— 1 —

Reminder: Principal G-bundle over M = G-manifold P & (*)

$$\pi : P \longrightarrow M$$

surjective submersion ... satisfying

$\begin{cases} \text{(1) the action is free} \\ \text{(2) the fibers of } \pi = \text{the orbits of the action} \end{cases}$

$\begin{cases} \text{(1') } \pi \text{ is } G\text{-invariant, i.e., } \pi(p \cdot g) = \pi(p) \ (\forall) p \in P, g \in G \\ \text{(2') each fiber } P_x = \pi^{-1}(x) \text{ is a } G\text{-torsor} \end{cases}$

(3) $P \times G \longrightarrow P \times_M P$, $(p, g) \longmapsto (p, pg)$ is a bijection/diffeom.

(4) the action is free and proper and π induces $P/G \simeq M$

(5) for each $x_0 \in M$, $\exists$ open neighbd. $U \subseteq M$ and an isomorphism

$$\begin{tikzcd} P|_U \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

— 2 —

(Local) sections of (*): $\sigma : M \longrightarrow P$ s.t. $\pi \circ \sigma = id_M$. Any such σ ($M \supseteq (U)$)

induces an isomorphism $F : M \times G \longrightarrow P$, $F(x, g) = \sigma(x) g$

... and conversely ... hence

always $\exists$ ⇐ (5) is a consequence

$\left\{ \begin{matrix} \text{(local) sections} \\ \text{of } P \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{(local) trivializations} \\ \text{of } P \end{matrix} \right\}$

Rk: Hence comparing (local) trivializations $\Leftrightarrow$ comparing (local) sections.

Immitate the discussion on transition functions $\Longrightarrow$

$\Longrightarrow$ a description in terms of open covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$

of M and G-valued cocycles — that is

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \qquad \text{smooth maps}$$

s.t. on $U_\alpha \cap U_\beta \cap U_\gamma \ni x$: $g_{\alpha\beta}(x)\, g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ ($g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma}$).

Explicitly, starting with (*): choose $\mathcal{U}$ s.t. over U_α there exists a local section $\sigma_\alpha : U_\alpha \to P$. For $x \in U_\alpha \cap U_\beta$: $\sigma_\alpha(x), \sigma_\beta(x) \in P_x \Longrightarrow$

$\Longrightarrow \sigma_\beta(x) = \sigma_\alpha(x) \cdot g_{\alpha\beta}(x)$ for some $g_{\alpha\beta}(x) \in G \Longrightarrow$ the $g_{\alpha\beta}$'s we are looking for

-1-

…undle over M = G-manifold P &

… $P \to M$ (*)

$\mathrm{Princ}_G(M)$ princ. G-bdls over M

… satisfying

… the orbits of the action

…t, i.e., $\pi(p\cdot g)=\pi(p)$ $(\forall) p\in P, g\in G$

… $\pi^{-1}(x)$ is a G-torsor

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…ree and proper and π induces $P/_G \cong M$

…M, $\exists$ open neighbd. $U\subseteq M$ and an isomorphism

$$\begin{tikzcd} \dots \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U\times G \arrow[dl, "pr_U"] \\ & U & \end{tikzcd}$$

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Example/constructions: Pull-backs: each $f: M\to N$ induces

$$f^*: \mathrm{Princ}_G(N)\to \mathrm{Princ}_G(M)$$

defined so that $(f^*P)_x = P_{f(x)}$

Hence, for $P\in \mathrm{Princ}_G(N)$

$$f^*P=\{\underbrace{(x,p)}_{\substack{\downarrow\\ x}}\in M\times P \mid f(x)=\pi(p)\}$$

$f^*P \downarrow M$

Action $(x,p)\cdot g=(x,pg)$

$$\begin{tikzcd} f^*P \arrow[r, "{(x,p)}"] \arrow[d] & P \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Ex/constr. Each $\iota: H\to G$ group homomorphism induces

$$\iota_*: \mathrm{Princ}_H(M)\to \mathrm{Princ}_G(M),\ \boxed{Q}\mapsto \iota_*(Q)$$

"by enlarging Q"

Explicitly:

→ start with $Q\times G$

→ act by H on the left $\quad h\cdot(q,a)=(qh^{-1}, \iota(h)\cdot a)$

→ notice the action is free & proper $\quad \iota_*(Q) := (Q\times G)/_H = \{[p,a\dots$

Take the quotient

The action of G on $\iota_*(Q)$: $[q,a]\cdot b=[q,ab]$ $\downarrow M$ $\quad \pi(p)$

…$M\to P$ s.t. $\pi\circ\sigma=\mathrm{id}_M$. Any such σ

…$\times G\to P$, $F(x,g)=\sigma(x)g$

…ce

… $\xleftrightarrow{1-1}$ {(local) trivializations of P}

…tions $\Leftrightarrow$ corresponding (local) sections.

…transition functions $\Longrightarrow$

…ms of open covers $\mathcal{U}=\{U_\alpha\}_{\alpha\in\Lambda}$

…cycles — that is

…$U_\beta\to G$ smooth maps

…$g_{\beta\alpha}^{-1}(x)=g_{\alpha\beta}(x)$ $(g_{\alpha\beta}\,g_{\beta\gamma}=g_{\alpha\gamma})$.

…choose $\mathcal{U}$ s.t over U_α there exists

…$x\in U_\alpha\cap U_\beta$: $\sigma_\alpha(x),\sigma_\beta(x)\in P_x$ $\Rightarrow$

…we $g_{\alpha\beta}(x)\in G \Rightarrow$ the $g_{\alpha\beta}$'s we're looking for

-4-

Def: Given $\boxed{P\in \mathrm{Princ}_G(M)}$ and $\boxed{H\subseteq G}$ a Lie subgroup an H-reduction of P is any $Q\in \mathrm{Princ}_H(M)$ s.t. $i_*(Q)\cong P$ where $i: H\hookrightarrow G$ is the inclusion.

ex: $H=\{e\}$ An e-reduction $\Leftrightarrow$ a trivialization of P $\quad Q\subseteq P \downarrow M$

Lemma: This is $\Leftrightarrow$ a submanifold $Q\subseteq P$ which is H-invariant ($q\cdot h\in Q$ $\forall q\in Q, h\in H$) and such that $\pi|_Q: Q\to M$ is a principal H-bdle.

"finding an H-red. of P = finding an H-principal bdle inside P"

Pf: Having such a Q we claim $i_*(Q)\cong P$.

Conversely: …

$$(Q\times G)/_H \xrightarrow{\sim} P$$

$$[q,a]\longmapsto q\cdot a \quad (q\in Q,\ a\in G)$$

$$[qh^{-1}, ha] \longrightarrow qh^{-1}ha$$

-3-

Example/constructions: Pull-backs: each $f : M \to N$ induces
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$$\begin{tikzcd} f^*P \arrow[r, "{(x,p)}"] \arrow[d] & P \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

$$f^*P = \{ \underbrace{(x,p)}_{x} \in M \times P \mid f(x) = \pi(p) \}$$
$$\downarrow$$
$$M$$

Action $(x,p)\cdot g = (x, pg)$

Ex/constr. Each $\imath : H \to G$ group homomorphism induces
$$\imath_* : \mathrm{Princ}_H(M) \longrightarrow \mathrm{Princ}_G(M), \quad \boxed{Q} \longmapsto \imath_*(Q)$$
"by enlarging Q"

Explicitly:
→ start with $Q \times G$
→ act by H on the left $h \cdot (q, a) = (q h^{-1}, \imath(h)\cdot a)$
→ notice the action is free & proper $\imath_*(Q) = (Q \times G)/_H = \{ [p, a] : p \in Q, a \in G \}$ where $[ph, a] = [p, \imath(h)a]$

Take the quotient

The action of G on $\imath_*(Q)$: $[q,a]\cdot b = [q, ab]$

$\downarrow$ M $\quad [p,a] \mapsto \pi(p)$

$\{ [p,a] : \pi(p) = x \} = \{ [p_0, a] : a \in G \}$

Fix p_0 s.t. $\pi(p_0) = x$

$M \ni x$

Rk: We could replace G by any H-manifold N ⟹
⟹ we produce $(Q \times N)/_H$ sitting above M with fibers copies of N.

when $Q = M \times H$

$$((M \times H) \times G)/_H = \{ [(\underbrace{x,h}_{p}), a] \mid x \in M, h \in H, a \in G \}$$
$$= \{ [(x,e), a] : a \in G, x \in M \} \quad [(x, hh^{-1}), a] = [(x,h), \imath(h^{-1})a]$$
$$\cong M \times G \quad [(x,h), a] = [(x,e), \imath(h)a]$$

-4-

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$Q \subseteq P$, $\downarrow$ M

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Lemma: This is ⟺ a submanifold $Q \subseteq P$ which is H-invariant ($q\cdot h \in Q$ $\forall q \in Q, h \in H$) and such that

$\to M = G$-manifol P &

$(*)$

$\mathrm{Princ}_G(M)$ princ. G-bdls over M

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a G-torsor

$\mapsto (p,pg)$ is a bijection/diffeom.

proper and π induces $P/_G \simeq M$

neighbd. $U\subseteq M$ and an isomorphism

$\to U\times G$

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---

[3]

Example/constructions: Pull-backs: each $f: M\to N$ induces

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defined so that $(f^*P)_x = P_{f(x)}$

Hence, for $P\in \mathrm{Princ}_G(N)$

$$f^*P = \{\underbrace{(x,p)}_{\downarrow\ x}\in M\times P \mid f(x)=\pi(p)\}$$

$\downarrow$

M

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

(x,p)

Action $(x,p)\cdot g = (x, pg)$

Ex/constr. Each $\imath: H\to G$ group homomorphism induces

$\imath_*: \mathrm{Princ}_H(M)\longrightarrow \mathrm{Princ}_G(M)$, $\boxed{Q}\longmapsto \imath_*(Q)$

"by enlarging Q"

Explicitly:

$\to$ start with $Q\times G$

$\to$ act by H on the left $h\cdot(q,a)=(qh^{-1}, \imath(h)\cdot a)$

$\to$ notice the action is free & proper $\imath_*(Q) := (Q\times G)/_H = \{[p\ldots$

Take the quotient

The action of G on $\imath_*(Q)$: $[q,a]\cdot b = [q,ab]$ $\downarrow$ M

$\pi(p)$

$\ldots$t $\pi\circ\sigma = id_M$. Any such σ

$F(x,g)=\sigma(x)g$

(local) trivializations of P

$\to$ comparing (local) sections.

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$g_{\alpha\beta}(x)$ ($g_{\alpha\beta}g_{\beta\gamma}=g_{\alpha\gamma}$).

s.t. over U_α there exists

$\cap U_\beta$: $\sigma_\alpha(x), \sigma_\beta(x)\in P_x \Longrightarrow$

$\in G \Longrightarrow$ the $g_{\alpha\beta}$'s we are looking for

---

[4]

Rk: The construction above can be applied more generally (replacing G) $\Longrightarrow$ $\left\{\begin{array}{l}\text{for any } P\in \mathrm{Princ}_H(M)\\ \text{for any } H\text{-manifold } N\end{array}\right\}\Longrightarrow$

$\Rightarrow$ can form $(P\times N)/_H \longrightarrow M$ with fibers copies of N!

ex: $H=\{e\}$

Lemma: (H-reduction of P) $\Longleftrightarrow$ a submanifold $\underline{Q\subseteq P}$ which is H-invariant ($q\cdot h\in Q$ $\forall q\in Q, h\in H$) and such that $\pi|_Q: Q\to M$ is a principal H-bdle.

"finding an H-red. of P = finding an H-principal bdle inside P"

Pf: Having such a Q we claim $\imath_*(Q)\simeq P$.

Conversely ....

$$(Q\times G)/_H \xrightarrow{\sim} P$$

$$[q,a] \longmapsto q\cdot a$$

$$\overset{\shortparallel}{[qh^{-1}, ha]} \longrightarrow qh^{-1}ha$$

($q\in Q$, $a\in G$)

Example: Vector bundle $\begin{array}{c} E \\ \downarrow \pi_E \\ M \end{array}$ of rank r $\overset{\llcorner 5 \lrcorner}{\rightsquigarrow}$ principal GL_r-bundle $\begin{array}{c} Fr(E) = \{ (x, e) \;\; x\in M, e \in Fr(E_x) \} \\ \downarrow \\ M \end{array}$ $\quad e = (e_1, \dots, e_r)$, GL_r

$\hat{e} : \mathbb{R}^r \xrightarrow{\sim} E_x$

Example: ______ " ______ & an inner product g on the v. bdle

$\Rightarrow$ we can focus on orthonormal frames of the fibers E_x $\left(g(e_i, e_j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \right)$ $\Rightarrow$ $\begin{array}{c} Fr(E,g) \\ \downarrow \\ M \end{array} = \{ (x, e) \in Fr(E) \,/\, e = \text{orthonormal} \}$

Here $O(r)$ acts

is a principal $O(r)$-bundle

$\Rightarrow$ $Fr(E,g)$ is an $O(r)$-reduction of $Fr(E)$

Example: Vector bundle $\begin{array}{c} E \\ \downarrow \pi_E \\ M \end{array}$ of rank $r = 2k$ & a complex structure on it $J : E \to E$

$\Rightarrow$ we can talk about complex frames of the real vector spaces E_x

i.e. of type $\{ e_1, \dots, e_k, Je_1, \dots, Je_k \}$. $\Rightarrow$ $\begin{array}{c} Fr(E, J) \\ \downarrow \\ M \end{array}$ principal $GL_k(\mathbb{C})$-bundle, $GL_k(\mathbb{C}) \subset GL_r(\mathbb{R})$

$\Rightarrow$ a $GL_k(\mathbb{C})$-reduction of $Fr(E)$.

Example: r $\begin{array}{c} E \\ \downarrow \\ M \end{array}$, $F \subseteq E$ sub-bundle of rank p $\quad$ $\llcorner G \lrcorner$ $\quad$ V $\quad$ $e_1, \dots, e_r, \dots e_1, \dots, e_r$

Talk frames of E adapted to F. $\quad e = (\underbrace{e_1, \dots, e_p}_{\text{frame of } F_x}, \dots, e_r)$

$\Rightarrow$ we get a principal bundle $Fr(E,F)$ w.r.t. $\quad G =$ exercise ...

From principal bundles to vector bundles.

Start with $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}$ principal G-bundle & $\underbrace{V \in Rep(G)}$

i.e. V = f dim. vector space + group homomorphism $\rho : G \to GL(V)$

$g \cdot v := \rho(g)(v)$

$\Updownarrow$

+ a left action $G \times V \longrightarrow V$ which is linear $g \cdot (v + w) = g \cdot v + g \cdot w$

The outcome is a vector bundle over M with "fibers ... copies of V"

Explicitely $E(P,V) := (P \times V)/G = \{ [p, v] : p \in P, v \in V \}$

$[pg, v] = [p, gv]$

$\begin{array}{ccc} E(P,V) & & [p,v] \\ \downarrow & & \downarrow \\ M & & \pi(p) \end{array}$

Vector bundle: $[p, v] + [p, w] = [p, v + w]$ $\quad (**)$

For the fiber above x, choose any $p_0 \in P_x$. Then any vector in the fiber above x is of type $[p_0, v]$ for some v. Use $(**)$ to define $+$.

$\pi : P \to M$ ... manifold P &

surjective submersion ... satisfying

(1) the action is free

(2) the fibers of π = the orbits of the action

(1') π is G-invariant, i.e., $\pi(p\cdot g) = \pi(p)$ $(\forall) p \in P, g \in G$

(2') each fiber $P_* = \pi^{-1}(*)$ is a G-torsor

(*)

$\mathrm{Princ}_G(M)$ princ. G-bdls over M

Example

defined so

Hence, fo

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Particular case: $G = GL_r$, $V = \mathbb{R}^r$ with the canonical action of GL_r

$\Rightarrow$ any principal GL_r-bundle $P \downarrow M$ induces a rank r vector bundle $E(P, \mathbb{R}^r) = (P \times \mathbb{R}^r)/GL_r \downarrow M$

What if $P = Fr(E)$ of a vector bundle E

$\Rightarrow$ $(Fr(E) \times \mathbb{R}^r)/GL_r \xrightarrow{\sim} E$

$[*, e, v] \mapsto \hat{e}(v)$

frame of E_*: $\hat{e} : \mathbb{R}^r \xrightarrow{\sim} E_*$

isom. of vector bundles

Big Conclusion:

$$\left\{\begin{array}{c}\text{rank } r\\ \text{vector}\\ \text{bundles}\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{c}\text{principal}\\ GL_r-\\ \text{bundles}\end{array}\right\}$$

Rk

gene

$\Rightarrow$ ca

ex:

Lem

"fin