-1- Lie groups & actions

$e \in G$ the unit of G

Reminder (?)

Def. A Lie group is a group $(G, \cdot)$ which is also a manifold in such a way that the group operations are smooth

multiplication $m : G \times G \to G$, $(g,h) \mapsto g \cdot h$ & inversion $i : G \to G$, $g \mapsto g^{-1}$

Rk. Such Gs are "homogeneous": each $a \in G$ gives rise to

$$R_a : G \to G, \quad R_a(g) = ga = m(g,a)$$

smooth & a diffeom. (its inverse is $R_{a^{-1}}$)

$$R_a(e) = a$$

For instance $(dR_a)_e : T_eG \to T_aG$ linear isomorphism

$T_eG = \mathfrak{g}$ ← called the "Lie algebra" of G

$\Rightarrow$ each T_aG is canonically isomorphic to $\mathfrak{g}$ $(\forall a \in G)$.

$\Rightarrow$ TG is (canonically) isomorphic to the trivial vector bundle $G \times \mathfrak{g}$ $\Rightarrow$ G is parallelizable

-2-

Ex • $(\mathbb{R}, +)$

• $(S^1, \cdot)$ — complex multiplication using $S^1 \subseteq \mathbb{C} \ni z$ of $|z| = 1$

• $(S^3, \cdot)$ — uses $S^3 \subseteq \mathbb{R}^4 = \mathbb{H} \ni \xi$ quaternions $|\xi| = 1$.

• $GL_n(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{R})$ ← n^2 dimensional Euclidean space

invertible matrices $A \Leftrightarrow \det A \in (-\infty, 0) \cup (0, \infty)$; open

$\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ continuous; $\mathbb{R}^* \subseteq \mathbb{R}$

• any subgroup $G \leq GL_n(\mathbb{R})$ which is an embedded submanifold is automatically a Lie group ($m : G \times G \to G$ is the restriction of the one of GL_n)

Prop: If $G \leq GL_n(\mathbb{R})$ is a closed subgroup $\Rightarrow$ G is automatically an embedded submanifold. (hence a Lie group)

(idea is to use exponential of matrices $\exp : M_{n\times n}(\mathbb{R}) \to GL_n(\mathbb{R})$, $0 \mapsto I$, restricted to "$\mathfrak{g}$" to build charts for G).

-3-

Ex: Symmet. of geometric structures

E.g. $O(n) = \{A \in GL_n(\mathbb{R}) \,/\, A \cdot A^T = I_n\} \subseteq GL_n(\mathbb{R})$ ← closed, hence Lie group.

Matrices $A \in GL_n(\mathbb{R})$ encode linear isomorphisms $\hat{A} : \mathbb{R}^n \to \mathbb{R}^n$.

The structure that is relevant: the standard inner product $\langle \cdot , \cdot \rangle$ on $\mathbb{R}^n$

$\hat{A}$ - symmetry of $(\mathbb{R}^n, \langle \cdot , \cdot \rangle)$: $\langle \hat{A}(u), \hat{A}(v) \rangle = \langle u, v \rangle$ $(\forall)\, u, v \in \mathbb{R}^n$

$(\Leftrightarrow \|\hat{A}(u)\| = \|u\| \;\; \forall\, u \in \mathbb{R}^n)$

in terms of matrices this $\Leftrightarrow A \cdot A^T = I_n$

$O(n)$ appears as the group of symmetries of $(\mathbb{R}^n, \langle \cdot , \cdot \rangle)$

Do the same for $(\mathbb{R}^{2n}, J)$ — the complex structure $J(x, y) = (y, -x)$

$\Rightarrow$ subgroup of $GL_{2n}(\mathbb{R})$

$$\left\{ \left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array} \right) : A, B \in GL_n(\mathbb{R}) \right\} \simeq GL_n(\mathbb{C}) \quad A \pm iB$$

Other example: $SO(n) = \{A \in O(n) : \det A = 1\}$

-4-

Right (similarly left actions) **Action** of a Lie group G on a manifold P: smooth map

$$P \times G \to P, \quad (x, g) \mapsto x \cdot g$$

such that $(x \cdot g) \cdot h = x \cdot (gh)$, $x \cdot e = x$ $(\forall)\, g, h \in G$, $x \in P$

say that $\boxed{P \text{ is a } G\text{-manifold}}$

Ex. $P = G$ carries a (right) action of G: given by multiplication.

Given an action: objects:

• for each $p \in P$: the isotropy group at p: $\boxed{G_p} = \{g \in G \,/\, p \cdot g = p\}$

• — " — the orbit through p: $\boxed{p \cdot G} = \{p \cdot g \,/\, g \in G\}$

• the orbit space (or quotient space). $\boxed{P/G} = \{p \cdot G \,/\, p \in P\}$

Properties that one can require:

• free action: if $p \cdot g = ph \Rightarrow g = h$

• transitive action: if $(\forall)\, p, q \in P$, $\exists\, g \in G : q = p \cdot g$

• proper action: if the map $P \times G \to P \times P$, $(p, g) \mapsto (p, pg)$ is a proper map.

• P is a G-torsor if action is free and transitive

(Secret: free and proper ... principal bundles)

The relevant maps between G-manifolds P, Q a...

$F : P \to Q$ smooth s.t. $F(p \cdot g) = \ldots$

$F = G$-e...

Reminder (?) : $\boxed{-1-}$ $\boxed{\text{Lie groups \& actions}}$ $e \in G$ the unit of G

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Right (simil…

Action of …

such that …

Ex. $\mathbb{P} = ($…

Given an a…

• for each …

• — " —

• the orbit…

Properties …

• free acti…

• transitiv…

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(Secret : f…

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Ex: S

E.g. (

Matrices

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$\hat{A}$ - s

in term

$O(n)$

Do t

$\Longrightarrow$ Su

Other ex

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[3]

Ex: Symmet of geometric structures

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G

Smooth

$\to G$,
$\to g^{-1}$

Right (similarly: left actions) $\lfloor -4- \rfloor$

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(Secret: free and proper ... principal bundles)

The relevant maps

$F: P \longrightarrow$

$\lfloor 3 \rfloor$

Ex: Symmet... of geometric structures

$\rightsquigarrow$ Claim: $\exists S' \subseteq S$ s.t. $f|_{S' \times G}$ = injective ④

$V \oplus W$

$V, W \subseteq U \quad U = V \oplus W$ Cand.

~~Pf~~: If not, find $(p_k, g_k) \neq (p_k', g_k')$ in $S \times G$ $(k \geq 1)$

$\gg g_k \neq g_k'$

$$\begin{cases} p_k g_k = p_k' g_k' \quad / (g_k')^{-1} \\ p_k, p_k' \to p_0 \end{cases} \Rightarrow \begin{cases} p_k' = p_k a_k \quad \text{with } \boxed{a_k \neq e} \\ p_k, p_k' \to p_0 \end{cases}$$

Rk: $\varsigma(p_k, a_k) = (p_k, p_k') \longrightarrow (p_0, p_0) \Rightarrow (p_k, a_k) \in \varsigma^{-1}(\text{closed ball}) = \text{cpt} \Rightarrow$

cpt; around (p_0, p_0); fork-large

$\Rightarrow$ may assume $(p_k, a_k) \longrightarrow (p_0, a_0)$ for some $a_0 \in G$. Passing to limit in $(*)$ $\Rightarrow p_0 a_0$ ...

Now: $f(p_k', a_k^{-1}) = p_k' a_k^{-1} = p_k = f(p_k, e)$ $\Rightarrow (p_k', a_k^{-1}) = (p_k, ...)$

$\Downarrow$

$\boxed{a_k = e}$

both sequences which $\to (p_0, e)$

Now use the action of G the property that $(dh)_{p_0,e}$ iso to all $(dh)_{p_0,g}$ $g \in G$

to extend the property $f: S' \times G \longrightarrow U \subseteq P$ $\quad f(p,a)\cdot g = f(p, ag)$.

Since f = G-equiv $\Rightarrow$ U G-invariant

... the ... Lie algebra of G.

$\longleftarrow$ consequence of freeness

...ction $\Leftrightarrow G_p=\{e\}$ $\forall p$

transitive action $\Leftrightarrow$ one single orbit $\Leftrightarrow P/_G$ = singleton

is isomorphic to $\mathbb{R}^n$ where $n=\dim V$

G-torsors play the same role as vector spaces for vector bundles

) is a proper map $\Leftarrow$ Reminder $f: P\to Q$ is called proper if $(\forall) K\subset Q$ cpt $\Rightarrow f^{-1}(K)$ = compact

$\xrightarrow{k\to\infty} A$

$\in O(n)$: $\underbrace{A_k\cdot A_k^T}=I_n$

$k\to\infty \Rightarrow A A^T=I_n$

...ce Lie group.

$\mathbb{R}^n\to\mathbb{R}^n$

...er product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$

$\langle u,v\rangle$ $\forall\, u,v\in\mathbb{R}^n$

$|\ \forall u\in\mathbb{R}^n\ |$

$(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$

...ure $J(x,y)=(y,-x)$

$\mathbb{Z}$ acting on $\mathbb{R}$

$\mathbb{R}/\mathbb{Z}$ looks like S^1

$\tilde f:\mathbb{R}\to S^1,\ r\mapsto e^{2\pi i r}$

$f:\mathbb{R}/\mathbb{Z}\to S^1$

$\hat r\longmapsto e^{2\pi i r}$

Applied to our map $\boxed{P\times G\xrightarrow{A} P\times P}$ For any $p_0\in P$

$A^{-1}(p_0,p_0)=\{(p,g)\ /\ (p,pg)=(p_0,p_0)\}$ $\quad (p_0,p_0)$

$=\{p_0\}\times G_{p_0}$

Hence proper action $\Rightarrow$ all isotropy groups must be compact
$\not\Leftarrow$

Does hold:

G = compact $\Rightarrow$ the action is proper. $\quad pr_1: P\times P\to P$, $\cup\!|\ K$

Pf: Let $K\subseteq P\times P$ be compact.

$A^{-1}(K)=\{(p,g)\ /\ (p,pg)\in K\}\subseteq pr_1(K)\times G$ ✓

Say that $P/_G$ looks like a more familiar set/space/(manifold) S if there is a bijection/homeom/(diffeom) $f: P/_G\longrightarrow S$.

Rk: Having such a bijection $\Leftrightarrow$ having a surjective map $\tilde f: P\to S$ s.t.: $\tilde f(p_1)=\tilde f(p_2)\Leftrightarrow p_2=p_1\cdot g$ for some $g\in G$
(the fibers of $\tilde f$ $\equiv$ the orbits of the action)

-6-

... y that
... is a G-manifold

...lication
closed subgroup
$\ldots p\} \subseteq G$

$\ldots G\}$

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$\ldots = \{e\} \; (\forall) p$

...on $\Leftrightarrow$ one single orbit $\Leftrightarrow P/_G$ = singleton

$p \leftarrow$

Prop: A G-manifold is a $\boxed{\text{G-torsor}}$ iff it is diffeomorphic, as a G-manifold to $\boxed{\text{G itself}}$.

pf: P = G-manifold, action: free & transitive.
$\boxed{\text{Choose any } p_0 \in P}$ and consider
$$h_{p_0}: G \longrightarrow P, \quad g \longmapsto p_0 \cdot g$$
This is a G-equivariant, smooth bijection

... (some work) ... this is actually a diffeomorphism.

(Compare with finite dimensional vector spaces: any V is isomorphic to $\mathbb{R}^n$ where $n = \dim V$)

G-torsors play the same role as vector spaces for vector bundles

Reminder: $f: P \to Q$ is called proper if
$$(\forall) K \subseteq Q \text{ cpt} \Rightarrow f^{-1}(K) = \text{compact}.$$
$q_0 \in Q$, $K = \{q_0\} \Rightarrow$ the fiber $f^{-1}(q_0)$ must be compact.

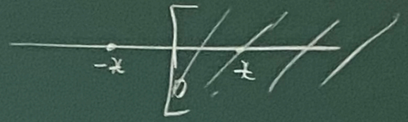
-5-

Applied to ...

-7-

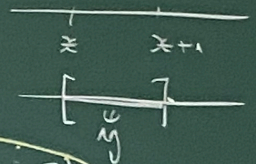
Ex 7.6: $G = \mathbb{Z}_2$ act on $P = \mathbb{R}$: $\underset{\mathbb{R}}{\overset{\cap}{x}} \cdot \underset{\mathbb{Z}_2}{\overset{\cap}{\hat{k}}} = (-1)^k x$

Action proper, not free. $G_x = \begin{cases} \{1\} & x \neq 0 \\ \mathbb{Z}_2 & x = 0 \end{cases}$



The orbit spaces "looks like" $[0, \infty)$

Ex 7.7: $G = \mathbb{Z}$ acts on $\mathbb{R}$: $x \underset{\text{action}}{\cdot} k := x + k$



Action still proper. Also free. Orbit space looks like S^1.

Similarly $G = \mathbb{Z}^2$ acts on $\mathbb{R}^2$ ... — " — the 2-torus

Ex 7.8: $G = \mathbb{Z}$, $P = S^1$ rotation by $\theta_0 = 2\pi r_0$ $r_0 \notin \mathbb{Q}$

$\underset{S^1}{\overset{\cap}{e^{2\pi i r}}} \cdot n := e^{2\pi i (r + n r_0)}$ ... action is free but quotient ....

Ex 7.9: $G = \mathbb{R}^*$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$ Prove that the quotient is non-Hausdorff.

$\underset{\mathbb{R}^{n+1}}{\overset{\cap}{v}} \cdot \underset{\mathbb{R}^*}{\overset{\cap}{\lambda}} := \lambda v$ free & proper

.... quotient: precisely $\mathbb{P}^n$.

Right (similarly left ac...

Action of a Lie gr...

$P \times G$ ...

Such that $(x \cdot g) \cdot h =$ ...

Ex. $P = G$ carrie...

Given an action: ob...

- for each $p \in P$: t...
- — " — th...
- the orbit space (or ...

Properties that one ...

- free action: if (p ...
- transitive action: if ($\forall$) ...
- proper action: if the ma...
- P is a G-torsor if ...

(Secret: free and pro...

When is P/G a smooth manifold? -8-

Remember the quotient map $\pi : P \to P/G$, $p \mapsto$ ...

When is P/G a smooth manifold ...

Ex: Symm...

$e^{2\pi i r} \cdot n := e^{2\pi i (r+ n\theta)}$ ... action is free but quotient ....

$e^{2\pi i r} \in S^1$

Prove that the quotien

Ex 7.9: $G = \mathbb{R}^*$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$ is non-Hausdorff.

$v \cdot \lambda := \lambda v$, $v \in \mathbb{R}^{n+1}$, $\lambda \in \mathbb{R}^*$ free & proper

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When is P/G a smooth manifold? [-8-]

Remember the quotient map $\pi: P \to P/G$, $p \mapsto p \cdot G$.

When is $\boxed{P/G \text{ a smooth manifold s.t. } \pi = \text{submersion}}$

Rk: if such a smooth structure on P/G exists, then it is unique!

(Hint: if I have such a smooth str.: a function $f: P/G \to \mathbb{R}$ is smooth $\iff$ $f \circ \pi$ is smooth

$$\begin{array}{ccc} P & \xrightarrow{\pi} & P/G \\ & \searrow^{f\circ\pi} & \downarrow f \\ & & \mathbb{R} \end{array}$$

~~$\pi^*: C^\infty(P/G) \to \{F \in C^\infty(P) \,/\, F \text{ is constant on each orbit}\}$, $f \mapsto f \circ \pi$~~

Theorem: Such a structure exists if the action is free and proper.

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in terms of matrices this $O(n)$ appears as the g...

Do the same for $(\mathbb{R}^{2n}$,

$\Rightarrow$ Subgroup of ...

-3-

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$A_k \xrightarrow{k\to\infty} A$

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$\hat{r} \longmapsto e^{2\pi i r}$

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- proper action: if the map $\mathbb{P} \times G \to \mathbb{P} \times \mathbb{P}$, $(p, g) \mapsto (p, pg)$ is a proper map
- $\mathbb{P}$ is a G-torsor if action is free and transitive

(Secret: free and proper . . . principal bundles)

-5-

Applied to our map $\boxed{P \times G \xrightarrow{A} P \times P}$

For any $p_0 \in$ [illegible]

$A^{-1}(p_0, p_0) = \{(p, g) / (p, pg) = (p_0, p_0)\}$ $\quad (p_0, p_0)$

$= \{p_0\} \times G_{p_0}$

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$\not\Leftarrow$

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$pr_1 : P \times P$ [illegible]

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Rk: Having such a bijection $\Leftrightarrow$ having a surjective map $\tilde{f}$ [illegible]

s.t. $\tilde{f}(p_1) = \tilde{f}(p_2) \Leftrightarrow p_2 = p_1 \cdot g$ for some $g \in G$

(the fibers of $\tilde{f}$ $\equiv$ the orbits of the action)

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-7-

Ex 7.6: $G = \mathbb{Z}_2$ act on $P = \mathbb{R}$: $\underset{\mathbb{R}}{x} \cdot \underset{\mathbb{Z}_2}{\hat{k}} = (-1)^k x$

Action proper, not free: $G_x = \begin{cases} \{1\} & x \neq 0 \\ \mathbb{Z}_2 & x = 0 \end{cases}$

The orbit space "looks like" $[0, \infty)$

Ex 7.7: $G = \mathbb{Z}$ acts on $\mathbb{R}$: $x \cdot k := x + k$ (action)

Action still proper. Also free. Orbit space looks like S^1

Similarly $G = \mathbb{Z}^2$ acts on $\mathbb{R}^2$ — " — the 2-torus

Ex 7.8: $G = \mathbb{Z}$, $P = S^1$: rotation by $\theta_0 = 2\pi r_0$, $r_0 \notin \mathbb{Q}$

$\underset{S^1}{e^{2\pi i r}} \cdot n := e^{2\pi i (r + n r_0)}$ . . . action is free but quotient . . .

Prove that the quotient is non-Hausdorff.

Ex 7.9: $G = \mathbb{R}^*$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$

$\underset{\mathbb{R}^{n+1}}{v} \cdot \underset{\mathbb{R}^*}{\lambda} = \lambda v$ free & proper

. . . quotient: precisely $\mathbb{P}^n$

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[W]hen is P/G a smooth manifold?

[R]emember the quotient map $\pi : P \to P/G$, $p \mapsto p \cdot G$

[Wh]en is $\boxed{P/G \text{ a smooth manifold s.t. } \pi = \text{submersion}}$

Rk: if such a smooth structure on P/G exists, then it is unique!

[H]int: if I have such a smooth str.

a function $f : P/G \to \mathbb{R}$ is smooth $\Leftrightarrow f \circ \pi$ is smooth

$P \xrightarrow{\pi} P/G \xrightarrow{f} \mathbb{R}$, $f \circ \pi$

~~$\pi^* : C^\infty(P/G) \longrightarrow \{h \in C^\infty(P) / h \text{ is constant on each orbit}\}$~~

~~$f \longmapsto f \circ \pi$~~

[Th]: Such a structure exists if the action [is free] and proper.