[-1-]

REMINDER: $E \longrightarrow M$ v.b

- connection ∇ on E: $\mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$, $(X, s) \longmapsto \nabla_X(s)$

which is bilinear and: $\nabla_{fX}(s) = f\nabla_X(s)$, $\nabla_X(fs) = f\nabla_X(s) + L_X(f)s$

- dual picture:

$$d_\nabla : \Gamma E \longrightarrow \Omega^1(M; E)$$

which is linear and Leibniz:

$$d_\nabla(fs) = f\, d_\nabla(s) + df \cdot s$$

Relation:

$$\boxed{d_\nabla(s)(X) = \nabla_X(s)}$$

$E = M \times \mathbb{R}$

$\nabla_X(f) = L_X(f) + f \cdot \omega(X)$

$\frac{\nabla u}{dt} = 0$ above γ:

- locally, w.r.t. frame $e = \{e_1, \dots, e_r\}$ of E over U: [-2-]

$\nabla_X(e_i) = \sum \omega_i^j(X)\, e_j$ for some ...

$\nabla_X(f) = L_X(f) + f\cdot\omega(X)$

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-2-

- locally, w.r.t. frame $e = \{e_1, \dots, e_r\}$ of E over U:

$$\nabla_X(e_i) = \sum_j \omega_i^j(X)\, e_j \quad \text{for some } \underset{\omega(\nabla, e)}{\omega} = (\omega_{i,j})_{1\le i,j\le r} \in M_r(\Omega^1(U))$$

Then, for arbitrary s, writing $s|_U = \sum f^i e_i \quad (f^i \in C^\infty(U))$

$$\Rightarrow \nabla_X(s) = \sum_i \underbrace{\left(L_X(f^i) + \sum_j f^j \omega_j^i(X)\right)}_{} e_i \quad \text{over } U$$

- for $u : \underset{\mathbb{R}}{\overset{\cup|}{I}} \longrightarrow E$ with base path $\gamma = \pi \circ u$ $\left(\text{i.e. } u(t) \in E_{\gamma(t)},\ u = \text{"path above } \gamma\text{"}\right)$

$\Rightarrow$ can talk about $\frac{\nabla u}{dt} : I \to E$, a path above the same γ

Locally: Writing $u(t) = \sum_i u^i(t) \underset{\in E_{\gamma(t)}}{e_i(\gamma(t))}$ $\quad (u^i : I \to \mathbb{R}) \Rightarrow$

$$\Rightarrow \boxed{\frac{\nabla u}{dt}(t) = \sum_i \left(\frac{du^i}{dt}(t) + \sum_j u^j(t)\, \omega_j^i(\dot\gamma(t))\right) e_i(\gamma(t))}$$

Def

(or

Rk:

If we

$\frac{dU}{dt} = A(t$

Prop

$\Rightarrow (\forall$

-1-

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Def: (...

(or ∇-...

Rk: Loca...

If we fix ...

$\frac{dU}{dt} = A(t)U(t)$

Prop: G...

$\Rightarrow (\forall)\, u_0 \in$ ...

$pr(u) = \sum_i \langle u, e_i \rangle e_i$

$u - pr(u) \perp V$

$\langle u - pr(u), e_i \rangle = 0$

$\langle u, e_i \rangle = \langle pr(u), e_i \rangle = \lambda^i(u)$

$pr: \mathbb{R}^n \to V$

$pr(u) = \lambda^1 e_1 + \dots + \lambda^r e_r$

Rk: || is ...

-3-

Def: Given ∇, a path $u: I \to E$ is called ∇-parallel (or ∇-horizontal, or ∇-constant) if $\frac{\nabla u}{dt} = 0$.

Rk: Locally: $u = \sum u^i(t) e_i(\gamma(t))$, the condition on the u^i's reads

$$\frac{du^i}{dt}(t) = -\sum_j u^j(t)\, \omega_j^i(\dot\gamma(t)) \qquad 1 \le i \le r$$

If we fix $\gamma \Rightarrow$ an equation of type

$$\frac{dU}{dt} = A(t)U(t) \qquad \underbrace{\frac{d}{dt}(u^1, \dots, u^r)}_{U} = A(t) \underbrace{(u^1(t), \dots, u^r(t))}_{U(t)}$$

always has many solution over the entire I on which γ is defined with sol

$$u(t) = \exp\left(\int_{t_0}^{t} A\right) \cdot u_0$$

for any t_0, u_0. $u(t_0) = u_0$.

Prop: Given ∇ and $\gamma: I \to E$, $t_0 \in I \Rightarrow$

$\Rightarrow (\forall)\, u_0 \in E_{\gamma(t_0)}$ $\exists!\, u: I \to E$ above γ s.t. $\begin{cases} u = \nabla\text{-parallel} \\ u(t_0) = u_0. \end{cases}$ (*)

-2-

$M_r(\Omega^1(U))$

$C^\infty(U)$

... U

... $E_{\gamma(t)}$

... above γ"

... same γ

$R) \Rightarrow$

$(\dot\gamma(t))\, e_i(\gamma(t))$

Def The ... t_0 to t_1

Propositio...

linear iso...

Pf: ① Linear ...

with ...

$\frac{\nabla u}{dt} = $

"∇ being compatible with a geom. structure"

d ∇-parallel

u's reads

$1 \le i \le r$

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$$u(t) = \exp\left(\int_{t_0}^{t} A\right) \cdot u_0$$

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$$T_\gamma^{t_0,t_1} : E_{\gamma(t_0)} \longrightarrow E_{\gamma(t_1)},$$

$$v_0 \quad u_0 \longmapsto u(t_1) \text{ where } u \text{ is the sol. of } (*)$$

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Pf: ① Linear: $\underbrace{T_\gamma^{t_0,t_1}(u_0)}_{u(t_1)} + \underbrace{T_\gamma^{t_0,t_1}(v_0)}_{v(t_1)} = \underbrace{T_\gamma^{t_0,t_1}(u_0+v_0)}_{(u+v)(t_1) = u(t_1)+v(t_1)}$

with u $\frac{\nabla u}{dt} = 0$, $u(t_0) = u_0$ $\quad \frac{\nabla v}{dt} = 0$, $v(t_0) = v_0$ $\Big| \Rightarrow \frac{\nabla(u+v)}{dt} = 0$, $(u+v)(t_0) = u_0 + v_0$

—5—

Corollary: Any vector bundle over a contractible space M is trivializable.

$$H: M \times [0,1] \to M \text{ smooth}$$

$H_t = H(\cdot, t) : M \to M$

$H_0 \sim H_1$

s.t. $\begin{cases} H(x,0) = x_0 & \text{for some } x_0 \in M \\ H(x,1) = x & (\forall)\, x \in M \end{cases}$

Equivalently, $\underbrace{\text{the identity map}}_{H_1 = H(\cdot,1)}$ is homotopic to $\underbrace{\text{the cst } x_0}_{H_0 = H(\cdot,0)}$

$\gamma_x = H(x, \cdot) : [0,1] \to M$

$0 \longmapsto x_0$

$1 \longmapsto x$

Do //-transport to compare E_x to E_{x_0}.

Rk: // is also useful to make sense of "∇ being compatible with a geom. structure"

$V \subseteq \mathbb{R}^k$ $\langle\cdot,\cdot\rangle_{\text{can}}$ on $\mathbb{R}^k$

$pr : \mathbb{R}^k \to V$ orthonormal fr. $e_1, \dots, e_r$ of V

$pr(u) = \lambda^1(u) e_1 + \dots + \lambda^r(u) e_r$

$\lambda^i(u)$

...alled ∇-parallel

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...)) $\quad 1 \le i \le r$

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...rallel

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—4—

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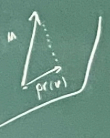

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-2-
• locally, w.r.t frame $e = \{e_1, \dots, e_r\}$ of E over U.
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Then, for arbitrary s, writing $s|_U = \sum f^i e_i$ $(f^i \in C^\infty(U))$
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-3-
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-5-
Corollary: Any vector bundle over a contractible space M is trivializable

The space of k-forms on M with coefficients in E, $\Omega^k(M,E)$

consists of maps

$$\omega : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k} \longrightarrow \Gamma(E)$$

which are
- skew symmetric
- $C^\infty(M)$-linear in each argument

Or: $\Omega^k(M,E) = \Gamma(\underbrace{\text{"}\Lambda^k T^*M \otimes E\text{"}})$

$\mathrm{Alt}^k(TM, E)$ = the vector bundle $/M$ whose fiber above $x \in M$

$$\mathrm{Alt}^k(T_xM, E_x) = \Big\{ \omega_x : \underbrace{T_xM \times \dots \times T_xM}_{k} \to E_x \,\Big|\, \begin{matrix}\text{multilin} \\ \text{skew symm}\end{matrix} \Big\}$$

$\boxed{-6'-}$

On manifolds M: $\quad \omega \in \Omega^k(M)$: $\boxed{\omega : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \to C^\infty(M)}$

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \cdots$$

$C^\infty(M) = \Omega^0(M)$ $\qquad d \circ d = 0$

De Rham complex

$$\Gamma E \xrightarrow{d_\nabla} \Omega^1(M,E)$$

E

$\omega \in \Omega^k(M)$ $\quad d\omega \in \Omega^{k+1}(M)$

Def: Give ∇ define

$$d_\nabla \omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i L_{X_i} \big(\underbrace{\omega(X_0, \dots, X_{i-1}, \hat{X}_i, X_{i+1}, \dots, X_k)}_{\in C^\infty(M) \quad \Gamma(E)}\big)$$

∇_{X_i}

$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \underbrace{\omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k)}_{\in C^\infty(M) \quad \Gamma(E)}$$

$$\Gamma E \xrightarrow{d_\nabla} \Omega^1(M,E) \xrightarrow{d_\nabla} \Omega^2(M,E) \xrightarrow{d_\nabla} \cdots$$

$\Gamma E = \Omega^0(M,E)$

Corollary

a contr

$H_t = H(\cdot, t) : M \to$

$H_0 \sim H_1$

Do //- to

Rk

Def: Given ∇ a path $\gamma: I \to E$ $\boxed{-3-}$

$\Omega^k(M, E)$ consists of $C^\infty(M)$-multilinear, skew-symmetric maps

—7—

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k} \longrightarrow \Gamma E$$

Classical wedge operations:

$E_1 \times E_2 \to E$

$$\begin{cases} \Omega^k(M) \times \Omega^\ell(M) \longrightarrow \Omega^{k+\ell}(M) \\ (\omega, \eta) \longmapsto \omega \wedge \eta \end{cases}$$

where $\underline{(\omega \wedge \eta)}(X_1, \ldots, X_{k+\ell}) =$

$$= \sum_{\sigma \in S_{k,\ell}} \operatorname{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)})$$

$\Gamma E \to \Gamma E$ $\qquad$ ΓE

$\underline{\sigma(1) < \ldots < \sigma(k)}$ $\qquad$ $\underline{\sigma(k+1) < \ldots < \sigma(k+\ell)}$

Variations:

$$\begin{cases} \Omega^k(M) \times \Omega^\ell(M, E) \longrightarrow \Omega^{k+\ell}(M, E) \\ \Omega^k(M, E) \times \Omega^\ell(M) \longrightarrow \Omega^{k+\ell}(M, E) \\ \Omega^k(M, \operatorname{End}(E)) \times \Omega^\ell(M, E) \longrightarrow \Omega^{k+\ell}(M, E) \\ \ldots \end{cases}$$

$\operatorname{End}(E) = \operatorname{Hom}(E, E)$

(B) $k = 2$

$\underline{\text{Cool}}$: (A) & (B) $\Rightarrow$ can use K_∇ to get:

$$\boxed{\begin{aligned} \Omega^\ell(M, E) &\longrightarrow \Omega^{\ell+2}(M, E) \\ \omega &\longmapsto K_\nabla \wedge \omega \\ &\ldots \text{ this is } d_\nabla^2 !! \end{aligned}}$$

-6'-

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$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \cdots$$

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Def: Give ∇ define

$$d_\nabla \omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \underset{\nabla_{X_i}}{L_{X_i}} \big(\underbrace{\omega(X_0, \ldots, X_{i-1}, \hat{X_i}, X_{i+1}, \ldots, X_k)}_{\in C^\infty(M) \quad \Gamma(E)}\big)$$

$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \underbrace{\omega([X_i, X_j], X_0, \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots, X_k)}_{\in C^\infty(M) \quad \Gamma(E)}$$

$$\Gamma E \xrightarrow{d_\nabla} \Omega^1(M,E) \xrightarrow{d_\nabla} \Omega^2(M,E) \xrightarrow{d_\nabla} \cdots$$

$\Gamma E = \Omega^0(M,E)$

Corollary

a (contr

$H_t = H(\cdot, t) : M \to$

$H_0 \sim H_1$

Do //-
to

Rk

Def: Given ∇ a path ... -3-

-8-

Def: Given ∇, the curvature of ∇ is

$$K_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma E \longrightarrow \Gamma E$$

$$(X, Y, s) \longmapsto \underbrace{K_\nabla(X,Y)}(s) =$$

$$= \nabla_X(\nabla_Y(s)) - \nabla_Y(\nabla_X(s)) - \nabla_{[X,Y]}(s)$$

Call ∇ flat if $K_\nabla = 0$.

Rk: Remember: $\boxed{L_{[X,Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f))}$

Interesting: $C^\infty(M)$-linear in X, Y & s !!

~~$K_\nabla(X,Y)$~~

$$K_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \Gamma(\mathrm{End}\, E).$$

$$(X, Y) \longmapsto K_\nabla(X,Y) : \Gamma E \to \Gamma E$$

$C^\infty(M)$-lin in s

$\boxed{K_\nabla \in \Omega^2(M, \mathrm{End}\, E)}$ (A)

$\Omega^k(M,E)$ consists of $C^\infty(M)$-multilinear, skew-symmetric maps -7-

$$\omega : \underbrace{\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)}_{k} \longrightarrow \Gamma E$$

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where $\underline{(\omega\wedge\eta)}(X_1,\dots,X_{k+l}) =$

$$= \sum_{\sigma\in S_{k,l}} sgn(\sigma)\, \overset{\Gamma E\to\Gamma E}{\omega}(X_{\sigma(1)},\dots,X_{\sigma(k)})\, \overset{\Gamma E}{\eta}(X_{\sigma(k+1)},\dots,X_{\sigma(k+l)})$$

$$\underline{\sigma(1)<\dots<\sigma(k)} \qquad \underline{\sigma(k+1)<\dots<\sigma(k+l)}$$

Variations:

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(B) $k=2$

Cool: (A) & (B) $\Rightarrow$ can use K_∇ to get:

$$\boxed{\begin{aligned} \Omega^l(M,E) &\longrightarrow \Omega^{l+2}(M,E) \\ \omega &\longmapsto K_\nabla\wedge\omega \\ &\dots \text{ this is } d_\nabla^2 !! \end{aligned}}$$