—1—

Reminder: Given connection $\nabla : \mathfrak{X}(M)\times \Gamma E \longrightarrow \Gamma E$

we have the curvature of ∇: $K_\nabla(X,Y)(s)=\nabla_X(\nabla_Y s)-\nabla_Y(\nabla_X s)-\nabla_{[X,Y]}(s)$

Locally, w.r.t. $e=\{e_1,\dots,e_r\}$ over $\mathcal{U}$:

connection matrix

$\omega=\omega(\nabla,e)\in \mathcal{M}_r(\Omega^1(U))$

$\boxed{\nabla_X(e_j)=\sum_i \omega^i_j(X)\, e}$

curvature matrix

$k=k(\nabla,e)\in \mathcal{M}_r(\Omega^2(U))$

$\boxed{K_\nabla(\cdot,\cdot)(e_j)=\sum k^i_j(\cdot,\cdot)\, e_i}$

$\boxed{k=d\omega+\omega\wedge\omega}$

if CHANGE FRAME

$e'=\{e'_1,\dots,e'_r\}$; CHANGE encoded in

$g=g(e,e')\in \mathcal{M}_r(C^\infty)$ (defined by $e'_p=\sum_i g^i_p\, e_i$)

$\Rightarrow$ $\boxed{\omega'=g^{-1}\omega g+g^{-1}dg}$

$\boxed{k'=g^{-1}k\, g}$ $\Longrightarrow$ can talk about $Tr(K_\nabla)\in\Omega^2(M)$ GLOBALLY!

$K_\nabla\in\Omega^2$

Frame e

$\Rightarrow$ ω'

$\Rightarrow$ Tr

For E^*

∇^*_X (

Frame

∇^*_X (

∇^*_X (

Rk: $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, vector bdles $E/\mathbb{F}$ —5—

Instead of Tr: use any map

$P: \mathcal{M}_r(\mathbb{F})\longrightarrow\mathbb{F}$

which is polynomial

$\Downarrow$

new

$\Downarrow$

the

$$\begin{Bmatrix} f^*: Vb(M) \to Vb(N) \\ f^*: H^*(M) \to H^*(N) \end{Bmatrix} \qquad c_1(f^*E) = f^*(c_1(E))$$

$$\begin{array}{ccc} Vb(M) & \xrightarrow{c_1} & H^2(M) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ Vb(N) & \xrightarrow[c_1]{} & H^2(N) \end{array}$$

Thm: Given E (over $\mathbb{R}$ or $\mathbb{C}$)

(a) $\forall \nabla$ on $E \Rightarrow Tr(K_\nabla) \in \Omega^2(M)$ is a closed form

(b) $[Tr(K_\nabla)] \in H^2(M)$ only depends on E and not on the ∇ used.

(c) over $\mathbb{R}$, this is 0

(d) over $\mathbb{C}$, (i.e. for complex vector bundles)

$$c_1(E) = \left[\frac{1}{2\pi i} Tr(K_\nabla)\right] \in H^2(M, \mathbb{R})$$

... called the 1st chern class of E

$(a+ib)\cdot u := (a-ib)u$

$\bar{E} \simeq E^*$ $\quad E \xrightarrow{\simeq} E^*$, $\quad u \mapsto h(\cdot, u)$

More properties:

A • E is trivializable $\Rightarrow$ E admits a flat connection $\Rightarrow$ $c_1(E) = 0$.

B • $E \simeq E'$ as v.b./M $\Rightarrow c_1(E) = c_1(E')$

C • $c_1(E \oplus E') = c_1(E) + c_1(E')$, $c_1(E \otimes E') = rk(E) \cdot c_1(E') + rk(E') c_1(E)$,

D • $f: N \to M$,

$$c_1(\bar{E}) = -c_1(E)$$

$$c_1(E^*) = -c_1(E)$$

—1—

$\nabla_{[X,Y]}(S)$

$K_\nabla \in \Omega^2(M, \underbrace{\mathrm{End}\,E})$

$\downarrow \mathrm{Tr}$

$\Omega^2(M, \text{trivial line bdle})$

$\| $

$\Omega^2(M)$

—3—

proofs: A, B ✓

C) $\left.\begin{matrix}\nabla \text{ conn on } E\\ \nabla' \text{ conn on } E'\end{matrix}\right\} \rightsquigarrow \nabla\oplus\nabla'$ on $E\oplus E'$

$\nabla\oplus\nabla' = \nabla''$

$\nabla''_X(s, s') = (\nabla_X(s), \nabla'_X(s'))$ $\forall$ $\underset{(s,s')}{s''} \in \Gamma(E\oplus E')$

Frame $\left.\begin{matrix} e = \{e_1, \dots, e_r\} \text{ of } E\\ e' = \{e'_1, \dots, e'_{r'}\} \text{ of } E'\end{matrix}\right\} \rightsquigarrow$ frame $e'' = \{e_1, \dots, e_r, e'_1, \dots, e'_{r'}\}$ of $E\oplus E'$

$\Rightarrow \omega'' = \begin{pmatrix}\omega & 0\\ 0 & \omega'\end{pmatrix} \in M_{r+r'}(\Omega^1(U))$, $k'' = \begin{pmatrix} k & 0 \\ 0 & k'\end{pmatrix}$

$\Rightarrow \mathrm{Tr}(k'') = \mathrm{Tr}(k) + \mathrm{Tr}(k') \Longrightarrow c_1(E\oplus E') = c_1(E) + c_1(E')$ ☒

For E^*: start with $\nabla \Rightarrow$ a connection ∇^* on E^*

$$\nabla^*_X(e^*)(e) = L_X(e^*(e)) - e^*(\nabla_X(e))$$

$\langle \nabla^*_X(e^*), e\rangle + \langle e^*, \nabla_X(e)\rangle = L_X\langle e^*, e\rangle$

Frame $e = \{e_1, \dots, e_r\}$ of $E \Rightarrow$ the dual frame $\{e^1, \dots, e^r\}$ $e^i(e_j) = \delta_{ij}$.

$\nabla^*_X(e^j)(e_p) = 0 - e^j\left(\sum_q \omega^q_p(X) e_q\right) = -\omega^j_p(X)$

$\nabla^*_X(e^j) = -\sum_p \omega^j_p(X) e^p \Longrightarrow (\omega^*)^p_j = -\omega^j_p \Rightarrow \omega^* = -\omega^T \Rightarrow k^* = -k^T \Rightarrow \mathrm{Tr}(K_{\nabla^*}) = -$

Over $\mathbb{R} \Rightarrow$ [Tr

But for real v.bdles

$\Rightarrow$ [Tr

Ex: L over
tautological

D) Similar: a
gives ris

Use Ch or c: depends o

—3—

$\rightsquigarrow \nabla \oplus \nabla'$ on $E \oplus E'$
$\quad = \nabla''$

$s), \nabla'_X(s'))$ $(\forall)$ $s'' \in \Gamma(E \oplus E')$ $= (s, s')$

$r, e'_1, \dots, e'_{r'}\}$ of $E \oplus E'$

$\begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}$

$E \oplus E') = c_1(E) + c_1(E')$ ☒

∇^* on E^*

$X(e)$ $\langle \nabla^*_X(e^*), e\rangle + \langle e^*, \nabla_X(e)\rangle = L_X(\langle e^*, e\rangle)$

frame $\{e^1, \dots, e^r\}$ $e^i(e_j) = \delta_{ij}$

$\omega^j_p(X)$

$\omega^j_p \Rightarrow \omega^* = -\omega^T \Rightarrow k^* = -k^T \Rightarrow \mathrm{Tr}(K_{\nabla^*}) = -\mathrm{Tr}(K_\nabla)$

—4—

Over $\mathbb{R}$ $\Rightarrow$ $[\mathrm{Tr}(K_\nabla)]_{E^*} = -[\mathrm{Tr}(K_\nabla)]_E$

But for real v.bdles $E^* \simeq E$
$\Rightarrow [\mathrm{Tr}(K_\nabla)] = -[\mathrm{Tr}(K_\nabla)] \Rightarrow [\mathrm{Tr}(K_\nabla)] = 0$! (i.e. c).

Ex: L tautological over $\mathbb{CP}^1 \Rightarrow \bar{L}$ cannot be isomorphic to L.

D) Similar: any ∇ on E $\quad N \xrightarrow{f} M \ni E$
gives rise to $f^*\nabla$ – a connection on f^*E

Thm: Given E (
(a) $\forall \nabla$ on E
(b) $[\mathrm{Tr}(K_\nabla)] \in H^2($
(c) over $\mathbb{R}$, this i
(d) over $\mathbb{C}$ (i.e. for
$c_1(E) = \left[\frac{1}{2\pi}\right.$

More properties:
A • : E is triviali
B • $E \simeq E'$ as v.b.
C • $c_1(E \oplus E') = c_1($
D • $f: N \to M$,

—7—

Use Ch or c : depends on the properties that are appropriate to use:
$\Sigma_d\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} = \Sigma_d(A) + \Sigma_d(A') \Rightarrow Ch_p(E \oplus E') = Ch_p(E) + Ch_p(E')$
$\Rightarrow c_d(E \oplus E') = \sum_{p+q=d} c_p(E)\, c_q(E')$

$Tr(K_\nabla) \in \Omega^2(M)$

GLOBALLY!

$k' = g^{-1} k g$

$\nabla^*_X(e^j)$

-5-

<u>Rk</u>: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, vector bdles $E / \mathbb{F}$

Instead of Tr: use any map

$$P : M_r(\mathbb{F}) \longrightarrow \mathbb{F}$$

which is polynomial in entries of the matrix } ⇝ $P(k)$ makes sense
say homogeneous of degree d. } ⟶ and is a form of degree $2d$

Such that: $\boxed{P(g^{-1} A g) = P(A)}$ $(\forall)\ g$ = invertible $(\forall)\ A$

Several possible & interesting choices.

① $P = \Sigma_d : A \longrightarrow Tr(A^d)$ $\quad 0 \le d \le r$

$\Sigma_d : M_r(\mathbb{F}) \to \mathbb{F}$

$\Sigma_d(A) = Tr(A^d)$

→ Any P which is invariant is actually a polynomial combination of $\Sigma_0, \Sigma_1, \dots, \Sigma_r$

$$P(A) = Q(\Sigma_0(A), \Sigma_1(A), \dots, \Sigma_r(A))$$

if $x_1, \dots, x_r$ - eigenvalues of A

$$\Sigma_d(A) = \sum_{i=1}^r x_i^d$$

$\begin{cases} x_1 + \dots + x_r \\ x_1^2 + \dots + x_r^2 \\ x_1^r + \dots + x_r^r \end{cases}$

$\left(\sum_{i<j} x_i x_j = \frac{(\sum x_i)^2 - \sum x_i^2}{2} \right.$

⇓

new inv...

⇓

the Chern...

$Ch_d($...

② $P = \sigma_d$

$\boxed{\det(I + t \dots}$

(or: $\sigma_d(A) = \dots$

Use σ_i

$c_d(E) = \dots$

the d-th Ch...

$C(E) = c_0(\dots$

talk about
$Tr(K_\nabla) \in \Omega^2(M)$
LOBALLY!

$\nabla^*_X(e^j)(e_p) = 0 - e^j\left(\sum_q \omega_p^q(X) e_q\right) = -\omega_p^j(X)$

$\nabla^*_X(e^j) = \sum_p \omega_p^j(X) e^p \implies (\omega^*)_j^p = -\omega_p^j \implies \omega^* = -\omega$

— 5 —

/ $\mathbb{F}$

trix } $\rightsquigarrow P(K)$ makes sense and is a form of degree $2d$

g = invertible $(\forall)$, A

oices.

$\Sigma_d : M_r(\mathbb{F}) \to \mathbb{F}$
$\Sigma_d(A) = Tr(A^d)$

if $x_1, \dots, x_r$ – eigenvalues of A
$\Sigma_d(A) = \sum_{p=1}^{r} x_p^d$

$\begin{cases} x_1 + \dots + x_r \\ x_1^2 + \dots + x_r^2 \\ x_1^r + \dots + x_r^r \end{cases}$ $\left(\sum_{i<j} x_i x_j = \frac{(\sum x_i)^2 - \sum x_i^2}{2}\right.$

— 6 —

⇓
new invariants
⇓
the Chern character classes of E

$$Ch_d(E) \in H^{2d}(E) \qquad 0 \le d \le r.$$

② $P = \sigma_d$ $\qquad \sigma_d : M_r(\mathbb{F}) \to \mathbb{F}$

$$\det(I + tA) = \underset{\sigma_0(A)}{1} + t\,\underset{\sigma_1(A)}{Tr(A)} + t^2 \sigma_2(A) + \dots + t^r \underset{\det(A)}{\sigma_r(A)}$$

$\left(\text{or: } \sigma_d(A) = \sum_{i_1 < \dots < i_d} x_{i_1} \cdots x_{i_d}\right)$

Use σ_i

$$c_d(E) = \left[\left(\frac{1}{2\pi i}\right)^d \sigma_d(K_\nabla)\right] \in H^{2d}(M).$$

the d-th Chern class of E.

$$C(E) = \underset{1}{c_0(E)} + \underset{H^2(M)}{c_1(E)} + c_2(E) + \dots$$ the total Chern class of E

Use Ch
more approp
E

$\det(I$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$x_1 + x_2 = Tr(A) = a + d$
$x_1 x_2 = \det(A) = ad - bc$

frame $\{e^1, \dots, e^r\}$ $\quad e^i(e_j) = \delta_{ij}$.

$\omega^g_p(x) e_g) = -\omega^j_p(x)$

$(\omega^*)^p_j = -\omega^j_p \Rightarrow \omega^* = -\omega^T \Rightarrow k^* = -k^T \Rightarrow Tr(K_{\nabla^*}) = -Tr(K_\nabla)$

-7-

Use Ch or c: depends on the properties that are more appropriate to use:

$E \oplus E'$ $\quad \Sigma_d \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} = \Sigma_d(A) + \Sigma_d(A') \Rightarrow$ $\boxed{Ch_p(E \oplus E') = ch_p(E) + ch_p(E')}$

$\sigma_d \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} = \sum_{p+q=d} \sigma_p(A)\sigma_q(A') \Rightarrow$ $\boxed{c_d(E \oplus E') = \sum_{p+q=d} c_p(E') c_q(E') \quad c(E \oplus E') = c(E)\, c(E')}$

$\det\left(I + \begin{pmatrix} tA & 0 \\ 0 & tA' \end{pmatrix}\right) = \det(I + tA)\det(I + tA')$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$x_1 + x_2 = Tr(A) = a + d$

$x_1 x_2 = \det(A) = ad - bc$

---

of E

$0 \le d \le r$.

$M_r(F) \to F$

$) + t^2 \sigma_2(A) + \dots + t^r \underbrace{\sigma_r(a)}_{\det(A)}$

$(K_\nabla)] \in H^{2d}(M)$.

of E.

$) + c_2(E) + \dots$ the total Chern class of E

–1–

Reminder: Given connection $\nabla: \mathfrak{X}(M)\times\Gamma E \longrightarrow \Gamma E$

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curvature matrix: $k=k(\nabla,e)\in\mathcal{M}_r(\Omega^2(U))$

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if CHANGE FRAME

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$$\boxed{k'=g^{-1}k\,g}$$

$\longrightarrow$ can talk about $\mathrm{Tr}(K_\nabla)\in\Omega^2(M)$ GLOBALLY!

$K_\nabla\in\Omega^2(M,\mathrm{End}\,E)$

$\downarrow \mathrm{Tr}$

$\Omega^2(M$, trivial line bdle$)$

$\parallel$

$\Omega^2(M)$

–3–

proofs A, B ✓

C) $\left.\begin{array}{l}\nabla \text{ conn on } E\\ \nabla' \text{ conn on } E'\end{array}\right\}\rightsquigarrow \nabla\oplus\nabla'$ on $E\oplus E'$ $\;\;(= \nabla'')$

$\nabla''_X(s,s')=(\nabla_X(s),\nabla'_X(s'))\quad \forall\; s''\in\Gamma(E\oplus E')$, $s''=(s,s')$

$\left.\begin{array}{l}\text{Frame } e=\{e_1,\dots,e_r\} \text{ of } E\\ e'=\{e'_1,\dots,e'_{r'}\} \text{ of } E'\end{array}\right\}\rightsquigarrow$ frame $e''=\{e_1,\dots,e_r,e'_1,\dots,e'_{r'}\}$ of $E\oplus E'$

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$$\Rightarrow \mathrm{Tr}(k'')=\mathrm{Tr}(k)+\mathrm{Tr}(k')\Longrightarrow c_1(E\oplus E')=c_1(E)+c_1(E') \;\square$$

For E^*: start with $\nabla\Longrightarrow$ a connection ∇^* on E^*

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–4–

Over $\mathbb{R}\Longrightarrow [\mathrm{Tr}(K_{\nabla^*})]=-[\mathrm{Tr}(K_\nabla)]$ $\quad$ (E^*, E)

But for real v.bdles $E^*\simeq E$

$$\Rightarrow [\mathrm{Tr}(K_\nabla)]=-[\mathrm{Tr}(K_\nabla)]\Rightarrow [\mathrm{Tr}(K_\nabla)]=0\;!$$

(i.e. C)

Ex: L (tautological) over $\mathbb{CP}^1\Longrightarrow \bar{L}$ cannot be isomorphic to L.

D) Similar: any ∇ on E $\qquad N\longrightarrow M$ (with $E\downarrow M$)

gives rise to $f^*\nabla$ – a connection on f^*E

–5–

$\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$, vector bdles $E/\mathbb{F}$

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which is polynomial in entries of the matrix } $\rightsquigarrow P(k)$ makes sense

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such that: $\boxed{P(g^{-1}Ag)=P(A)}$ $(\forall)\,g$ = invertible $(\forall)\,A$

Several possible & interesting choices.

① $P=\Sigma_d$: $A\longrightarrow\mathrm{Tr}(A^d)$, $\;0\le d\le r$

$$\boxed{\begin{array}{l}\Sigma_d:\mathcal{M}_r(\mathbb{F})\to\mathbb{F}\\ \Sigma_d(A)=\mathrm{Tr}(A^d)\end{array}}$$

if $x_1,\dots,x_r$ – eigenvalues of A: $\quad \Sigma_d(A)=\sum_{p=1}^r x_p^d$

$\left\{\begin{array}{l}x_1+\dots+x_r\\ x_1^2+\dots+x_r^2\\ x_1^r+\dots+x_r^r\end{array}\right.$ $\quad\Big(\sum_{i<j}x_ix_j=\dfrac{(\sum x_i)^2-\sum x_i^2}{2}\Big)$

– Any P which is invariant is actually a polynomial combination of $\Sigma_0,\Sigma_1,\dots,\Sigma_r$

$$P(A)=Q(\Sigma_0(A),\Sigma_1(A),\dots,\Sigma_r(A))$$

–6–

$\Downarrow$ new invariants

$\Downarrow$ the Chern character classes of E

$$Ch_d(E)\in H^{2d}(E)\qquad 0\le d\le r$$

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$$\boxed{\det(I+tA)}=\underbrace{1}_{\sigma_0(A)}+t\underbrace{\mathrm{Tr}(A)}_{\sigma_1(A)}+t^2\sigma_2(A)+\dots+t^r\underbrace{\sigma_r(A)}_{\det(A)}$$

(or $\sigma_d(A)=\sum_{i_1<\dots<i_d}x_{i_1}\cdots x_{i_d}$)

Use σ_i

$$c_d(E)=\Big[\Big(\frac{1}{2\pi i}\Big)^d\sigma_d(K_\nabla)\Big]\in H^{2d}(M).$$

the d-th Chern class of E.

$$C(E)=\underbrace{c_0(E)}_{1}+\underbrace{c_1(E)}_{\in H^2(M)}+c_2(E)+\dots\quad\text{the total Chern class of } E$$

–7–

Use Ch or c: depends on the properties that are more appropriate to use

$$E\oplus E'\quad \Sigma_d\begin{pmatrix}A & 0\\ 0 & A'\end{pmatrix}=\Sigma_d(A)+\Sigma_d(A')\Rightarrow\boxed{Ch_p(E\oplus E')=Ch_p(E)+Ch_p(E')}$$

$$\sigma_d\begin{pmatrix}A & 0\\ 0 & A'\end{pmatrix}=\sum_{p+q=d}\sigma_p(A)\sigma_q(A')\Rightarrow\boxed{\begin{array}{l}c_d(E\oplus E')=\sum_{p+q=d}c_p(E)\,c_q(E')\\ c(E\oplus E')=c(E)\,c(E')\end{array}}$$

$$\det\Big(I+\begin{pmatrix}tA & 0\\ 0 & tA'\end{pmatrix}\Big)=\det(I+tA)\det(I+tA')$$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$x_1+x_2=\mathrm{Tr}(A)=a+d$

$x_1x_2=\det(A)=ad-bc$

Locally, ...

connection matrix

$\omega = \omega(\nabla, e) \in M_r(\Omega^1(U))$

$\nabla_X(e_j) = \sum_i \omega^i_j(X)\, e_i$

if CHANGE FRAME

$k = k(\nabla, e) \in M_r(\Omega^2(U))$

$K(\cdot,\cdot)(e_j) = \sum k^i_j(\cdot,\cdot)\, e_i$

$k = d\omega + \omega \wedge \omega$

$e' = \{e'_1, \ldots, e'_r\}$; CHANGE encoded in

$g = g(e, e') \in M_r(C^\infty)$ (defined by $e'_p = \sum_i g^i_p e_i$)

$\Rightarrow \omega' = g^{-1}\omega g + g^{-1} dg$ ... can talk about

$\Rightarrow \omega'' = \ldots$

$\Rightarrow Tr(k'') = Tr(k) + Tr(k')$

$\Rightarrow$ a

For E^*: start with $\nabla \Rightarrow$ a

$\nabla^*_X(e^*)(e) = L_X(e^*(e))$

Frame $e = \{e_1, \ldots, e_r\}$ of $E \Rightarrow$

— 8 —

Metric connections: E vector bundle $/M$, g = inner product on E
∇ = conn on E

① Geometric/global: all parallel transports w.r.t. ∇, along curves $\gamma: I \to M$

$$T^{t_0,t_1}_\gamma : (E_{\gamma(t_0)}, g_{\gamma(t_0)}) \longrightarrow (E_{\gamma(t_1)}, g_{\gamma(t_1)})$$

isometries.

② Algebraic/infinitesimal. $(\forall)\ X \in \mathfrak{X}(M)$, $s, s' \in \Gamma E$

$$L_X(\underbrace{g(s,s')}_{C^\infty(M)}) = g(\nabla_X s, s') + g(s, \nabla_X s')$$

③ $(\forall)$ orthonormal frame (w.r.t. g) $e = \{e_1, \ldots, e_r\}$ of E, the resulting connection matrix:

$$\omega^i_j = -\omega^j_i \quad (\forall)\ i,j$$

Prop: These are equiv. Mo...
$\exists \nabla$ compatible with g

Pf of ② $\Rightarrow$ ③: Apply

$$\underbrace{L_X(\underbrace{g(e_i, e_j)}_{0 \text{ or } 1})}_{0} = g(\nabla_X e \ldots$$

$\sum_p \omega^p_i (\ldots$

$$\frac{d}{dt} g(u, v) = g\left(\frac{\nabla u}{dt} \ldots\right.$$

See: u, v = horizontal $\Rightarrow$ $\frac{d}{d\ldots}$

$u(t) = T^{t_0,t}_\gamma(u_0)$

$v(t) = T^{t_0,t}_\gamma(v_0)$

$Tr(k'') = Tr(k) + Tr(k') \implies c_1(E \oplus E') = c_1(E) + c_1(E')$ ☒

E^*: start with $\nabla \implies$ a connection ∇^* on E^*

$\nabla^*_X(e^*)(e) = L_X(e^*(e)) - e^*(\nabla_X(e))$

$\langle \nabla^*_X(e^*), e\rangle + \langle e^*, \nabla_X(e)\rangle = L_X(\langle e^*, e\rangle)$

frame $e = \{e_1, \dots, e_r\}$ of $E \Rightarrow$ the dual frame $\{e^1, \dots, e^r\}$ $\quad e^i(e_j) = \delta_{ij}$

D) Similar: any ∇ on E $\quad N \to M$ gives rise to $f^*\nabla$ - a connection on

Prop: These are equiv. Moreover $(\forall) E, (\forall) g$ $\exists \nabla$ compatible with g (i.e. ①-③ ✓).

X, Y, Z

$[\nabla_X Y - \nabla_Y X = \dots$

$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = L_X(g(Y, Z))$

Pf of ② ⇒ ③: Apply 2 to $s = e_i$, $s' = e_j$ $(\forall) i, j$

$$L_X(g(e_i, e_j)) = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X(e_j)) = \sum_p \omega_i^p(X)\, g(e_p, e_j) + \sum_q \omega_j^q(X)\, g(e_i, e_q)$$

($g(e_i,e_j)$: 0 or 1, so LHS $= 0$; $\nabla_X e_i = \sum_p \omega_i^p(X) e_p$; $\nabla_X(e_j) = \sum_q \omega_j^q(X) e_q$; $p = j$, $q = i$)

$$0 = \omega_i^j(X) + \omega_j^i(X) \implies \boxed{\omega_i^j = -\omega_j^i}$$

$$\frac{d}{dt} g(u, v) = g\left(\frac{\nabla u}{dt}, v\right) + g\left(u, \frac{\nabla v}{dt}\right) \quad (\forall) u, v : I \to E \text{ sitting above some } \gamma : I \to M.$$

See: u, v = horizontal $\implies \frac{d}{dt} g(u, v) = 0 \implies g(u(t), v(t))$ = constant in t.

$u(t) = T_\gamma^{t_0, t}(u_0)$

$v(t) = T_\gamma^{t_0, t}(v_0)$

$g(u(t_1), v(t_1)) = g(u(t_0), v(t_0))$

$g(T_\gamma^{t_0, t_1}(u_0), T_\gamma^{t_0, t_1}(v_0)) = g(u_0, v_0)$!

∇

$\nabla^A_X(Y) = \nabla \dots$

$T_{\nabla^A}(X, Y) = \nabla^A_X(Y) - \nabla^A_Y(X) - [X, Y]$

$A_X(Y) = \frac{1}{2} T(X, Y)$

$= \nabla_X(Y) + A_X(Y) - \nabla_Y(X) \dots$

$= A_X(Y) - A_Y(X)$

Next: move to Riemannian manifolds (M, g) ← inner product on TM

Novelty when looking at connections ∇ on TM:

$$\nabla : \mathfrak{X}(M) \times \underbrace{\Gamma(TM)}_{\mathfrak{X}(M)} \longrightarrow \underbrace{\Gamma(TM)}_{\mathfrak{X}(M)}$$

$\Longrightarrow$ make sense to talk about the torsion T_∇ of such a connection

$$T_\nabla(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y] \in \mathfrak{X}(M) \qquad X, Y \in \mathfrak{X}(M)$$

$$T_\nabla \in \Omega^2(M, TM)$$

Ex: T_∇ is $C^\infty(M)$-linear in X & Y

Call ∇ torsion free if $T_\nabla = 0$

THM: Given $(M, g) \Rightarrow \exists!$ connection ∇ on TM that is torsion free & compatible with g.

Levi-Civita

$\langle \ldots(e^*), e\rangle + \langle e^*, \nabla_X(e)\rangle = L_X(\langle e^*, e\rangle)$

$f^*\nabla$ - a connection on f^*E

$e^i(e_j) = \delta_{ij}$

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X, Y, Z

$\nabla_X Y - \nabla_Y X = [X, Y]$

$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = L_X(g(Y, Z))$

$= \sum_p \omega_i^p(x)\, g(e_p, e_j) + \sum_q \omega_j^q(x)\, g(e_i, e_q)$

$p = j$, $q = i$

$= \omega_i^j(x) + \omega_j^i(x) \Rightarrow \boxed{\omega_i^j = -\omega_j^i}$

E sitting above some $\gamma: I \to M$

$I \to E$

$\nabla^A_X(Y) = \nabla_X(Y) + A_X(Y)$, $C^\infty(M)$-lin in X, Y, $A \in \Omega^1(M, TM)$

$T_{\nabla^A}(X,Y) = \nabla^A_X(Y) - \nabla^A_Y(X) - [X,Y] = \nabla_X(Y) + A_X(Y) - \nabla_Y(X) - A_Y(X) - [X,Y] = A_X(Y) - A_Y(X)$

$A_X(Y) = \frac{1}{2} T(X,Y)$