Reminder:

-0-

- $E \xrightarrow{\pi} M$ vector bundle $\Rightarrow$ space of sections $\Gamma(E)$
- $\Gamma(E) = C^\infty(M)$-module
- locally, over $U \subseteq M$ small enough: $\exists$ $e = \{e_1, \dots, e_r\}$ local frame of E over U

  & any $s \in \Gamma(E)$ is, over U: $s|_U = f^1 \cdot e_1 + \dots + f^r e_r$ $\quad (\Gamma(E|_U) \cong C^\infty(U)^r)$
- for $E = M \times \mathbb{R}$ (trivial line bundle) $\Rightarrow$ we get $\underline{C^\infty(M)}$
- but(!) there are useful operations on $C^\infty(M)$ that do not make sense on $\Gamma(E)$

  e.g. Lie derivatives $L_X : C^\infty(M) \to C^\infty(M)$, one for each $X \in \mathfrak{X}(M)$

  $L_X(f) = df(X)$

-1-

Def: A connection on a v.b. E: a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma E \to \Gamma E, \quad (X, s) \longmapsto \nabla_X(s)$$

satisfying: $\nabla_{fX}(s) = f\,\nabla_X(s)$, $\underbrace{\nabla_X(fs) = f\,\nabla_X(s) + L_X(f)\cdot s}_{\text{the Leibniz identity}}$

$(\forall)\ X \in \mathfrak{X}(M),\ s \in \Gamma(E),\ f \in C^\infty(M)$

Ex: When $E = M \times \mathbb{R}$, $\nabla^{fl} : \mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M),\ (X, f) \longmapsto L_X(f)$

is a connection (the flat connection on $M \times \mathbb{R}$)

Ex: When $E = M \times \mathbb{R}$, ∇ = arbitrary, Leibniz $\Rightarrow$ ∇ is determined by

$$\boxed{\omega \in \Omega^1(M),} \qquad \omega(X) = \nabla_X(1) \in C^\infty(M)$$

Indeed, by Leibniz $\qquad$ $C^\infty(M)$-linear in X

$$\nabla_X(f) = L_X(f) + f\cdot\omega(X). \qquad d_\nabla(f) = d(f) + f\cdot\omega$$

Ex

Th

$\Rightarrow$

-2-

-3-

Ex: When $E = M \times \mathbb{R}^r$: for arbitrary ∇ we can write $\left(\Gamma E = C^\infty(M)^r\right)$

$$\nabla_X(0,\dots,0,\underset{j^{th}}{1},0,\dots,0) = (\underset{C^\infty(M)}{\omega_j^1(X)},\dots,\underset{C^\infty(M)}{\omega_j^r(X)})$$

THEN $\omega_j^i \in \Omega^1(M)$

$$\left(\underline{\text{OR}}:\ \boxed{\nabla_X(e_j^{can}) = \sum_i \omega_j^i(X)\, e_i^{can}}\right)$$

These encode completely ∇: for $s = f^1 e_1^{can} + \dots + f^r e_r^{can}$

$$\nabla_X(s) = \sum_i \Big(L_X(f^i) + \sum_j f^j \omega_j^i(X)\Big)\cdot e_i^{can}$$

$$\Rightarrow \boxed{\left\{\begin{matrix}\text{connections}\\ \nabla\\ \text{on } M\times\mathbb{R}^r\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{matrices of 1-forms}\\ \omega = (\omega_j^i)_{1\le i,j\le r} = \begin{pmatrix}\omega_1^1 & \dots & \omega_r^1\\ \vdots & & \vdots\\ \omega_1^r & \dots & \omega_r^r\end{pmatrix} \in \mathcal{M}_r(\Omega^1(M))\end{matrix}\right\}}$$

$$\boxed{d_\nabla(e_j^{can}) = \sum_i \omega_j^i\, e_i^{can}}$$

-5-

Dual view point:

$$\nabla_X(s) \in \Gamma E$$

When moving from $M \times \mathbb{R}$ $\xrightarrow{\text{to}}$ general v.bdle. $E \to M$

$$C^\infty(M) \xrightarrow[\text{by}]{\text{replaced}} \Gamma(E)$$

$$C^\infty(M) = \Omega^0(M)$$

$$\omega \in \boxed{\Omega^1(M)} \rightsquigarrow \Omega^1(M,E) \ni \omega$$

$\boxed{\Omega^1(M,E) = \Gamma(T^*M \otimes}$

$\omega : \mathfrak{X}(M) \to C^\infty(M)$, $C^\infty(M)$-linear

$\updownarrow$ 1-1

$\omega \in \Gamma(T^*M)$ $\left(x \in M : \omega_x : T_xM \to \mathbb{R} \text{ linear, smooth in } x \right)$

$\omega : \mathfrak{X}(M) \to \Gamma E$, $C^\infty(M)$-linear

$\updownarrow$ bijection

$\omega \in \Gamma(\underbrace{\mathrm{Hom}(TM, E)}_{T^*M \otimes E})$

(for each $x \in M$ we have a $\omega_x : T_xM \to E$

Locally: chart (U, x) & local frame $e_1, \dots, e_r$

$\omega = \sum f_{ij}\, dx_i \cdot e_j$ E-valued forms on M (over U)

$= \sum \underbrace{\omega^j}_{\text{ordinary 1-forms}} \cdot e_j$

-4-

The space of 1-forms on M with coefficients in E:

$$\Omega^1(M,E) := \Gamma(T^*M \otimes E)$$ where $T^*M \otimes E = Hom(TM, E)$ is the vector bundle $/M$ wose fiber above $x \in M$ is $T^*_x M \otimes E_x = Hom(T_x M, E_x) = \{ w_x : T_x M \to E_x \,/\, \text{linear} \}$

Hence: any $w \in \Omega^1(M,E)$ gives rise to

$$w : \mathfrak{X}(M) \to \Gamma E, \quad w(X)(x) := w_x(X_x)$$

which is $C^\infty(M)$ ... and conversely.

<u>Remark</u>: Any $\eta \in \Omega^1(M), s \in \Gamma(E)$ give rise to $\eta \cdot s \in \Omega^1(M,E)$ (sends $X \longmapsto w(X) \cdot s$)

An arbitrary $w \in \Omega^1(M,E)$ is always a sum $\sum \eta_i \cdot s_i$ with η_i, s_i as above.

(Locally: may get $s_i = e_i$ members of a local frame)

-5-

Dual view point:

$$\nabla_X(s) \in \Gamma E, \quad X \in \mathfrak{X}(M), \ s \in \Gamma(E)$$

For $s \in \Gamma E \Rightarrow$ a map $\begin{cases} \mathfrak{X}(M) \to \Gamma(E) \\ X \longmapsto \nabla_X(s). \end{cases}$ $C^\infty(M)$-linear, i.e.

$\nabla_\cdot(s) \in \Omega^1(M,E)$
$\overset{!!}{=} d_\nabla(s)$

Therefore we re-interpret ∇

as a map $\boxed{\begin{array}{l} d_\nabla : \Gamma E \longrightarrow \Omega^1(M,E) \\ d_\nabla(s)(X) = \nabla_X(s). \end{array}}$

<u>Check</u>: $\nabla =$ conn $\Leftrightarrow d_\nabla$ is linear & $\boxed{\begin{array}{l} d_\nabla(fs) = f\, d_\nabla(s) + df \cdot s \\ (\forall) f \in C^\infty(M),\ s \in \Gamma(E). \end{array}}$

—6—

Locality of ∇: For any ∇, given x:

$$\nabla_X(s)(x) \in E_x$$

only depends on X_x and $s|_U$ for some open $U \subseteq M$, $x \in U$

I.e., if $X_x = 0$ OR $s|_U = 0$ for some U THEN $\nabla_X(s)(x) = 0$

$s = f \cdot s$ ← use bump functions & Leibniz

$d_\nabla(s)(\underset{0}{X_x}) = 0$ ☒

In particular

(∀) $U \subseteq M$ open we can restrict ∇ to U:

$$\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$$ connection on $E|_U$

uniquely such that (∀) $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$: $\nabla^U_{X|_U}(s|_U) = \nabla_X(s)|_U$

—7—

Now, if U is the domain of a local frame $e = \{e_1, \dots, e_r\}$ over U

we can write

over U (*) $\boxed{\nabla_X(e_j) = \sum_i \omega^i_j(X)\, e_i}$ where ω^i_j are now 1-forms on U

Hence, over U: ∇ is completely encoded in

$$\omega = \omega(\nabla, e) = (\omega^i_j)_{1 \le i,j \le r} \in \mathcal{M}_r(\Omega^1(U))$$

called the connection matrix of ∇ w.r.t. e

Given ω, for $s \in \Gamma(E)$, writing $s|_U = \sum f^i e_i \Rightarrow$

$$\Rightarrow \underset{\text{over } U}{\nabla_X(s)} = \sum_i \Big(L_X(f^i) + \sum_p f^p \omega^i_p(X) \Big) \cdot e_i \quad (**)$$

$$\nabla_X\big(\sum f^i e_i\big) = \sum \nabla_X(f^i e_i) = \sum \big(f^i \nabla_X(e_i) + L_X(f^i) e_i \big) =$$

$$= \sum L_X(f^i) e_i + \sum_{i,p} f^p \omega^i_p(X) e_i$$

Satisfying: $\nabla_{fX}(s) = f\nabla_X(s)$, $\underbrace{\nabla_X(fs) = f\nabla_X(s) + L_X(f)\cdot s}_{\text{the Leibniz identity}}$

$(\forall)\, X\in\mathfrak{X}(M),\ s\in\Gamma(E),\ f\in C^\infty(M)$

Ex: When $E = M\times\mathbb{R}$, $\nabla^{E}: \mathfrak{X}(M)\times C^\infty(M)\longrightarrow C^\infty(M),\ (X,f)\longmapsto L_X(f)$

These

Lemma: Given ∇ & $X\in\mathfrak{X}(M)$, if $\gamma: I\longrightarrow M$ is an integral curve of X $\left(\dot\gamma(t) = X_{\gamma(t)}\right)$ then $\underbrace{s \text{ along } \gamma}_{\text{i.e. } s\circ\gamma:\, I\longrightarrow E}$ determine $\underbrace{\nabla_X(s) \text{ along } \gamma}_{\nabla_X(s)\circ\gamma:\, I\longrightarrow M}$

Start with formula (**) and evaluate at arbitrary $x\in U$:

$$\nabla_X(s)(x) = \sum_i\left((df^i)(X_x) + \sum_p f^p(x)\,\omega_p^i(X_x)\right)e_i(x)$$

$\Vert \longleftarrow$ if $x=\gamma(t)$ with γ integral curve of X $\left(X_x = \frac{d\gamma}{dt}(t)\right)$

$$\sum_i\left(\frac{d(f^i\circ\gamma)}{dt}(t) + \sum_p f^p(\gamma(t))\,\omega_p^i\left(\frac{d\gamma}{dt}\right)\right)e_i(\gamma(t))$$

$s=\sum f^i e_i \Rightarrow s\circ\gamma = \sum f^i\circ\gamma\cdot e_i$ & this proves the Lemma