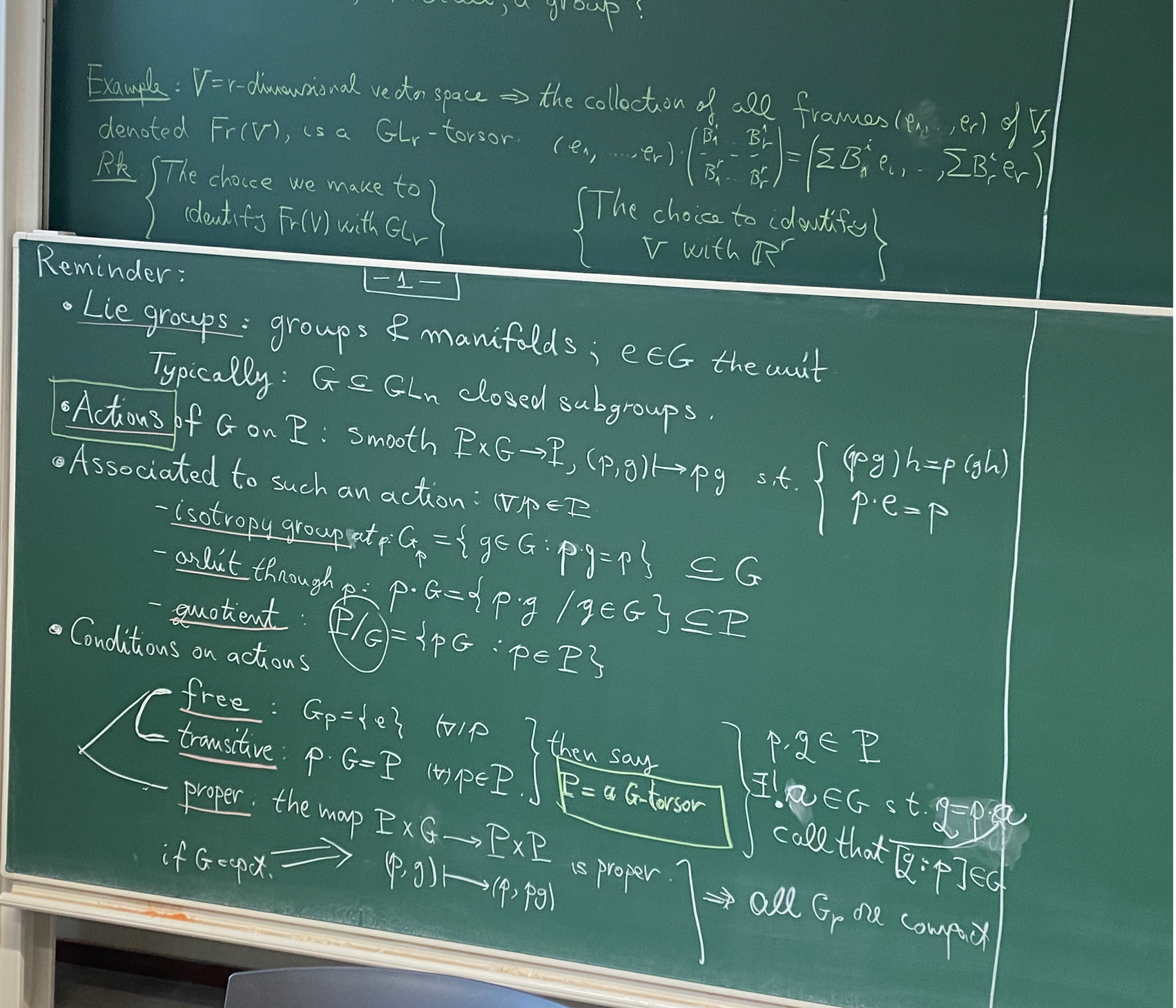

, a group ?

Example: $V = r$-dimensional vector space ⇒ the collection of all frames $(e_1, \dots, e_r)$ of V, denoted $Fr(V)$, is a GL_r-torsor. $(e_1, \dots, e_r) \cdot \begin{pmatrix} B^1_1 & \dots & B^1_r \\ \vdots & & \vdots \\ B^r_1 & \dots & B^r_r \end{pmatrix} = \left(\sum B^i_1 e_i, \dots, \sum B^i_r e_r \right)$

Rk {The choice we make to identify $Fr(V)$ with GL_r} {The choice to identify V with $\mathbb{R}^r$}

Reminder:

−1−

- **Lie groups**: groups & manifolds; $e \in G$ the unit
  Typically: $G \subseteq GL_n$ closed subgroups.
- **Actions** of G on P: smooth $P \times G \to P$, $(p,g) \mapsto pg$ s.t. $\begin{cases} (pg)h = p(gh) \\ p \cdot e = p \end{cases}$
- Associated to such an action: $(\forall) p \in P$
  - **isotropy group** at p: $G_p = \{ g \in G : p \cdot g = p \} \subseteq G$
  - **orbit** through p: $p \cdot G = \{ p \cdot g \, / \, g \in G \} \subseteq P$
  - **quotient**: $P/G = \{ pG : p \in P \}$
- Conditions on actions
  - **free**: $G_p = \{e\}$ $(\forall) p$
  - **transitive**: $p \cdot G = P$ $(\forall) p \in P$.

  } then say $\boxed{P = \text{a } G\text{-torsor}}$ } $p, q \in P$ $\exists! \, a \in G$ s.t. $q = p \cdot a$; call that $[q : p] \in G$

  - **proper**: the map $P \times G \to P \times P$, $(p,g) \mapsto (p, pg)$ is proper. ⇒ all G_p are compact

  if G cpct. $\Longrightarrow$

-2-

Theorem: Free & proper action $\Rightarrow \exists!$ smooth structure on P/G s.t the quotient map $\pi_{can} : P \to P/G$, $p \mapsto pG$ is a submersion.

$\Updownarrow$

Given a set S, assume we have $\begin{cases} \text{a surjective map } \pi : P \to S \\ \text{a free \& proper action of } G \text{ on } P \text{ s.t} \end{cases}$ (P ← manifold)

the orbits are precisely the fibers of π

Then $\exists!$ smooth st on S s.t π = submersion

$(P/G \cong S)$

G-Torsors: $\Rightarrow (\forall) p \in P$, $\begin{cases} h_p : \underline{G} \to \underline{P} \\ g \mapsto pg \end{cases}$ is a bijection ... actually a diffeomorphism... really an isomorphism of G-torsors.

So: is P, after all, a group?

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Def: A principal G-

π

consists of a

(1)

(2)

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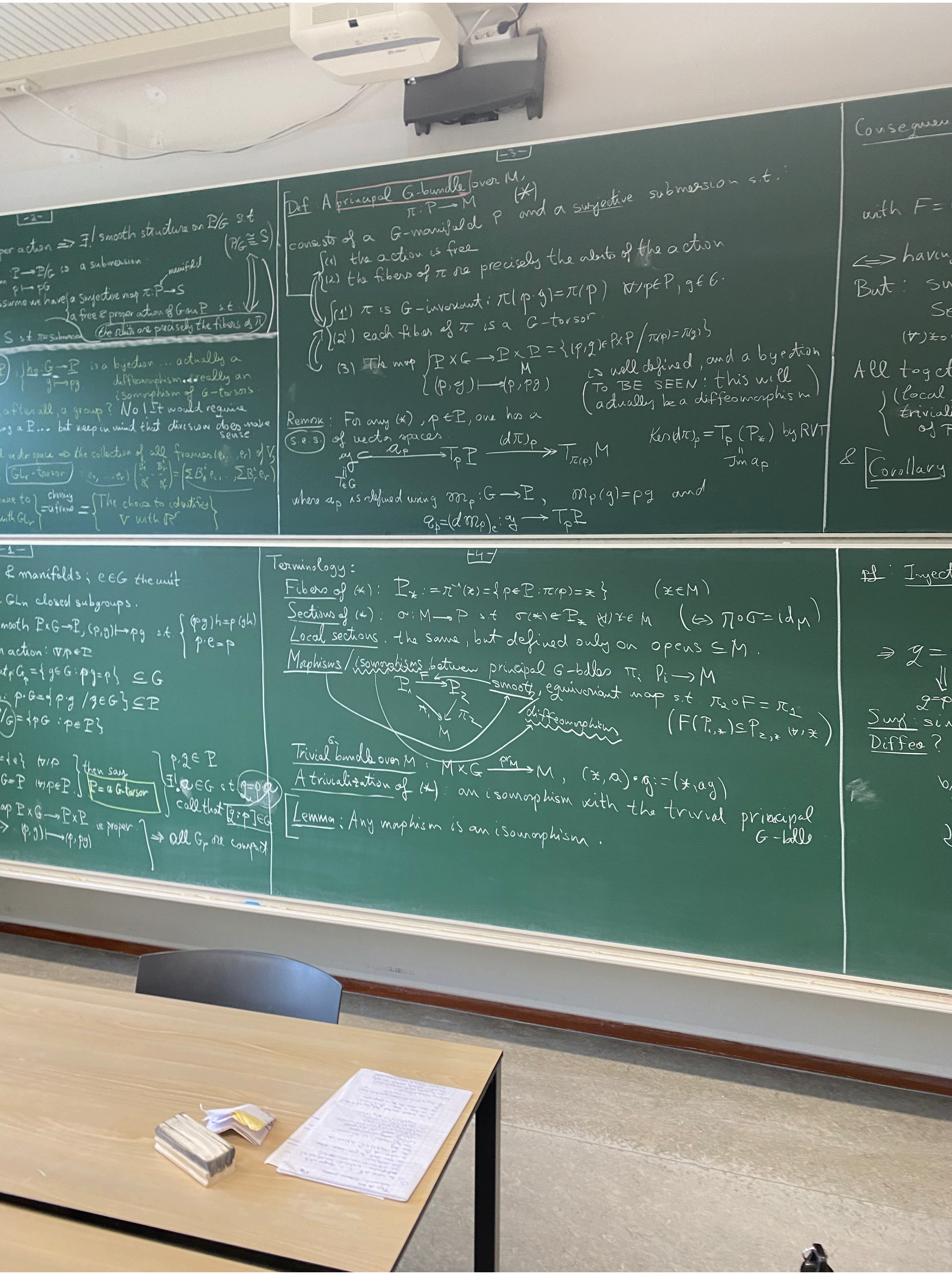

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ap $P \times G \to P \times P$, $(p,g) \mapsto (p, pg)$ is proper ⇒ all G_p are compact

-2-

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$P \to P/G$ is a submersion
$p \mapsto pG$

ssume we have $\begin{cases} \text{a surjective map } \pi : P \to S \text{ (manifold)} \\ \text{a free \& proper action of } G \text{ on } P \text{ s.t.} \end{cases}$ the orbits are precisely the fibers of π

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-4-

Terminology:

Fibers of $(*)$: $P_x := \pi^{-1}(x) = \{ p \in P : \pi(p) = x \}$ $(x \in M)$

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Local sections. the same, but defined only on opens $\subseteq M$.

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Trivial G-bundle over M: $M \times G \xrightarrow{pr_M} M$, $(x,a) \cdot g := (x, ag)$

A trivialization of $(*)$: an isomorphism with the trivial principal G-bdle

Lemma: Any morphism is an isomorphism.

Consequen

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Pf: Inject

$\Rightarrow q =$
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$q = p$

Surj: si

Diffeo?

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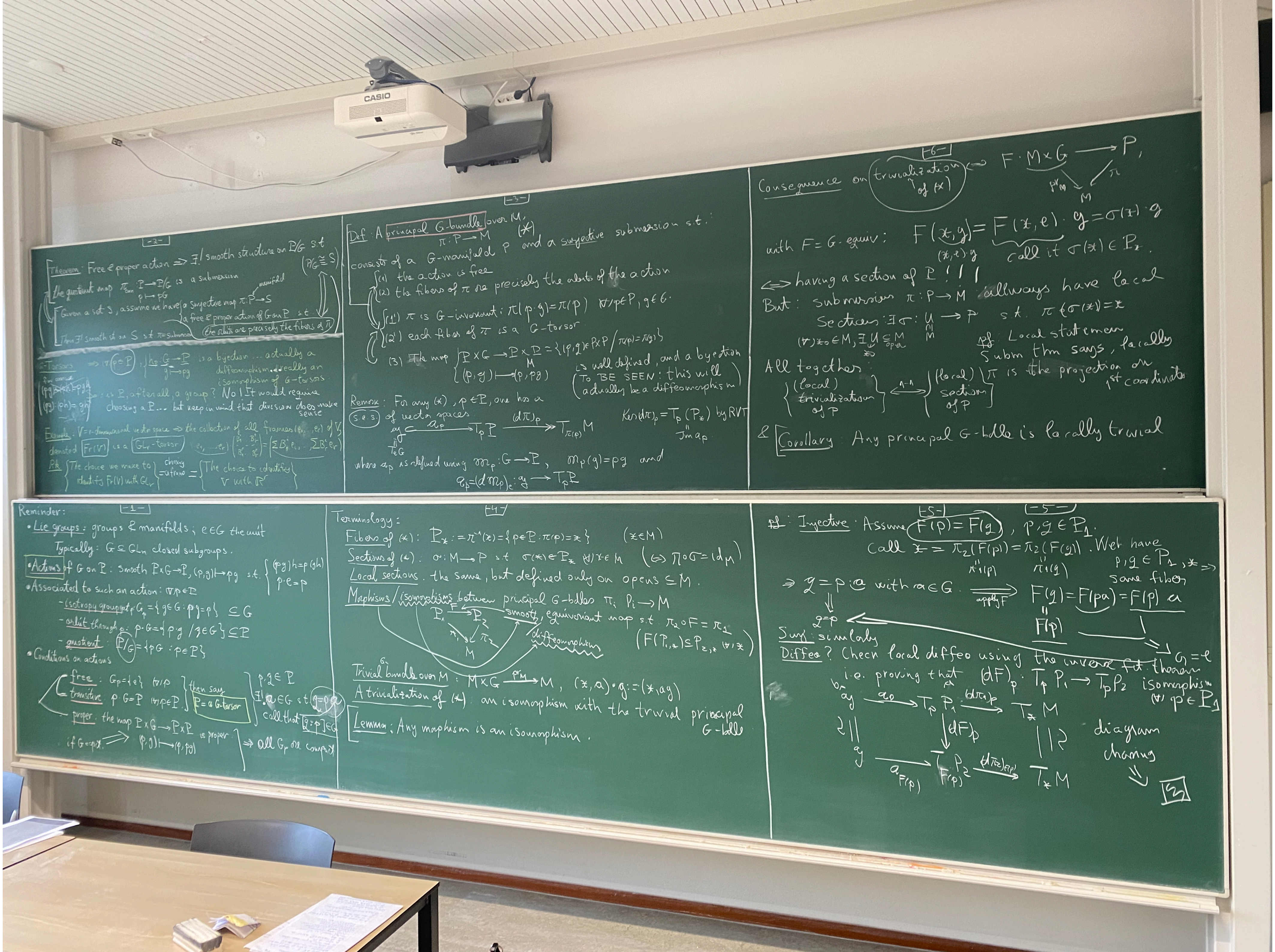

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-5-

Pf: **Injective**: Assume $F(p) = F(q)$, $p, q \in P_1$.
Call $x = \pi_2(F(p)) = \pi_2(F(q))$. We have $p, q \in P_{1,x} \Rightarrow$ same fiber
$\pi_1(p) \qquad \pi_1(q)$

$\Rightarrow q = p \cdot a$ with $a \in G$. $\xRightarrow{\text{apply } F}$ $F(q) = F(pa) = F(p)\,a$, $F(q) = F(p)$ $\Rightarrow a = e$ $\Rightarrow q = p$

Surj: similarly

Diffeo? Check local diffeo using the inverse fct. theorem i.e. proving that $(dF)_p : T_p P_1 \to T_p P_2$ isomorphism $(\forall) p \in P_1$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "a_p"] \arrow[d, "\wr\|"] & T_p P_1 \arrow[r, "(d\pi_1)_p"] \arrow[d, "(dF)_p"] & T_x M \arrow[d, "\|\wr"] \\ \mathfrak{g} \arrow[r, "a_{F(p)}"'] & T_{F(p)} P_2 \arrow[r, "(d\pi_2)_{F(p)}"'] & T_x M \end{tikzcd}$$

diagram chasing $\Rightarrow$ ☑

-6-

Consequence on trivialization of (*) $\leftrightarrow$ $F : M \times G \to P$,

$$\begin{tikzcd} M \times G \arrow[rr] \arrow[dr, "pr_M"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

with F = G-equiv: $F(x,g) = F(x,e) \cdot g = \sigma(x) \cdot g$; $(x,g) = (x,e) \cdot g$; $\underbrace{F(x,e)}_{\text{call it } \sigma(x) \in P_x}$.

$\Longleftrightarrow$ having a section of P !!!

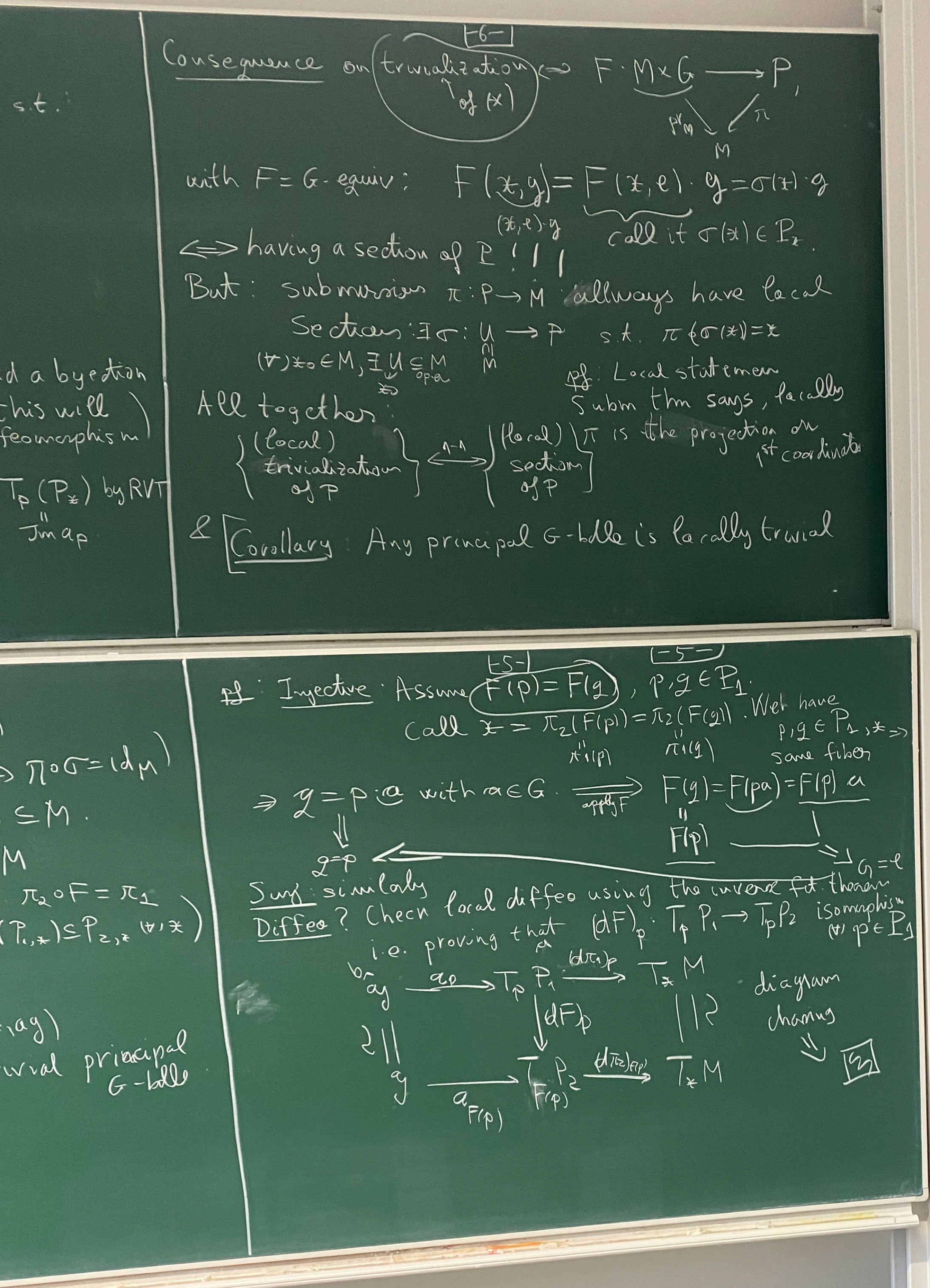
But: submersion $\pi : P \to M$ allways have local sections: $\exists \sigma : U \to P$ s.t. $\pi(\sigma(x)) = x$, $U \subseteq M$; $(\forall) x_0 \in M, \exists U \ni x_0, U \subseteq_{\text{open}} M$

Pf: Local statement. Subm thm says, locally π is the projection on 1st coordinate

All together:

$$\begin{Bmatrix} \text{(local)} \\ \text{trivializations} \\ \text{of } P \end{Bmatrix} \overset{1-1}{\longleftrightarrow} \begin{Bmatrix} \text{(local)} \\ \text{sections} \\ \text{of } P \end{Bmatrix}$$

& **Corollary**: Any principal G-bdle is locally trivial



s.t.

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$T_p(P_*)$ by RVT

$\overset{\shortparallel}{\mathrm{Im}\, a_p}$

$\boxed{-6-}$

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$\Downarrow$

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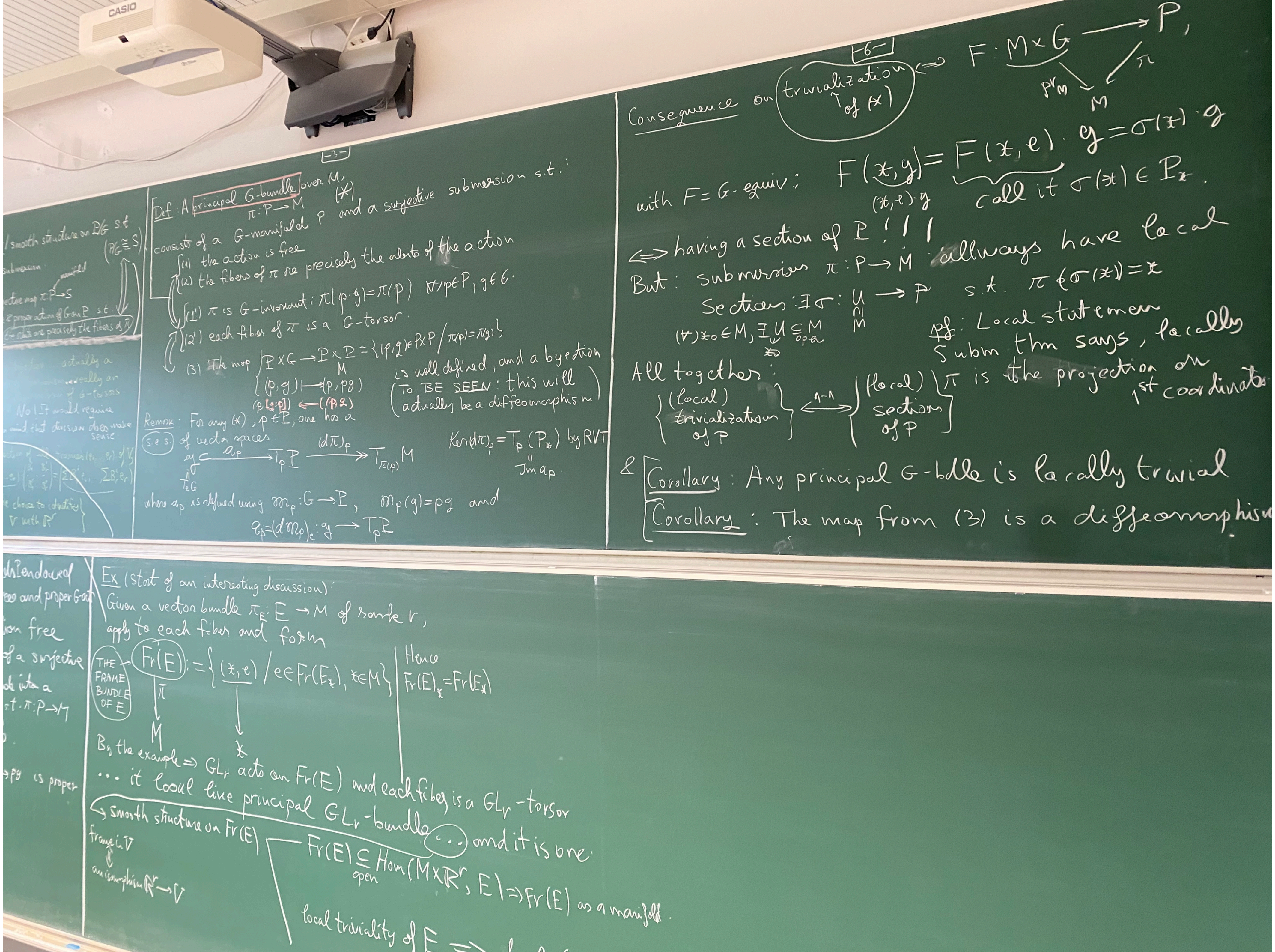

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<u>Ex</u> (start of an interesting discussion):

Given a vector bundle $\pi_E: E \to M$ of rank r, apply to each fiber and form

THE FRAME BUNDLE OF E $\to$ $Fr(E) := \{\underline{(x, e)} / e \in Fr(E_x), x \in M\}$

$$\begin{tikzcd} Fr(E) \arrow[d, "\pi"] & (x,e) \arrow[d, mapsto] \\ M & x \end{tikzcd}$$

Hence $Fr(E)_x = Fr(E_x)$

By the example $\Rightarrow$ GL_r acts on $Fr(E)$ and each fiber is a GL_r-torsor

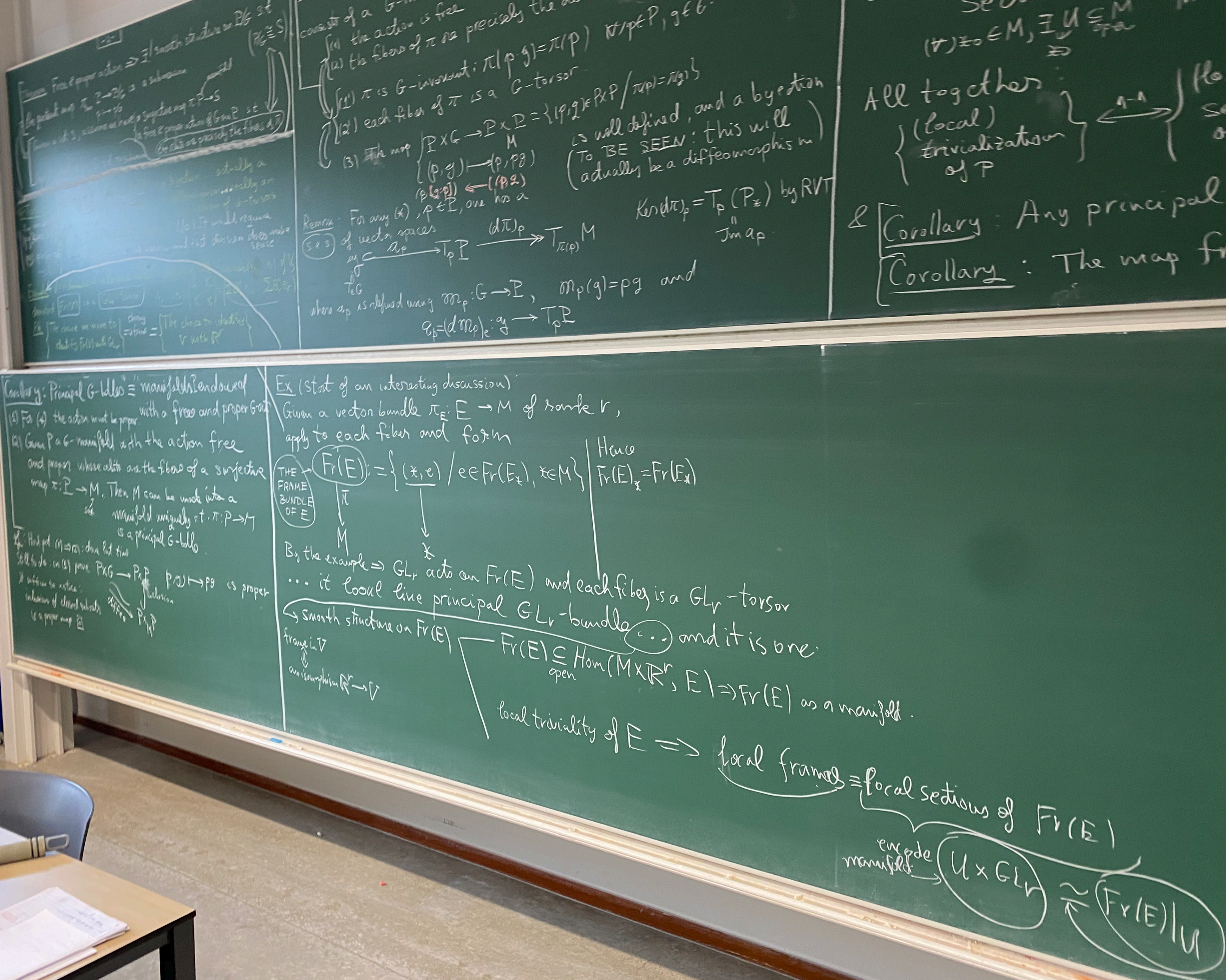
... it look like principal GL_r-bundle (...) and it is one.

$\hookrightarrow$ smooth structure on $Fr(E)$

frame in V $\Updownarrow$ an isomorphism $\mathbb{R}^r \to V$

$Fr(E) \underset{\text{open}}{\subseteq} Hom(M \times \mathbb{R}^r, E) \Rightarrow Fr(E)$ as a manifold.

local triviality of E $\Longrightarrow$



Theorem: Free & proper action ⇒ ∃! smooth structure on P/G s.t. the quotient map $\pi_{can}: P \to P/G$, $p \mapsto pG$ is a submersion

Given a set S, assume we have a surjective map $\pi: P \to S$ [smooth] & a free & proper action of G on P s.t. the orbits are precisely the fibers of π $(P/G \cong S)$

[illegible] actually a [illegible] really an [illegible] of G-torsors

No! It would require [illegible] and that discussion does make sense

Ex $\left\{\begin{array}{l}\text{The choice we make to} \\ \text{identify } Fr(V) \text{ with } GL_r\end{array}\right\} \overset{\text{strong}}{=} \left\{\begin{array}{l}\text{The choice to identify} \\ V \text{ with } \mathbb{R}^r\end{array}\right\}$

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(1') π is G-invariant: $\pi(p\cdot g)=\pi(p)$ $\forall p \in P, g \in G$

(2') each fiber of π is a G-torsor.

(3) The map $\begin{cases} P\times G \to P \times_M P = \{(p,q) \in P\times P / \pi(p)=\pi(q)\} \\ (p,g) \mapsto (p,pg) \end{cases}$ is well defined, and a bijection

$(p,[q:p]) \longleftarrow (p,q)$

(To BE SEEN: this will actually be a diffeomorphism)

Remark: For any (*), $p \in P$, one has a s.e.s. of vector spaces

$$\mathfrak{g} \xhookrightarrow{a_p} T_pP \xrightarrow{(d\pi)_p} T_{\pi(p)}M$$

$$\mathfrak{g} \cong T_eG$$

$$\operatorname{Ker}(d\pi)_p = T_p(P_x) \text{ by RVT} = \operatorname{Im} a_p$$

where a_p is defined using $m_p: G \to P$, $m_p(g)=pg$ and

$$a_p=(dm_p)_e: \mathfrak{g} \to T_pP$$

Se

$(\forall)\, x_0 \in M,\ \exists\, U \underset{open}{\subseteq} M \ni x_0$

All together: $\left\{\begin{array}{c}\text{(local)} \\ \text{trivialization} \\ \text{of } P\end{array}\right\} \xleftrightarrow{1-1} \Big\{$ lo se o

& **Corollary**: Any principal

Corollary: The map fr

Corollary: Principal G-bdles ≡ "manifolds P endowed with a free and proper G-act

(1) For (*) the action must be proper.

(2) Given P a G-manifold with the action free and proper whose orbits are the fibers of a surjective map $\pi: P \to M$. Then M can be made into a manifold uniquely s.t. $\pi: P \to M$ is a principal G-bdle.

Pf: Hard part (1)⇒(2): done last time.
Still to do: in (1) prove $P\times G \to P\times_M P$, $(p,g) \mapsto pg$ is proper
It suffices to notice: inclusion of closed subsets is a proper map ☐

$$\begin{tikzcd} P\times G \arrow[r] \arrow[dr, "\text{diffeo}"'] & P\times P \\ & P\times_M P \arrow[u, hook, "\text{inclusion}"'] \end{tikzcd}$$

Ex (start of an interesting discussion):
Given a vector bundle $\pi_E: E \to M$ of rank r, apply to each fiber and form

THE FRAME BUNDLE OF E → $Fr(E) := \{(x,e) / e \in Fr(E_x), x \in M\}$

$$\begin{tikzcd} Fr(E) \arrow[d, "\pi"] & (x,e) \arrow[d, mapsto] \\ M & x \end{tikzcd}$$

Hence $Fr(E)_x = Fr(E_x)$

By the example ⇒ GL_r acts on $Fr(E)$ and each fiber is a GL_r-torsor
... it looks like principal GL_r-bundle (...) and it is one.

→ smooth structure on Fr(E)

frame in V ⇕ an isomorphism $\mathbb{R}^r \to V$

$Fr(E) \underset{open}{\subseteq} Hom(M\times\mathbb{R}^r, E) \Rightarrow Fr(E)$ as a manifold.

local triviality of E ⟹ local frames ≡ local sections of Fr(E)

encode manifold → $U \times GL_r \cong Fr(E)|_U$

a surjective map

a free & proper action of G on P s.t.

the orbits are precisely the fibers of π

(1') π is a

(2') each fiber of π is a

$P \times_M P = \{(p,q) \in P \times P \,/\, \pi(p) = \pi(q)\}$

(3) The map $\begin{cases} P \times G \longrightarrow P \times_M P \\ (p,g) \longmapsto (p, pg) \\ (p, [q:p]) \longleftarrow (p,q) \end{cases}$ is well defined, and a bijection (TO BE SEEN: this will actually be a diffeomorphism)

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where a_p is defined using $m_p : G \to P$, $m_p(g) = pg$ and

$a_p = (dm_p)_e : \mathfrak{g} \longrightarrow T_pP$

All together:

$$\left\{\begin{matrix}\text{(local)}\\ \text{trivializations}\\ \text{of } P\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{(local)}\\ \text{sections}\\ \text{of } P\end{matrix}\right\}$$

& Corollary: Any principal G-bdle is locally trivial

Corollary: The map from (3) is a diffeomorphism.

…les ≡ manifolds P endowed with a free and proper G-action

…be proper.

…d with the action free

…orbits are the fibers of a surjective

Then M can be made into a manifold uniquely s.t. $\pi: P \to M$ is a principal G-bdle

last time

$P \times G \longrightarrow P \times P$, $(p,g) \longmapsto pg$ is proper

inclusion, diffeo, $P \times_M P$

Ex (start of an interesting discussion)

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local triviality of E ⟹ local frames ≡ local sections of Fr(E)

encode manifold → $U \times GL_r \simeq Fr(E)|_U$

transfer the smooth str to Fr(E)