

Reminder: Principal G -bundle over $M = G$ -manifold $P \xrightarrow{\pi} M$ (*)

surjective submersion ... satisfying

- (1) the action is free
- (2) the fibers of $\pi =$ the orbits of the action
- (1') π is G -invariant, i.e., $\pi(p \cdot g) = \pi(p) \quad \forall p \in P, g \in G$
- (2') each fiber $P_x = \pi^{-1}(x)$ is a G -torsor
- (3) $P \times G \rightarrow P \times_M P, (p, g) \mapsto (p, pg)$ is a bijection/diffeom.
- (4) the action is free and proper and π induces $P/G \cong M$.
- (5) for each $x_0 \in M, \exists$ open neighb. $U \subset M$ and an isomorphism

$$\begin{array}{ccc}
 P|_U & \xrightarrow{\sim} & U \times G \\
 \pi \searrow & & \swarrow \text{pr}_U \\
 & U &
 \end{array}$$

(Local) sections of (*): $\sigma: M \rightarrow P$ s.t. $\pi \circ \sigma = \text{id}_M$. Any such σ

induces an isomorphism $F: M \times G \rightarrow P, F(x, g) = \sigma(x) \cdot g$

... and conversely ... hence

$$\left. \begin{array}{l} \text{always} \\ \exists \\ (5) \text{ is a consequence} \end{array} \right\} \text{(local) sections of } P \xleftrightarrow{1-1} \text{(local) trivializations of } P$$

Rk: Hence comparing trivializations \Leftrightarrow comparing (local) sections.

Imitate the discussion on transition functions \Rightarrow

\Rightarrow a description in terms of open covers $U = \{U_\alpha\}_{\alpha \in \Lambda}$ of M and G -valued cocycles - that is

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G \quad \text{smooth maps}$$

s.t. on $U_\alpha \cap U_\beta \cap U_\gamma, g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \quad (g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma})$.

Explicitly, starting with (*): choose U s.t. over U_α there exists a local section $\sigma_\alpha: U_\alpha \rightarrow P$. For $x \in U_\alpha \cap U_\beta: \sigma_\alpha(x), \sigma_\beta(x) \in P_x \Rightarrow$

$\Rightarrow \sigma_\beta(x) = \sigma_\alpha(x) \cdot g_{\alpha\beta}(x)$ for some $g_{\alpha\beta}(x) \in G \Rightarrow$ the $g_{\alpha\beta}$'s were looking for

(1)
 bundle over $M = G$ -manifold $P \rightarrow M$
 satisfying $\text{Princ}_G(M)$ princ G -bdl over M

the orbits of the action
 i.e., $\pi(p \cdot g) = \pi(p) \quad \forall p \in P, g \in G$
 $\pi^{-1}(x)$ is a G -torsor

$\pi: P \rightarrow M, (p, g) \mapsto (p, pg)$ is a bijection/diffeom.
 free and proper and π induces $P/G \cong M$

\exists open neighb. $U \subseteq M$ and an isomorphism
 $U \xrightarrow{\sim} U \times G$
 $\pi|_U \cong \text{pr}_U$

Example/constructions Pull-backs: each $f: M \rightarrow N$ induces
 $f^*: \text{Princ}_G(N) \rightarrow \text{Princ}_G(M)$
 defined so that $(f^*P)_x = P_{f(x)}$
 Hence, for $P \in \text{Princ}_G(N)$
 $f^*P = \{ (x, p) \in M \times P \mid f(x) = \pi(p) \}$
 Action $(x, p) \cdot g = (x, pg)$

Ex/constr. Each $i: H \rightarrow G$ group homomorphism induces
 $i_*: \text{Princ}_H(M) \rightarrow \text{Princ}_G(M), [Q] \mapsto i_*(Q)$
 "by enlarging Q "

Explicitly:
 → start with $Q \times G$
 → act by H on the left $h \cdot (g, a) = (hg^{-1}, i(h) \cdot a)$
 → notice the action is free & proper $i_*(Q) = (Q \times G)/H = \{ [g, a] \}$
 Take the quotient
 The action of G on $i_*(Q): [g, a] \cdot b = [g, ab]$

$\sigma: M \rightarrow P$ s.t. $\pi \circ \sigma = \text{id}_M$. Any such σ
 $\times G \rightarrow P, F(x, g) = \sigma(x)g$
 (local) trivializations of P

sections \Leftrightarrow comparing (local) sections.
 transition functions \Rightarrow
 atlas of \mathbb{R}^n covers $U = \{U_\alpha\}_{\alpha \in A}$
 cycles - that is
 $U_\beta \rightarrow U_\alpha$ smooth maps
 $f_{\beta\alpha}(x) = g_{\beta\alpha}(x) (g_{\beta\alpha} g_{\alpha\beta} = g_{\alpha\alpha})$
 choose U s.t. over U_α there exists
 for $x \in U_\alpha \cap U_\beta: (x, \sigma_\beta(x)) \in P_x \Rightarrow$
 we $g_{\beta\alpha}(x) \in G \Rightarrow$ the $g_{\beta\alpha}$'s we were looking for

Def. Given $[P \in \text{Princ}_G(M)]$ and $[H \subseteq G]$ a Lie subgroup
 an H -reduction of P is any $[Q \in \text{Princ}_H(M)]$
 s.t. $i_*(Q) \cong P$ where $i: H \hookrightarrow G$ is the inclusion.

$i_*: H = \{e\}$ An e -reduction \Leftrightarrow a trivialization of P
Lemma: This is \Leftrightarrow a submanifold $Q \subseteq P$
 which is H -invariant ($g \cdot h \in Q \forall g \in Q, h \in H$) and such that
 $\pi|_Q: Q \rightarrow M$ is a principal H -bdl.

"finding an H -red. of P = finding an H -principal bdl" inside P
 \Leftarrow Having such a Q we claim $i_*(Q) \cong P$

Conversely:
 $(Q \times G)/H \xrightarrow{\sim} P$
 $[g, a] \mapsto g \cdot a$
 $[gh^{-1}, ha] \mapsto gh^{-1} \cdot ha$



Example/constructions ^[3-] Pull-backs: each $f: M \rightarrow N$ induces $f^*: \text{Princ}_G(N) \rightarrow \text{Princ}_G(M)$ defined so that $(f^*P)_x = P_{f(x)}$. Hence, for $P \in \text{Princ}_G(N)$

$$f^*P = \{ (x, p) \in M \times P \mid f(x) = \pi(p) \}$$

Action $(x, p) \cdot g = (x, pg)$

Qk: We could replace G by any H -manifold $N \Rightarrow \Rightarrow$ we produce $(Q \times N)/H$ sitting above M with fibers copies of N .

Ex/constr: Each $i: H \rightarrow G$ group homomorphism induces $i_*: \text{Princ}_H(M) \rightarrow \text{Princ}_G(M)$, $[Q] \mapsto i_*(Q)$ "by enlarging Q "

When $Q = M \times H$

$$(M \times H) \times G / H = \{ [(x, h), a] \mid x \in M, h \in H, a \in G \}$$

$$= \{ [(x, e), a] \mid a \in G, x \in M \} \cong M \times G$$

$$[(x, h), a] = [(x, e), i(h)a]$$

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$\{ [p, a] \mid \pi(p) = x \} = \{ [p, a] \mid a \in G \}$
 Fix p_0 s.t. $\pi(p_0) = x$
 $M \ni x$

Explicitly:
 \rightarrow start with $Q \times G$
 \rightarrow act by H on the left $h \cdot (g, a) = (gh^{-1}, i(h) \cdot a)$
 \rightarrow notice the action is free & proper $i_*(Q) = (Q \times G)/H = \{ [p, a] \mid p \in Q, a \in G \}$
 Take the quotient
 The action of G on $i_*(Q)$: $[g, a] \cdot b = [g, ab]$

Def: Given $[P \in \text{Princ}_G(M)]$ ^[4-] and $[H \subseteq G]$ a Lie subgroup an H -reduction of P is any $Q \in \text{Princ}_H(M)$ s.t. $i_*(Q) \cong P$ where $i: H \hookrightarrow G$ is the inclusion.

$H = \{e\}$ An e -reduction \Leftrightarrow a trivialization of P

lemma: This is \Leftrightarrow a submanifold $Q \subseteq P$ which is H -invariant ($g \cdot h \in Q \iff g \in Q, h \in H$) and such that

$M = G$ -manifold P

$(*)$
 $\text{Princ}_G(M)$ $\text{Princ } G\text{-bdl}$
 $\text{over } M$

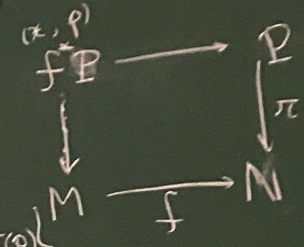
of the action
 $\pi(p \cdot g) = \pi(p) \quad (p \in P, g \in G)$
 P a G -torsor

$\pi: P \rightarrow M$ is a bijection/diffeomorphism

proper and π induces $P/G \cong M$

weighted $U \subseteq M$ and an isomorphism
 $U \times G \rightarrow P_U$

Example/constructions: Pull-backs: each $f: M \rightarrow N$ induces
 $f^*: \text{Princ}_G(N) \rightarrow \text{Princ}_G(M)$



defined so that $(f^*P)_x = P_{f(x)}$

Hence, for $P \in \text{Princ}_G(N)$
 $f^*P = \{ (x, p) \in M \times P \mid f(x) = \pi(p) \}$
 $\downarrow \quad \downarrow$
 $M \quad x$
 Action $(x, p) \cdot g = (x, pg)$

Ex/constr: Each $\iota: H \rightarrow G$ group homomorphism induces
 $\iota_*: \text{Princ}_H(M) \rightarrow \text{Princ}_G(M), [Q] \mapsto \iota_*(Q)$
 "by enlarging Q "

Explicitly:
 \rightarrow start with $Q \times G$
 \rightarrow act by H on the left $h \cdot (g, a) = (hg^{-1}, \iota(h) \cdot a)$
 \rightarrow notice the action is free & proper $\iota_*(Q) = (Q \times G)/H = \{ [g, a] \}$

Take the quotient
 The action of G on $\iota_*(Q): [g, a] \cdot b = [g, ab]$
 \downarrow
 $M \quad \pi(p)$

$\pi \circ \sigma = \text{id}_M$. Any such σ

$F(x, g) = \sigma(x)g$

local trivializations of P
 composing (local) sections

on functions \Rightarrow
 $\mathcal{U} = \{ U_\alpha \}_{\alpha \in \Lambda}$
 that is

smooth maps
 $(g_\alpha, g_\beta) = g_\alpha g_\beta^{-1}$

st over U_α there exists
 $U_\beta: \sigma_\alpha(x), \sigma_\beta(x) \in P_x \Rightarrow$
 $g \in G \Rightarrow$ the g_α 's we're looking for

Rk: The construction above can be applied more generally replacing $G \Rightarrow$ for any $P \in \text{Princ}_H(M)$
 \Rightarrow can form $(P \times N)/H \rightarrow M$ with fibers copies of N for any H -manifold M

ex: $H = \mathbb{Z}_2$ (H -reduction)
 Lemma: of $P \iff$ a submanifold $Q \subseteq P$
 which is H -invariant ($g \cdot h \in Q \iff g \in Q, h \in H$) and such that
 $\pi|_Q: Q \rightarrow M$ is a principal H -bundle

"finding an H -red. of $P =$ finding an H -principal bundle" inside P

Having such a Q we claim $\iota_*(Q) \cong P$
 Conversely:
 $(Q \times G)/H \cong P$
 $[g, a] \mapsto g \cdot a$
 $[gh^{-1}, ha] \mapsto gh^{-1} \cdot ha$

$\pi: P \rightarrow M$
 surjective submersion ... satisfying

- (1) the action is free
 - (2) the fibers of $\pi =$ the orbits of the action
- (1') π is G -invariant, i.e., $\pi(p \cdot g) = \pi(p) \quad (\forall p \in P, g \in G)$
- (2') each fiber $P_* = \pi^{-1}(*)$ is a G -torsor

(*)
 $\text{Princ}_G(M) \text{ princ. } G\text{-tbl. over } M$

Example
 defined so
 Hence, for

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Particular case: $G = GL_r$, $V = \mathbb{R}^r$ with the canonical action of GL_r

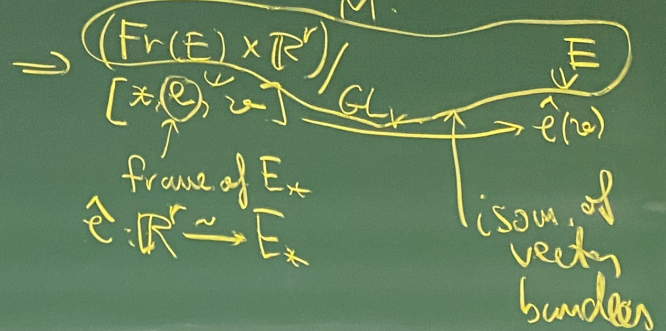
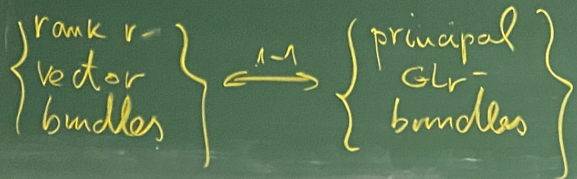
\Rightarrow any principal P GL_r -bundle \downarrow M

induces a rank r vector bundle

$E(P, \mathbb{R}^r) = (P \times \mathbb{R}^r) / GL_r$
 \downarrow
 M

What if $P = \text{Fr}(E)$ of a vector bundle E

Big Conclusion:



Rk
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