

Reminder on connections  $\nabla: \mathfrak{X}(M) \times \Gamma E \rightarrow \Gamma E$  ( $E = \text{real/complex v. bundle}/M$ )

- any  $E$  admits one (at least)
- locally, w.r.t. local frame  $e = \{e_1, \dots, e_r\}$  encoded in the connection matrix  $\omega = \omega(\nabla, e) = (\omega_{ij}^k)_{i,j,k=1,\dots,r} \in M_r(\mathcal{R}^1(U))$

defined by:  $\nabla_X(e_j) = \sum_{i=1}^r \omega_{ij}^k(X) e_i$

- the curvature of  $\nabla: K_\nabla \in \Omega^2(M, \text{End } E)$ , defined by

$$K_\nabla(X, Y)(s) = \nabla_X(\nabla_Y(s)) - \nabla_Y(\nabla_X(s)) - \nabla_{[X, Y]}(s) \in \Gamma E \text{ (for } X, Y \in \mathfrak{X}(M), s \in \Gamma E)$$

Locally we can write

$$K_\nabla(X, Y)(e_j) = \sum_i \underbrace{k_{ij}^k(X, Y)}_{\in C^\infty(U)} e_i$$

$\Rightarrow K_\nabla$  over  $U$  is completely encoded in

$k = (k_{ij}^k)_{i,j,k=1,\dots,r} \in M_r(\mathcal{R}^2(U))$  the curvature matrix of  $\nabla$  w.r.t.  $e$ .

Comput 3  
 $k' = dw' + w' \wedge w$

$$= \cancel{d(g^{-1})} \cdot dg + g^{-1} \cdot (dw + w \wedge w)$$

$$= g^{-1} (dw + w \wedge w)$$

Lemma:  $k = dw + w \wedge w = \begin{pmatrix} dw_1^1 & \dots & dw_r^1 \\ \vdots & \ddots & \vdots \\ dw_1^r & \dots & dw_r^r \end{pmatrix} + \begin{pmatrix} w_1^1 & \dots & w_r^1 \\ \vdots & \ddots & \vdots \\ w_1^r & \dots & w_r^r \end{pmatrix} \wedge \begin{pmatrix} w_1^1 & \dots & w_r^1 \\ \vdots & \ddots & \vdots \\ w_1^r & \dots & w_r^r \end{pmatrix}$

i.e.:  $k_{ij}^k = dw_j^k + \sum_p w_i^p \wedge w_p^k$

Rk: Dual picture:  $d_\nabla(e_j) = \sum_i w_j^i e_i$  1-form with coeff. in  $E$

Very-very-  
 How does  
 change



$$+ w^i \wedge w^i = d(g^i dg + g^i \omega g) + (g^i dg + g^i \omega g) \wedge (g^i dg + g^i \omega g)$$

$$dg + g^i d(\omega g) + (d(g^i) \omega g + g^i d(\omega) g - g^i \omega dg) + g^i dg g^i dg + g^i dg g^i \omega g + g^i \omega g g^i dg + (g^i \omega g g^i \omega g)$$

$$(dw + w \wedge w) g^i$$

Comp 1

pd:

$$k_{\nabla}(x, y)(e_j) = \nabla_x(\nabla_y(e_j)) - \nabla_y(\nabla_x(e_j)) - \nabla_{[x, y]}(e_j) =$$

$$d_{\nabla}^2(e_j) = \nabla_x \left( \sum_i w_j^i(y) e_i \right) - \nabla_y \left( \sum_i w_j^i(x) e_i \right) - \sum_i w_j^i(x, y) e_i$$

$$= \sum_i d w_j^i \cdot e_i - \sum_i w_j^i d_{\nabla}(e_i) - \sum_i w_j^i e_i$$

$$= \sum_i (d w_j^i + \sum_k w_k^i \omega_j^k) e_i = \sum_i (L_x(w_j^i(y)) - L_y(w_j^i(x)) - w_j^i(x, y)) e_i + \sum_{l, i} \frac{w_l^i(y) w_j^i(x) e_i - \sum_{l, i} w_j^i(x) w_l^i(y) e_i}{(dw_j^i)(x, y)}$$

$$\Rightarrow k_{j,i}^i(x, y) = d w_{j,i}^i(x, y) + \sum_i (w_i^i \wedge w_j^i)(x, y) \quad \square$$

$$\sum_{l, i} (w_l^i \wedge w_j^i)(x, y) \cdot e_i$$



$K_{\nabla}(X, Y)(e_j) = \sum_i \underbrace{k_{ij}^{\nabla}(X, Y)}_{\in C^{\infty}(U)} e_i \iff K_{\nabla} = \sum_j k_j \cdot \underbrace{e_i^j}_{\in \text{End}(E)}$   
 $\Rightarrow K_{\nabla}$  over  $U$  is completely encoded in  $k = (k_{ij})_{1 \leq i, j \leq r} \in M_r(\mathcal{R}^2(U))$  the curvature matrix of  $\nabla$  w.r.t.  $e$ .  
 $g \mapsto e_i$  rest to 0

Lemma:  $k = d\omega + \omega \wedge \omega = \begin{pmatrix} dw^1 & \dots & dw^r \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} + \begin{pmatrix} w^1 & \dots & w^r \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \wedge \begin{pmatrix} w^1 & \dots & w^r \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \quad (-2)$

i.e.:  $k_i^i = dw^i + \sum_p w_p^i \wedge w_j^p$

Rk: Dual picture:  $d_{\nabla}(e_j) = \sum_i w_j^i e_i$  1-form with coeff. in  $E$   
 $\Rightarrow d_{\nabla}^2(e_j) = \sum_i k_j^i e_i = K \wedge e_j \Rightarrow d_{\nabla}^2 = \text{wedging with } K_{\nabla}$ .

Changing frames:  $e \rightsquigarrow$  new one  $e'$

$e'_i = \sum_p g_i^p e_p$  over  $U \cap U'$  with  $g_i^p \in C^{\infty}(U \cap U')$

$(\Rightarrow e_p = \sum_i (g^{-1})_p^i e'_i)$

$g = (g_i^p)_{1 \leq p, i \leq r} \in M_r(C^{\infty})$   
 invertible matrix

$\Rightarrow \omega' = g^{-1} \cdot dg + g^{-1} \omega g$

$\Rightarrow k' = g^{-1} \cdot k \cdot g$



$$+ \underbrace{dg^i w_j + g^i dw_j}_{-g^i dg^j} - \underbrace{g^i w_j dg^j}_{-g^i dg^j} + \underbrace{g^i dg^j g^k w_l}_{-g^i dg^j} + \underbrace{g^i w_j g^k dg^l}_{\frac{1}{i \Delta}} + \underbrace{g^i w_j g^k w_l}_{\frac{1}{i \Delta}}$$

$$d_{\nabla}(e'_i) = d_{\nabla}\left(\sum_p g_{ip}^p e_p\right) = \sum_p dg_{ip}^p \cdot e_p + \sum_p g_{ip}^p \underbrace{d_{\nabla}(e_p)}_{\sum_q w_{pq}^q e_q} = \text{Comput 2}$$

$$= \sum_q \left( dg_{iq}^q + \sum_p g_{ip}^p w_{pq}^q \right) e_q$$

$$= \sum_p \left( dg_{ip}^q (g^{-1})^j_q \right) + \sum_p g_{ip}^p w_{pq}^q (g^{-1})^j_q \Bigg) e'_j$$

$$(w')^q_i = \sum_j (g^{-1})^j_q (dg)_{ij}^q$$

$$(g^{-1} \cdot dg)_i^j$$

$$\sum_{q,p} g_{ip}^p w_{pq}^q (g^{-1})^j_q = (g^{-1} w g)_i^j$$





real / complex  
v. bddle / M

$\mathcal{R}^2(U)$

d by

$\mathcal{K} \in \mathcal{X}(M), \mathcal{S} \in \mathcal{P}(E)$

End (E)  
rest to 0  
matrix

Comput 3

$$k' = dw + w' \wedge w = d \left( \underbrace{g^{-1}dg}_{w'} + \underbrace{g^{-1}wg}_{w'} \right) + \underbrace{(g^{-1}dg + g^{-1}wg)}_{w'} \wedge \underbrace{(g^{-1}dg + g^{-1}wg)}_{w'} =$$

$$= \cancel{d(g^{-1})dg} + \underbrace{g^{-1}d(dg)}_0 + \cancel{d(g^{-1})wg} + \underbrace{g^{-1}d(w)g}_{w'} - \cancel{g^{-1}wdg} + \cancel{g^{-1}dg} \underbrace{g^{-1}dg}_{-g \cdot dg^{-1}} + \underbrace{g^{-1}dg}_{-g \cdot dg^{-1}} \underbrace{g^{-1}wg}_{Id} + \underbrace{g^{-1}wg}_{Id} \underbrace{g^{-1}dg}_{Id} + \underbrace{g^{-1}wg}_{Id} \underbrace{g^{-1}wg}_{Id}$$

$$= g^{-1} \underbrace{(dw + w' \wedge w)}_k g^{-1}$$

$$g \cdot g^{-1} = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$dg \cdot g^{-1} + g \cdot d(g^{-1}) = 0$$

$$d(w \cdot \eta) = dw \cdot \eta + (-1)^k w \cdot d\eta$$

Very-very-very-very interesting consequence / remark:  
How does  $\text{Tr}(k) = k_1 + k_2 + \dots + k_r \in \mathcal{R}^2(U)$   
change? So... ?



Reminder on connections  $\nabla: \mathfrak{X}(M) \times PE \rightarrow PE$  ( $E = \text{real/complex}$  or  $\mathbb{R}/\mathbb{C}$ )

- any  $E$  admits one (at least)
- locally, wrt local frame  $e = \{e_1, \dots, e_r\}$  encoded in the connection matrix  $\omega = \omega(\nabla, e) = (\omega^i_j)_{i,j=1,r} \in \mathcal{M}_r(\mathbb{R}/\mathbb{C}(U))$

defined by:  $\nabla_X(e_j) = \sum_{i=1}^r \omega^i_j(X) e_i$

the curvature of  $\nabla: K_\nabla \in \Omega^2(M, \text{End } E)$ , defined by  $K_\nabla(X,Y)(s) = \nabla_X(\nabla_Y(s)) - \nabla_Y(\nabla_X(s)) - \nabla_{[X,Y]}(s) \in PE$  (for  $X,Y \in \mathfrak{X}(M), s \in PE$ )

Locally we can write  $K_\nabla(X,Y)(e_j) = \sum_{i=1}^r k^i_j(X,Y) e_i$  (C<sup>∞</sup>-linear in  $X,Y$ )

$\Rightarrow K_\nabla$  over  $U$  is completely encoded in  $k = (k^i_j)_{i,j=1,r} \in \mathcal{M}_r(\mathbb{R}/\mathbb{C}(U))$  the curvature matrix of  $\nabla$  wrt  $e$

Comput  $\nabla$

$$k^i = d\omega^i + \omega^i \wedge \omega^i = d(g^i dg + g^i \omega g) + (g^i dg + g^i \omega g) \wedge (g^i dg + g^i \omega g) = d(g^i) dg + g^i d(dg) + (dg^i) \omega g + g^i d(\omega g) - g^i \omega dg + g^i dg g^i dg + g^i dg g^i \omega g + g^i \omega g g^i dg + g^i \omega g g^i \omega g$$

$$= g^i (d\omega + \omega \wedge \omega) g^i$$

$$d(\omega \eta) = d\omega \eta + (-1)^k \omega d\eta$$

$$g_i g^i = \text{id} = \begin{pmatrix} 1 & \\ & \ddots \\ & & 1 \end{pmatrix}$$

$$dg \cdot g^i = g \cdot d(g^i) = 0$$

Lemma:  $k = d\omega + \omega \wedge \omega = \begin{pmatrix} d\omega^1 & \dots & d\omega^1 \omega^2 - \omega^1 d\omega^2 \\ \vdots & \ddots & \vdots \\ d\omega^r & \dots & d\omega^r \omega^1 - \omega^r d\omega^1 \end{pmatrix}$

$\Rightarrow k^i_j = d\omega^i_j + \sum_{k=1}^r \omega^i_k \omega^k_j$

$R^k$ : Dual picture:  $d\eta_j(e_i) = \sum_j \omega^j_i e_i$  1-form with coeff in  $E$

$\Rightarrow d\eta^i(e_j) = \sum_j k^i_j e_i \Rightarrow K \wedge \eta_j \Rightarrow d\eta^i = \text{wedging with } K_\nabla$

Changing frame:  $e \rightsquigarrow$  new one  $e'$

$e'_i = \sum_j g^j_i e_j$  over  $U$  with  $g^j_i \in C^\infty(U)$

$\Rightarrow e_j = \sum_i (g^i_j)^{-1} e'_i$   $g = (g^i_j)_{i,j=1,r} \in \mathcal{M}_r(C^\infty)$  invertible matrix

$\Rightarrow \omega' = g^i dg + g^i \omega g$

$\Rightarrow k' = g^i k g$

Very-very-very-very interesting consequence/remark

How does  $\text{Tr}(k) = k^1_1 + k^2_2 + \dots + k^r_r \in \Omega^2(U)$  change? So...?

$k^i_j(X,Y)(e_j) = \nabla_X(\nabla_Y(e_j)) - \nabla_Y(\nabla_X(e_j)) - \nabla_{[X,Y]}(e_j)$

$d\eta^i(e_j) = \sum_j d\omega^j_i e_j = \sum_j (L_X \omega^j_i) e_j$

$\Rightarrow k^i_j(X,Y) = d\omega^j_i(X,Y) + \sum_k \omega^k_j(X) \omega^i_k(Y)$



Very-very-very-very interesting consequence / remark:  
How does  $\text{Tr}(k) = k_1^1 + k_2^2 + \dots + k_r^r \in \mathcal{S}^2(U)$   
change? So....?



$$\begin{aligned} \text{Tr}(k) - \text{Tr}(k) &= \text{Tr}(d(w+\alpha) + (w+\alpha)\wedge(w+\alpha) - d\bar{w} - \bar{w}\wedge\bar{w}) \\ &= \text{Tr}(d\alpha) + \text{Tr}(\cancel{w\wedge\alpha} + \cancel{\alpha\wedge w} + \alpha\wedge\alpha) \\ &= d(\text{Tr}(\alpha)) \quad \square \end{aligned}$$

The simula argument /  $\mathbb{C} \Rightarrow \text{Tr}(w)$  is pure  
 $\Rightarrow \text{Tr}(w) = 0 \Rightarrow c$   
 $\Rightarrow \text{Tr}(w)$  is pure  
 $\Rightarrow \text{Tr}(w)$  is pure  
 $\Rightarrow \text{Tr}(w)$  is pure

Very-very-very-very interesting consequence/remark:  
 How does  $\text{Tr}(k) = k^1 + k^2 + \dots + k^r \in \mathcal{O}(\mathbb{R}^2(M))$   
 change? So...?

Does not change:

$$\text{Tr}(k') = \text{Tr}(g^{-1} k g) = \text{Tr}(k g g^{-1}) = \text{Tr}(k)!$$

$$\text{Tr}(GAG^{-1}) = \text{Tr}(A)$$

$$\text{Tr}(A) = \text{Tr}(A^2)$$

$\mathbb{R}^2$  For any v.space  $V \Rightarrow$  a canonical map  $\text{Tr}: \text{End}(V) \rightarrow \mathbb{R}$  : using a basis you take traces of matrices  
 $\Rightarrow$  for any vector bundle  $E/M$  one has  $\text{Tr}: \text{End}(E) \rightarrow M \times \mathbb{R}$  morphism of vector bundles  
 $K_{\nabla} \in \mathcal{O}^2(M, \text{End } E) \xrightarrow{\text{Tr}} \text{Tr}(K_{\nabla}) \in \mathcal{O}^2(M)$   
 this does not depend on the basis

$$\begin{aligned} \text{Tr}(AB) &= \text{Tr}(BA) \\ \sum_i (AB)_i^i &= \sum_i (BA)_i^i \\ \sum_i A_i^j B_j^i &= \sum_i B_i^j A_j^i \end{aligned}$$



$\Rightarrow K_\nabla$  over  $U$  is completely encoded in  $\{k_j \cdot e_i\} \in \text{End}(E)$   
 $k = (k_{ij})_{1 \leq i, j \leq r} \in M_r(\mathcal{O}^2(U))$  the curvature matrix of  $\nabla$  w.r.t.  $e$ .

Lemma:  $k = dw + w \wedge w$

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Changing frames  $e \rightsquigarrow e'$ : encoded in matrix  $(g_{ij})_{i,j} \in M_r(\mathcal{O}^0(U \cup U'))$ :  $e'_i = \sum_p g_{ip} e_p$  (over  $U \cap U'$ )

$\Rightarrow w$  and  $k$  are changed to

$w' = g^{-1} dg + g^{-1} w g$ ,  $k' = g^{-1} k g$

$\Rightarrow$  we get a well defined, global 2-form on  $M$ :  
 $\text{Tr}(K_\nabla) \in \mathcal{O}^2(M)$

uniquely determined by:  $(\forall)$  frame  $e$  over  $U$  as above:

$\text{Tr}(K_\nabla)|_U = \text{Tr}(k(\nabla, e))$

Ver  
 Ho  
 ch  
 Doe  
 Tr  
 $J(A) =$

$R_k$  For  
 $\Rightarrow$  for a  
 $K_\nabla \in$



Thm:

(a)  $\text{Tr}(K_\nabla) \in \Omega^2(M)$  is a closed 2-form

(b)  $[\text{Tr}(K_\nabla)] \in H^2(M)$  does not depend on  $\nabla$ !

(c) When working over  $\mathbb{R}$  ... this is always 0

(d) When working over  $\mathbb{C}$ :

$$c_1(E) := \left[ \frac{1}{2\pi i} \text{Tr}(K_\nabla) \right] \in H^2(M; \mathbb{R}) \quad (\text{real coefficients!})$$

the first Chern class of  $E$ .

Rk: If  $c_1(E) \neq 0 \Rightarrow E$  cannot be isomorphic to a trivial vector bundle

$$C^{\infty}(M) \text{-mod } \Omega^2(M)$$

$$\lambda \cdot [w] = [\lambda w]$$
~~$$f \cdot [w] = [fw]$$~~

$$\left\{ \frac{\omega \in \Omega^2(M) : d\omega = 0}{\omega \in \Omega^2(M) : \omega = d\eta \text{ for some } \eta \in \Omega^1(M)} \right\} = \{ [\omega] / \omega \in \Omega^2(M) \text{ closed} \}$$

$$[\omega] = [\omega'] \Leftrightarrow$$

$$\Leftrightarrow \omega' - \omega \text{ is exact} \\ (\omega' - \omega = d\eta \text{ for some } \eta)$$



can easily be fixed  
by using other  
replacement of Tr.

metric  $g$ . Choose  $g$   
compatible with  $g$ , i.e.  
 $\langle \nabla_X S, S' \rangle + g(S, \nabla_X S') = 0$

normal frames  
 $\langle e_i, e_j \rangle = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) = 0$

$\Rightarrow \text{Tr}(w) = 0$

traces of matrices  
of vector bundles



$\mathcal{X}(M) \times \Gamma E \rightarrow \Gamma E$  ( $E = \text{real / complex}$   
v. bundle /  $M$ )

$\{e_1, \dots, e_r\}$  encoded in the  
 $\omega(\nabla, e) = (\omega_{ij}^k)_{i,j \in \mathcal{I}} \in \mathcal{M}_r(\Omega^1(U))$

$\nabla \in \Omega^2(M, \text{End } E)$ , defined by  
 $(\nabla_X(s) - \nabla_{[X,Y]}(s)) \in \Gamma E$  (for  $X, Y \in \mathcal{X}(M), s \in \Gamma E$ )

$K_\nabla = \sum_j k_j \cdot e_i^{\otimes 2}$   
 $\in \text{End}(E)$   
 $\text{rest to } 0$   
 $(\Omega^2(U))$  the curvature matrix  
of  $\nabla$  w.r.t.  $e$ .

proofs

(a) Locally:  $k = d\omega + \omega \wedge \omega \Rightarrow dk = d\omega \wedge \omega + \omega \wedge d\omega$

$$d(\text{Tr}(k)) = \text{Tr}(dk) = \text{Tr}\left(\underbrace{d\omega}_{\frac{d\omega}{z}} \cdot \underbrace{\omega}_{\frac{\omega}{i}}\right) - \text{Tr}(\omega \cdot d\omega) = 0$$

(b) Assume we have two conn.  $\nabla, \bar{\nabla}$  on  $E$

Locally  $\begin{cases} \omega = \omega(\nabla, e) \\ \bar{\omega} = \omega(\bar{\nabla}, e) \end{cases}$

Look at  $\alpha := \bar{\omega} - \omega \in \mathcal{M}_r(\Omega^1(U))$

If we use another frame  $\Rightarrow$

$$\begin{aligned} \Rightarrow \alpha' &= \bar{\omega}' - \omega' = (g^{-1} \bar{\omega} g + g^{-1} \bar{\omega} g) - (g^{-1} \omega g + g^{-1} \omega g) \\ \Rightarrow \alpha' &= g^{-1} \alpha g \end{aligned}$$

$\Rightarrow$  we have a global  $\text{Tr}(\alpha) \in \Omega^1(M)$ .

$$\begin{aligned} \text{Tr}(\bar{k}) - \text{Tr}(k) &= \text{Tr}\left(\underbrace{d(\omega + \alpha)}_{d\bar{\omega} + \bar{\omega} \wedge \bar{\omega}} + \underbrace{(\omega + \alpha) \wedge (\omega + \alpha)}_{d\omega + \omega \wedge \omega}\right) - \text{Tr}(d\omega + \omega \wedge \omega) \\ &= \text{Tr}(d\alpha) + \text{Tr}(\cancel{\omega \wedge \alpha} + \cancel{\alpha \wedge \omega} + \alpha \wedge \alpha) = d(\text{Tr}(\alpha)) \quad \square \end{aligned}$$

(c)  $E$  admits

Look for conn.  $L_X$

Given  $g \Rightarrow$   
If we use  $\Rightarrow L_X$

The siml

-3-

-4-

Very-very-very-very interesting consequence / remark:  
How does  $\text{Tr}(k) = k_1^1 + k_2^2 + \dots + k_r^r \in \Omega^2(U)$



Thm:

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$$\lambda \cdot [w] = [\lambda w]$$
~~$$f[w] = [fw]$$~~

$$\frac{\{w \in \Omega^2(M) : dw=0\}}{\{w \in \Omega^2(M) : [w]=d\eta \text{ for some } \eta \in \Omega^1(M)\}} = \{[w] / w \in \Omega^2(M) \text{ closed}\}$$

$$[w] = [w'] \Leftrightarrow w' - w \text{ is exact} \\ (w' - w = d\eta \text{ for some } \eta)$$

$$H^2(M; \mathbb{C}) = H^2(M; \mathbb{R}) + i H^2(M; \mathbb{R})$$

by using other replacements of Tr.

metric  $g$ . Choose  $g$   
no compatible with  $g$ , i.e.  
 $\langle \nabla_X S, S' \rangle + g(S, \nabla_X S') = 0$

$\nabla$  do exist  
normal frames  
 $\langle e_i, e_j \rangle = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) = 0$

$\Rightarrow [w]$  is  
 $\Rightarrow \text{Tr}(w) = 0$

mat /  $\mathbb{C} \Rightarrow \text{Tr}$

the traces of matrices of vector bundles