Reminder on connections [-1-] $\nabla : \mathfrak{X}(M)\times \Gamma E \longrightarrow \Gamma E$ (E = real/complex v. bdle $/M$)

- any E admits one (at least)
- locally, w.r.t. local frame $e=\{e_1,\dots,e_r\}$: encoded in the **connection matrix** $\omega=\omega(\nabla,e)=(\omega^i_j)_{1\le i,j\le r}\in \mathcal{M}_r(\Omega^1(U))$

$$\begin{pmatrix}\omega^1_1 & \cdots & \omega^1_r\\ \cdots & & \cdots \\ \omega^r_1 & \cdots & \omega^r_r\end{pmatrix}$$

defined by: $\boxed{\nabla_X(e_j)=\sum_{i=1}^{r} \omega^i_j(X)\, e_i}$

- **the curvature of ∇**: $K_\nabla \in \Omega^2(M,\mathrm{End}\,E)$, defined by

$$\boxed{K_\nabla(X,Y)(s)=\nabla_X(\nabla_Y(s))-\nabla_Y(\nabla_X(s))-\nabla_{[X,Y]}(s)}\in\Gamma E \quad (\text{for } X,Y\in\mathfrak{X}(M),\ s\in\Gamma E)$$

$C^\infty(M)$-linear in X, Y, s.

Locally we can write

$$\boxed{K_\nabla(X,Y)(e_j)=\sum_i \underbrace{k^i_j(X,Y)}_{\in C^\infty(U)}\, e_i} \quad \left(\Leftrightarrow K_\nabla=\sum_{i,j} k^i_j\, e^j_i\right)$$

$e^j_i \in \mathrm{End}(E)$, $e_j\mapsto e_i$, rest to 0

$\Rightarrow K_\nabla$ over U is completely encoded in

$k=(k^i_j)_{1\le i,j\le r}\in\mathcal{M}_r(\Omega^2(U))$ the curvature matrix of ∇ w.r.t. e.

$= k(\nabla,e)$

Comput 3

$k' = d\omega' + \omega'\wedge\omega \dots$

$= \cancel{d(g^{-1})\cdot dg} + \dots$

$= \underbrace{g^{-1}(d\omega + \dots}_{k}$

Lemma: $\boxed{k=d\omega+\omega\wedge\omega} = \begin{pmatrix} d\omega^1_1 & \cdots & d\omega^1_r\\ \cdots & & \cdots\end{pmatrix} + \begin{pmatrix}\omega^1_1 & \cdots & \omega^1_r\\ \cdots & & \end{pmatrix}\wedge\begin{pmatrix}\omega^1_1 & \cdots & \omega^1_r\\ & \cdots & \end{pmatrix}$ [-2-]

i.e.: $k^i_j = d\omega^i_j + \sum_l \omega^i_l\wedge\omega^l_j$

Rk: Dual picture: $d_\nabla(e_j)=\sum_i \omega^i_j\, e_i$ 1-form with coeff. in E

$\Rightarrow d^2 \dots$

Very-very-...

How does ...

change ...

$$+\omega'\wedge\omega' = d(\underbrace{g^{-1}dg + g^{-1}\omega g}_{\omega'}) + (\underbrace{g^{-1}dg+g^{-1}\omega g}_{\omega'})\wedge(\underbrace{g^{-1}dg+g^{-1}\omega g}_{\omega'})$$

$$dg + g^{-1}\underbrace{d(dg)}_{0} + d(g^{-1})\omega g + g^{-1}d(\omega)g - g^{-1}\omega\, dg + g^{-1}dg\,\underbrace{g^{-1}dg}_{-g\cdot dg^{-1}} + g^{-1}\underbrace{dg\, g^{-1}}_{-g\cdot dg^{-1}}\omega g + g^{-1}\omega \underbrace{g\, g^{-1}}_{Id} dg + g^{-1}\omega \underbrace{g\, g^{-1}}_{Id}\omega g$$

$$(\underbrace{d\omega+\omega\wedge\omega}_{k})\, g$$

$\eta) = d\omega \cdot \eta + \dots$

Comp 1

pf:

$$K_\nabla(X,Y)(e_j) = \nabla_X(\nabla_Y(e_j)) - \nabla_Y(\nabla_X(e_j)) - \nabla_{[X,Y]}(e_j) =$$

$$= \nabla_X\Big(\sum_l \omega_j^l(Y)e_l\Big) - \nabla_Y\Big(\sum_l \omega_j^l(X)e_l\Big) - \sum_i \omega_j^i([X,Y])\, e_i$$

$$\sum_l \Big(L_X(\omega_j^l(Y))e_l + \omega_j^l(Y)\nabla_X(e_l)\Big) - \sum_l \Big(L_Y(\omega_j^l(X))e_l + \omega_j^l(X)\nabla_Y(e_l)\Big) -$$

$$\sum_i \underbrace{\Big(L_X(\omega_j^i(Y)) - L_Y(\omega_j^i(X)) - \omega_j^i([X,Y])\Big)}_{(d\omega_j^i)(X,Y)} \cdot e_i + \sum_{l,i}\omega_j^l(Y)\,\omega_l^i(X)e_i - \sum_{l,i}\omega_j^l(X)\,\omega_l^i(Y)e_i$$

$$\sum_{l,i}(\omega_l^i\wedge\omega_j^l)(X,Y)\cdot e_i$$

$$\Rightarrow K_j^i(X,Y) = d\omega_j^i(X,Y) + \sum_l(\omega_l^i\wedge\omega_j^l)(X,Y) \quad \blacksquare$$

$d_\nabla^2(e_j) =$

$= \sum_i d\omega_j^i \cdot e_i - \sum_l \omega_j^l \wedge \underbrace{d_\nabla(e_l)}_{\sum_i \omega_l^i e_i}$

$= \sum_i(\underbrace{d\omega_j^i + \sum_l \omega_l^i\wedge\omega_j^l}) e_i$

$= \sum_i K_j^i e_i$

y-very-v…

…oes Tr(…

…e?

$$K_\nabla(X,Y)(e_j)=\sum_i \underbrace{k^i_j(X,Y)}_{\in C^\infty(U)}\, e_i \quad (\leftrightarrow K_\nabla=\sum_{i,j} k^i_j\cdot e^j_i)$$

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-2-

Lemma: $\boxed{k=d\omega+\omega\wedge\omega}=\begin{pmatrix} d\omega^1_1 & \dots & d\omega^1_r\\ \dots & & \dots\end{pmatrix}+\begin{pmatrix}\omega^1_1 & \dots & \omega^1_r\\ & \dots & \end{pmatrix}\wedge\begin{pmatrix}\omega^1_1 & \dots & \omega^1_r\\ & \dots & \end{pmatrix}$

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$\Rightarrow d^2_\nabla(e_j)=\sum_i k^i_j\, e_i = K\wedge e_j \Rightarrow d^2_\nabla=$ wedging with K_∇.

Changing frames: $e \rightsquigarrow$ new one e'

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$(\Leftrightarrow e_p=\sum_j (g^{-1})^j_p\, e'_j)$

$\Rightarrow \boxed{\omega'=g^{-1}dg+g^{-1}\omega g.}$

$\Rightarrow \boxed{k'=g^{-1}k\cdot g.}$

$$d(g^{-1})\omega g + g^{-1}d(\omega)g - g^{-1}\omega\, dg + g^{-1}\underbrace{dg\, g^{-1}}_{-g\cdot dg^{-1}}dg + g^{-1}\underbrace{dg\, g^{-1}}_{-g\cdot dg^{-1}}\omega g + g^{-1}\omega \underbrace{g\, g^{-1}}_{Id} dg + g^{-1}\omega \underbrace{g\, g^{-1}}_{Id}\omega g$$

Comput 2

$$d_\nabla(e'_i) = d_\nabla\Big(\sum_p g_i^p e_p\Big) = \sum_p dg_i^p \cdot e_p + \sum_p g_i^p \underbrace{d_\nabla(e_p)}_{\sum_q \omega_p^q e_q} =$$

$$= \sum_q \Big(dg_i^q + \sum_p g_i^p \omega_p^q\Big)\underbrace{e_q}_{\sum_j (g^{-1})_q^j e'_j}$$

$$= \sum_{q,j}\Big(dg_i^q \cdot (g^{-1})_q^j + \sum_p g_i^p \omega_p^q (g^{-1})_q^j\Big) e'_j$$

$$(\omega')_i^j = \underbrace{\sum_q (g^{-1})_q^j (dg)_j^q}_{(g^{-1}\cdot dg)_i^j}$$

$$\sum_{q,p} g_i^p \omega_p^q (g^{-1})_q^j = (g^{-1}\omega g)_i^j$$

= real /complex)
v.bdle /M

$\Omega^1(U)$

d by

$\in \mathfrak{X}(M), s \in \Gamma E$

End(E)
rest to 0
atrix

Compat 3:

$$k' = d\omega' + \omega' \wedge \omega' = d\big(\underbrace{g^{-1}dg + g^{-1}\omega g}_{\omega'}\big) + \big(\underbrace{g^{-1}dg + g^{-1}\omega g}_{\omega'}\big) \wedge \big(\underbrace{g^{-1}dg + g^{-1}\omega g}_{\omega'}\big) =$$

$$g \cdot g^{-1} = Id = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$dg \cdot g^{-1} + g \cdot d(g^{-1}) = 0$$

$$= \cancel{d(g^{-1})\, dg} + g^{-1} \underbrace{d(dg)}_{0} + d(g^{-1})\omega g + g^{-1} d(\omega) g - g^{-1}\omega\, dg + \cancel{g^{-1} \underbrace{dg\, g^{-1}}_{-g \cdot dg^{-1}} dg} + g^{-1} \underbrace{dg\, g^{-1}}_{-g \cdot dg^{-1}} \omega g + g^{-1}\omega \underbrace{g\, g^{-1}}_{Id} dg + g^{-1}\omega \underbrace{g\, g^{-1}}_{Id} \omega g$$

$$= g^{-1} \big(\underbrace{d\omega + \omega \wedge \omega}_{k}\big) g$$

$$d(\omega \eta) = d\omega \cdot \eta + (-1)^k \omega \cdot d\eta$$

Very-very-very-very interesting consequence/remark:

How does $Tr(k) = k_1^1 + k_2^2 + \ldots + k_r^r \in \Omega^2(U)$ change? So.... ?

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i.e. $k^i_j = d\omega^i_j + \sum_l \omega^i_l \wedge \omega^l_j$

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$\left(\Rightarrow e_p = \sum_j (g^{-1})^j_p\, e'_j\right)$

$\Rightarrow \boxed{\omega' = g^{-1} dg + g^{-1} \omega g}$

$\Rightarrow \boxed{k' = g^{-1} k \cdot g}$

Very-very-very-very interesting consequence/remark

How does $\mathrm{Tr}(k) = k^1_1 + k^2_2 + \dots + k^r_r \in \Omega^2(U)$ change? So... ?

Comput 3

$k' = d\omega' + \omega' \wedge \omega' = d\left(g^{-1}dg + g^{-1}\omega g\right) + (\underbrace{g^{-1}dg + g^{-1}\omega g}_{\omega'}) \wedge (\underbrace{g^{-1}dg + g^{-1}\omega g}_{\omega'}) =$

$= \cancel{d(g^{-1})\, dg} + g^{-1} \underbrace{d(dg)}_{0} + \cancel{d(g^{-1})\, \omega g} + g^{-1} d(\omega) g - \cancel{g^{-1} \omega\, dg} + \cancel{g^{-1} \underbrace{dg\, g^{-1}}_{-g\, dg^{-1}} dg} + g^{-1} \underbrace{dg\, g^{-1}}_{-g\, dg^{-1}} \omega g + g^{-1} \omega \underbrace{g\, g^{-1}}_{\mathrm{Id}} dg + g^{-1} \omega \underbrace{g\, g^{-1}}_{\mathrm{Id}} \omega g$

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Pf: $K_\nabla(X,Y)(e_j) = \nabla_X(\nabla_Y(e_j \dots$

$d^2_\nabla(e_j) =$

$= \sum_i d\omega^i_j \cdot e_i -$

$- \sum_l \omega^l_j \wedge \underbrace{d_\nabla(e_l)}_{\sum_i \omega^i_l e_i}$

$= \sum_i \left(d\omega^i_j + \sum_l \omega^i_l \wedge \omega^l_j\right) e_i$

$= \sum_i k^i_j\, e_i$

$= \nabla_X\left(\sum_l \omega \dots\right.$

$\sum_l \left(L_X(\omega^l_j(Y \dots\right.$

$\sum \left(L_X(\omega^i_j(Y) \dots\right.$

$(d\omega^i_j) \dots$

$\Rightarrow k^i_j(X,Y) = d\omega^i_j(X,Y) + \sum_l (\omega^i_l \dots$

Very-very-very-very interesting consequence/remark:

How does $Tr(k) = k^1_1 + k^2_2 + \dots + k^r_r \in \Omega^2(U)$

change? So.... ?

$\ldots(\alpha) \in \Omega^2(M).$

$$\underset{d\tilde{\omega}+\tilde{\omega}\wedge\tilde{\omega}}{\mathrm{Tr}(\tilde{k})} - \underset{d\omega+\omega\wedge\omega}{\mathrm{Tr}(k)} = \mathrm{Tr}\big(d(\omega+\alpha) + (\omega+\alpha)\wedge(\omega+\alpha) - d\omega - \omega\wedge\omega\big) =$$

$$= \mathrm{Tr}(d\alpha) + \mathrm{Tr}(\cancel{\omega\wedge\alpha} + \cancel{\alpha\wedge\omega} + \alpha\wedge\alpha)$$

$$= d(\mathrm{Tr}(\alpha)) \quad \boxtimes$$

$\Rightarrow [\omega]$ is skew symmet…

$\Rightarrow \mathrm{Tr}(\omega) = 0 \Rightarrow$ c).

The similar argument $/\mathbb{C} \Rightarrow \mathrm{Tr}(\omega)$ is pur… i.e. d)

-4-

Very-very-very-very interesting consequence/remark:

How does $\mathrm{Tr}(k) = k^1_1 + k^2_2 + \ldots + k^r_r \in \ldots(\Omega^2(U))$ change? So.... ?

Does not change:

$$\mathrm{Tr}(k') = \mathrm{Tr}(g^{-1} k g) = \mathrm{Tr}(k g \cdot g^{-1}) = \mathrm{Tr}(k)!$$

$$J(GAG^{-1}) = J(A)$$

$$J(A) = \mathrm{Tr}(A^2)$$

$\mathrm{Tr}(A\cdot B) = \mathrm{Tr}(B\cdot A)$: $\sum (AB)^i_i = \sum A^i_l B^l_i$

$V^* \otimes V$

Rk: For any v.space $V \Rightarrow$ a canonical map $\mathrm{Tr}: \mathrm{End}(V) \to \mathbb{R}$: using a basis you take traces of matrices | this does not depend on the basis

For any vector bundle E/M one has $\mathrm{Tr}: \mathrm{End}(E) \to M\times\mathbb{R}$ morphism of vector bundles

$\Rightarrow$ for any vector …

$$K_\nabla \in \Omega^2(M, \mathrm{End}\,E) \xrightarrow{\mathrm{Tr}} \mathrm{Tr}(K_\nabla) \in \Omega^2(M)$$

$\Rightarrow K_\nabla$ over U is completely encoded in

$$\underset{k(\nabla,e)}{k} = (k^i_j)_{1\le i,j\le r} \in \mathcal{M}_r(\Omega^2(U))$$ the curvature matrix of ∇ w.r.t. e.

Lemma: $\boxed{k = d\omega + \omega\wedge\omega}$

— 3 —

Changing frames $e \rightsquigarrow e'$: encoded in matrix

$(g^i_j)_{i,j} \in \mathcal{M}_r(C^\infty(U\cap U'))$: $e'_i = \sum_p g^p_i e_p$ (over $U\cap U'$)

$\Rightarrow$ ω and k are changed to

$$\boxed{\omega' = g^{-1}dg + g^{-1}\omega g}, \quad \boxed{k' = g^{-1}k\,g}$$

$\Rightarrow$ we get a well defined, global 2-form on M:

$$\text{“}Tr(K_\nabla)\text{”} \in \Omega^2(M)$$

uniquely determined by: $(\forall)$ frame e over U as above:

$$Tr(K_\nabla)\big|_U = Tr(k(\nabla,e))$$

Thm:

(a) $\mathrm{Tr}(K_\nabla) \in \Omega^2(M)$ is a closed 2-form

(b) $[\mathrm{Tr}(K_\nabla)] \in H^2(M)$ does not depend on ∇ ! !

$$\frac{\{\omega \in \Omega^2(M) : d\omega = 0\}}{\{\omega \in \Omega^2(M) : \boxed{\omega = d\eta} \text{ for some } \eta \in \Omega^1(M)\}} = \left\{ [\omega] \,/\, \omega \in \Omega^2(M) \text{ closed} \right\}$$

$[\omega] = [\omega'] \iff$

$\iff \omega' - \omega$ is exact

$(\omega' - \omega = d\eta$ for some $\eta)$

(c) When working over $\mathbb{R}$ ... this is always 0 ⟵ can easily be fixed by using other replacement of Tr.

$H^2(M,\mathbb{C}) = H^2(M,\mathbb{R}) + i\, H^2(M,\mathbb{R})$

(d) When working over $\mathbb{C}$:

$$\underbrace{c_1(E)}_{} := \left[\left(\frac{1}{2\pi i} \right) \mathrm{Tr}(K_\nabla) \right] \in H^2(M;\mathbb{R}) \quad \text{(real coefficients!)}$$

the first Chern class of E.

$\Sigma \subseteq M$, $\int_\Sigma \omega$

Rk: If $c_1(E) \neq 0 \Rightarrow E$ cannot be isomorphic to a trivial vector bundle

$\lambda \cdot [\omega] = [\lambda \omega]$

$\cancel{f[\omega] = [f\omega]}$

$C^\infty(M)$-linear $\Omega^2(M)$

$\mathfrak{X}(M)\times\Gamma E \longrightarrow \Gamma E$ (E = real/complex v.bdle/M)

st)

$\{e_1,\dots,e_r\}$: encoded in the

$\omega(\nabla,e) = (\omega^i_j)_{1\le i,j\le r} \in \mathcal{M}_r(\Omega^1(U))$

$\begin{pmatrix} \omega^1_1 & \dots & \omega^1_r \\ \dots & & \dots \\ \omega^r_1 & \dots & \omega^r_r \end{pmatrix}$

$\boxed{(X)\, e_i}$

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$(\Omega^2(U))$ the curvature matrix of ∇ w.r.t. e.

$\boxed{-5-}$

<u>proofs</u>:

(a) Locally: $k = d\omega + \omega\wedge\omega \Rightarrow dk = d\omega\wedge\omega - \omega\wedge d\omega$

$d(\mathrm{Tr}(k)) = \mathrm{Tr}(dk) = \mathrm{Tr}(\underbrace{d\omega}_{2}\cdot\underbrace{\omega}_{1}) - \mathrm{Tr}(\omega\cdot d\omega) = 0$

(b) Assume we have two conn. $\nabla, \bar\nabla$ on E

Locally: $\begin{cases} \omega = \omega(\nabla, e) \\ \bar\omega = \omega(\bar\nabla, e) \end{cases}$

Look at $\alpha := \bar\omega - \omega \in \mathcal{M}_r(\Omega^1(U))$

If we use another frame $\Rightarrow$

$\Rightarrow \alpha' = \bar\omega' - \omega' = (\cancel{g^{-1}dg} + g^{-1}\bar\omega g) - (\cancel{g^{-1}dg} + g^{-1}\omega g)$

$\Rightarrow \alpha' = g^{-1}\alpha g$.

$\Rightarrow$ we have a global $\underline{\mathrm{Tr}(\alpha) \in \Omega^1(M)}$.

$\underbrace{\mathrm{Tr}(\bar k)}_{d\bar\omega+\bar\omega\wedge\bar\omega} - \underbrace{\mathrm{Tr}(k)}_{d\omega+\omega\wedge\omega} = \mathrm{Tr}\big(d(\omega+\alpha) + (\omega+\alpha)\wedge(\omega+\alpha) - d\omega - \omega\wedge\omega\big) =$

$= \mathrm{Tr}(d\alpha) + \mathrm{Tr}(\cancel{\omega\wedge\alpha} + \cancel{\alpha\wedge\omega} + \alpha\wedge\alpha) \qquad = d(\mathrm{Tr}(\alpha))$ ∎

(c) E admit

Look for conn

$L_X($

Given $g \Rightarrow$

If we use

$\Rightarrow L_X$

The sim

$\boxed{-3-}$

$\boxed{-4-}$

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$C^\infty(M)$-module $\Omega^2(M)$

$\omega + i\eta$