Vect^r_{\mathbb{R}}(M)$:

  fiberwise bilinear, skew-symmetric, non-degenerate $\omega_x : E_x \times E_x \longrightarrow \mathbb{R}$ —— " ——

- hermitian product h on $F \in Vect^r_{\mathbb{C}}(M)$:

  fiberwise, conjugate symmetric, linear in 1st, + definite $h_x : F_x \times F_x \longrightarrow \mathbb{C}$ —— " ——

- hermitian structure on $E \in Vect^r_{\mathbb{R}}(M)$: (J, h) with $\begin{cases} J = \text{complex str. } J : E \to E \\ h \text{ as above} \end{cases}$

  g, ω

Relationship: $h(u,v) = g(u,v) + i\,\omega(u,v)$ being hermitian:

$\rightarrow h(v,u) = \overline{h(u,v)} \iff g = \text{symmetric}, \omega = \text{skew}$

$\rightarrow h = \mathbb{R}\text{-bilinear} \iff g, \omega \text{ both } \mathbb{R}\text{-bilinear}$

$\rightarrow h = \mathbb{C}\text{-linear in } u \iff \begin{cases} \omega(u,v) = -g(Ju,v) \\ g(u,v) = \omega(Ju,v) \end{cases}$

$\rightarrow h(u,u) > 0 \ (\forall u \neq 0) \iff g = \text{positive definite} \Rightarrow \omega = \text{non-degenerate}.$

$h(iu,v) = i\,h(u,v)$

(J,h) on E

$g(iu,v) + i\,\omega(iu,v) = i\,g(u,v) - \omega(u,v)$

…morphism $J: E \to E$ s.t. $J^2 = -\mathrm{Id}$.

We denote by E …

(When does it come from a complex v.b.?)

Geometric structures: use $E = TM$. E.g.

- **Riemannian structures** on M: inner product on $E = TM$.
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    $\Leftrightarrow$ $\boxed{\text{— " — chart } (U, \chi) \text{ foliated}}$
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Def: An almost complex structure on M is any $J: TM \to TM$, complex structure on its tangent bundle TM.

We say that J is integrable if $J = J_{\mathcal{C}}$ for some holom. structure $\mathcal{C}$ (will be unique!)

Def: A almost symplectic str. on M is a sympl. str. on TM

i.e., a 2-form $\omega \in \Omega^2(M)$ which is fiberwise nondegenerate.

It is called integrable (Ω: symplectic str.) if: …

Standard example. $M = \mathbb{R}^{2n}$ $(\mathbb{C}^n)$

Use standard g and J $\Rightarrow$

$$\omega_{\text{standard}} := \sum_{j=1}^{n} dy_j \wedge dx_j \in \Omega^2(\mathbb{R}^{2n})$$

Call a general $\omega \in \Omega^2(M)$ integrable if around any $p \in M$ $\exists$ coord. $(x_1, \ldots, y_n)$ s.t.

$$\boxed{\omega|_U = \chi^*(\omega_{\text{standard}}) = \sum_{j=1}^{n} dy_j \wedge dx_j}$$

In this setting:

Theorem (Darboux): An almost symplectic $\omega \in \Omega^2(M)$ is integrable if and only if $d\omega = 0$.

$(\omega([X,Y],Z) + \ldots = 0)$

Standard examplex: $M = \mathbb{R}^{2n}$ $(\mathbb{C}^n)$ $\boxed{-11-}$

Use standard g and J $\Rightarrow$

$$\omega_{\text{standard}} := \sum_{j=1}^{n} dy_j \wedge dx_j \in \Omega^2(\underbrace{\mathbb{R}^{2n}}_{(x_1,\dots,x_n,y_1,\dots,y_n)})$$

Call a general $\omega \in \Omega^2(M)$ integrable if around any $p \in M$ $\exists$ coord $\underset{x}{(x_1,\dots,y_n)}$ s.t.

$$\boxed{\omega|_U = x^*(\omega_{\text{standard}}) = \sum_{j=1}^{n} dy_j \wedge dx_j}$$

In this setting:

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  g, ω

Relationship: $h(u,v) = g(u,v) + i\,\omega(u,v)$ being hermitian

$\to h(v,u) = \overline{h(u,v)} \Leftrightarrow g =$ symmetric, $\omega =$ skew

$\to h = \mathbb{R}$-bilinear $\Leftrightarrow g, \omega$ both $\mathbb{R}$-bilinear

$\to h = \mathbb{C}$-linear in $u \Leftrightarrow \begin{cases} \omega(u,v) = -g(Ju,v) \\ g(u,v) = \omega(Ju,v) \end{cases}$ (J,h) on E

$\to h(u,u) > 0$ $(\forall u \neq 0) \Leftrightarrow g$ positive definite $\Rightarrow \omega =$ non-deg