–1–

Reminder

→ for general connections ∇ on E:

- local description: $\nabla_X(e_j)=\sum_i \omega_j^i(X)\, e_i$ $\quad$ ($e=\{e_1,\dots,e_r\}$ local frame)
- for any $\begin{cases}\gamma: I\to M\\ u: I\to E \text{ above } \gamma\end{cases}$ we have $\frac{\nabla u}{dt}: I\to E$ (above γ)

  Locally, writing $u(t)=\sum_i u^i(t)\, e_i(\gamma(t)) \Rightarrow \frac{\nabla u}{dt}=\sum_i\left(\frac{du^i}{dt}+\sum_j u_j(t)\,\omega_j^i(\dot\gamma(t))\right)e_i(\gamma(t))$
- compatibility with an inner product g on E:

  $L_X(g(s,s'))=g(\nabla_X s,s')+g(s,\nabla_X s')$ $\quad$ ($\Leftrightarrow$ ... $\Leftrightarrow$ ...)

→ for ∇ on $E=TM$: can also talk about ∇ being torsion free:

$$\nabla_X Y-\nabla_Y X=[X,Y]$$

→ Riemannian manifold: (M,g) with g - inner product on TM

THM: $\exists!\ \nabla$ on TM which is both — COMPATIBLE WITH g — TORSION FREE.

THE LEVI-CIVITA CONNECTION OF (M,g)

$(M,g$

Loca

Funk

$\Longrightarrow$

Check

Geode

Hen

Gain: from the general theory of vector fields: –5–

→ $(\forall)\ (p, v)\in TM$ ($p\in M$, $v\in T_pM$) we have a maximal geodesic

$$\gamma_{p,u}: I_{p,u}\longrightarrow M \quad \text{with } \begin{cases}\gamma_{p,u}(0)=p\end{cases}$$

→ arranged all ...

$e = \{e_1, \dots, e_r\}$ local frame)

$\to E$ (above γ)

$\sum y_j(t)\, \omega^i_j(\dot\gamma(t))\big)\, e_i(\gamma(t)))$

E:

$\Rightarrow \dots \Leftrightarrow \dots$)

torsion free:

on TM

IBLE WITH g

FREE.

(M, g) fixed. Look at ∇.

-2-

Locally Now $E = TM$. Use chart (U, x) and $e_i = \frac{\partial}{\partial x^i}$

Furthermore, each $\omega^i_j = \sum \underbrace{\text{coeff}_p}_{\text{call them } \Gamma^i_{pj}} dx^p$ $\quad \left(= \omega^i_j\left(\frac{\partial}{\partial x^p}\right)\right)$

$\Rightarrow \nabla$ is encoded in $\Gamma^i_{pj} \in C^\infty(U)$.

$$\nabla_{\frac{\partial}{\partial x^p}} \left(\frac{\partial}{\partial x^j}\right) = \sum_i \Gamma^i_{pj} \frac{\partial}{\partial x^i}$$

Check: ∇ = torsion free $\Leftrightarrow \Gamma^i_{pj} = \Gamma^i_{jp}$ | ∇ = compatible with g $\Leftrightarrow$ $\Leftrightarrow$ beautiful formulas

Geodesics: For $E = TM$, above any $\gamma : I \to M$ sits a natural u

$$u = \frac{d\gamma}{dt} : I \longrightarrow TM$$

Hence it makes sense

$$\frac{\nabla\left(\frac{d\gamma}{dt}\right)}{dt} =: \frac{\nabla^2 \gamma}{dt^2}$$

"2^{nd} derivative of γ" (acceleration)

Def: Given $(M$

Ex: $M = \mathbb{R}^n$, $g =$

$\left(\nabla_X\left(\frac{\partial}{\partial x^i}\right) = 0\right) \Rightarrow$

In general loc

$\Rightarrow \frac{\nabla^2\gamma}{dt^2} =$

2^{nd} order ... n

$$\begin{cases} \frac{d\gamma^i}{dt} = y^i \\ \frac{dy^i}{dt}(t) = -\sum_{j,k} \end{cases}$$

s: -5-

This is al

v

—2—

x) and $e_i = \frac{\partial}{\partial x^i}$

$\Gamma^i_{pj} \left(= \omega^i_j\left(\frac{\partial}{\partial x^p}\right)\right)$

mpatible with g ⟺

⟹ beautiful formulas

→M sits a natural u

nd derivative of γ"

acceleration)

Def: Given (M, g), a geodesic is any γ s.t. $\frac{\nabla^2 \gamma}{dt^2} = 0$

—3—

Ex: $M = \mathbb{R}^n$, $g = g_{can}$ $\left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}\right) \Rightarrow \nabla = \nabla^{can}$

$\left(\nabla_X \left(\frac{\partial}{\partial x^i}\right) = 0 \right) \Rightarrow \frac{\nabla^2 \gamma}{dt^2} = \frac{d^2\gamma}{dt^2} \Rightarrow$ geodesics are straight lines

In general locally $u = \frac{d\gamma}{dt} = \sum \underbrace{\frac{d\gamma^i}{dt}}_{u^i} \frac{\partial}{\partial x_i} \Rightarrow$

$$\Rightarrow \frac{\nabla^2 \gamma}{dt^2} = \sum_i \underbrace{\left(\frac{d^2\gamma^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk}(\gamma(t)) \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} \right)}_{\text{should be } 0 \ (\forall i)} \frac{\partial}{\partial x_i}$$

2^{nd} order ... make it 1^{st} order by the "usual trick"

$$\begin{cases} \frac{d\gamma^i}{dt} = y^i \\ \frac{dy^i}{dt}(t) = -\sum_{j,k} \Gamma^i_{jk}(\gamma(t))\, y^j(t)\, y^k(t) \end{cases}$$

$\frac{d\gamma}{dt} = \dot{\gamma}$

—4—

This is about the integral curves of a vector field on $\mathbb{R}^{2n}$

$$V = \sum y^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma^i_{jk} y^j y^k \frac{\partial}{\partial y^i}$$

$(\gamma(t), y(t))$ with $y = \frac{d\gamma}{dt}$

—4"—

Reminder: Manifold charts for $TM = \{(p, v) / p \in M, v \in T_pM\}$

Key idea: $(\forall)$ chart (U, x) of M $\Rightarrow$ each $v \in T_pM$ with $p \in U$ is written as

$$v = \sum_{i=1}^{m} \underbrace{\text{coeff}_i}_{\text{denoted } y_x^i(v)} \left(\frac{\partial}{\partial x_i}\right)_p$$

$\Longrightarrow$ chart $(\tilde{U}, \tilde{x})$ of TM where

$$\tilde{U} = TU = \{(p, v) / p \in U\}$$

$$\tilde{x} : \tilde{U} \longrightarrow \mathbb{R}^{2m}, \quad \tilde{x}(p, v) = (\underbrace{x^1(p), \ldots, x^m(p)}_{x(p)}, y_x^1(v), \ldots, y_x^m(v))$$

...celeration

$$\frac{d^2\gamma}{dt^2} ... \sum_{j,k} \Gamma^i_{jk}(\gamma(t))\, \dot\gamma^j(t)\, \dot\gamma^k(t)$$

$\frac{d\gamma}{dt} = \dot\gamma$

- 4 -

This is about the integral curves of a vector field on $\mathbb{R}^{2n}$

$$V = \sum_i y^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma^i_{jk}\, y^j y^k \frac{\partial}{\partial y^i}$$

$(\gamma(t), y(t))$, $y(t) = \frac{d\gamma}{dt}$

... intrinsic on $\mathbb{R}^{2n} = T\mathbb{R}^n$

THE GEODESIC VECTOR FIELD

Prop: Given (M,g), $\exists!\ V \in \mathfrak{X}(TM)$ s.t.

$$\left\{ \begin{array}{c} \text{Integral curves} \\ u : I \to TM \\ \text{of } V \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Geodesics} \\ \gamma : I \to M \\ \text{of } (M,g) \end{array} \right\}$$

$$(\gamma, \dot\gamma) \longleftarrow\!\shortmid\ \gamma$$

$$u \longmapsto \mathrm{pr} \circ u$$

$$\begin{array}{ccc} & & TM \\ & \overset{u}{\nearrow} & \downarrow \mathrm{pr} \\ I & \longrightarrow & M \end{array}$$

THE LEVI-CIVITA CONNECTION OF (M,g) TORSION FREE.

—5—

Gain: from the general theory of vector fields:

→ $(\forall)\ (p,v)\in TM$ ($p \in M$, $v \in T_pM$) we have a maximal geodesic

$$\gamma_{p,v} : I_{p,v} \longrightarrow M \quad \text{with } \begin{cases} \gamma_{p,v}(0)=p \\ \dot\gamma_{p,v}(0)=v. \end{cases}$$

→ arranged all together in the "geodesic flow"

$$\gamma : (t,p,v) \longmapsto \gamma_{p,v}(t)$$

$(t,p,v) \in$ domain of the geodesic flow; $\gamma_{p,v}(t) \in M$

$$\{0\}\times TM \subseteq \mathcal{D}(\mathcal{V}) = \{(t,p,v) \,/\, (p,v)\in TM,\ t\in I_{p,v}\} \underset{\text{open}}{\subseteq} \mathbb{R}\times TM$$

$$\gamma : \mathcal{D}(\mathcal{V}) \longrightarrow M$$

→ Exercise: $\gamma\left(\frac{1}{c}t, p, cv\right) = \gamma(t,p,v)\ (\forall) c$ $\left(\& \ I_{p,cv} = \frac{1}{c} I_{p,v}\right)$

HENCE: all the info in γ is contained in what happens at $t=1$!


Remainder
for general connections ∇ on E
compatibility with an inner product g on E
for ∇ on E = TM can also talk about ∇ being torsion free
Riemannian manifold (M, g) with g inner product on TM
THM ∃! ∇ on TM which is both
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HENCE all the info in γ is contained in what happens at t = 1
This is about the integral curves of a vector field on
Prop Given (M, g), ∃!
Integral curves
of
Geodesics

Def: $\exp(p,v) := \gamma(1,p,v)$ -6-

In more detail:

$$\begin{cases} \mathcal{D}_{\exp} = \{ (p,v) \in TM \,/\, \gamma_{p,v} \text{ is defined up } t=1 \} \overset{\text{open}}{\subseteq} TM \\ \boxed{\exp : \mathcal{D}_{\exp} \longrightarrow M}, \quad \exp(p,v) := \gamma_{p,v}(1) \text{ . smooth map} \end{cases}$$

"sloppy notation: $\exp : T\!\!\!/M \longrightarrow M$"

For general $(t,p,v) \in \mathcal{D}(V)$ $\mathcal{D}_{\exp}$ (open) $\supseteq O_M = \{(p,0) : p \in M\}$

$\gamma(t,p,v) = \exp(p,tv)$.

$\Rightarrow$ hence $\boxed{\left.\frac{d}{dt}\right|_{t=0} \exp(p,tv) = v}$ $\boxed{\exp(p,0) = p}$

THE EXPONENTIAL MAP OF (M,g).

$\times TM$

$\frac{1}{c} I_{p,v}$

at $t=1$

This is ab

V

... intrinsic

Prop: Give

{Integral c
$u : I \longrightarrow$
of v

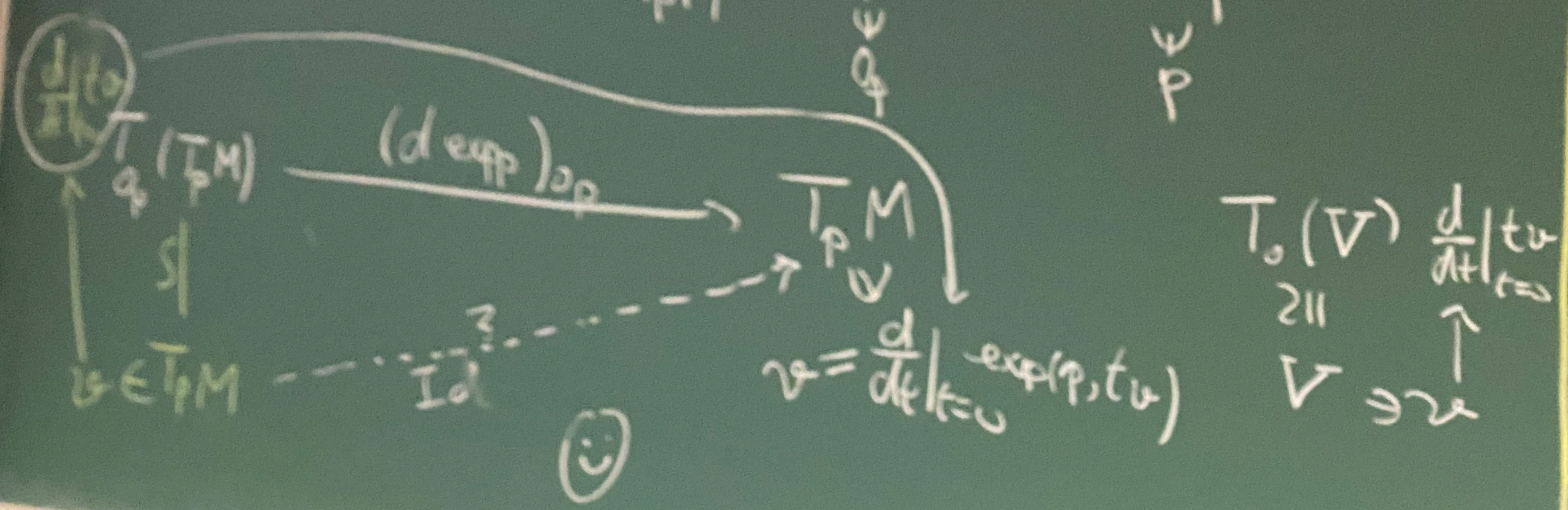
$(\gamma,$

u

-7-

Prop: For any $p \in M$, $\exists\, U_p \subseteq T_pM$ open containing 0_p such that:
$$\exp|_{U_p} : U_p \longrightarrow M$$
is a local diffeom between $U_p \subseteq T_pM$ and an open neighborhood of p in M.

proof: Use the inverse function theorem for
$$\exp_p := \exp|_{T_pM} : T_pM \longrightarrow M$$
$$0_p \mapsto p$$



Tubular

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Answer

like

→ whic

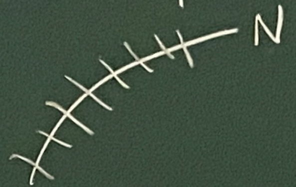
Gain:

-8-

Tubular neighborhoods : $N^n \subseteq M^m$ $\quad k = m-n$

→ When $N = \{p\}$: M around p looks like a vector space

→ Arbitrary N :

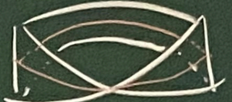


N × vector space ?

$N \times \mathbb{R}^k$

Answer : M will look like a rank k vector bundle over N.

No.



N = S1

→ which vector bundle over N ? The normal bundle!

TN

$TM|_N$

- Abstract version :

$$T_N M = TM|_N \quad \text{\& take quotient}$$

$$\cup\!\!\!\mid \quad TN$$

$$v \bmod TN \in \boxed{\mathcal{N} = T_N M / TN}$$

$$\begin{array}{c} TM \\ \downarrow \\ N \hookrightarrow M \end{array}$$

- Concrete version : use g

$$\cong$$

$$(TN)^\perp = \{ (p,v) \in TM \,/\, p \in N,\ g(v,w) = 0 \ \forall w \in T_pN \}$$

$$\downarrow$$

$$N$$

containing p

an

(V) $\frac{d}{dt}\big|_{t=0}$

≅ ↑

V ∋ v

→ Wh

& a

Claim

is

Expla

$E_p \hookrightarrow T$

-6-

Def : $\exp(p,v) := \gamma(1,p,v)$

This

→ What if we forget about M & we start with N [-9-]
& a vector bundle $E \to N$ of rank r?

$$E \xleftarrow{\;0\;} N \qquad 0(p) = 0_p \in E_p$$

E is a manifold of dimension $n+r$

Claim: the resulting normal bundle of N in E (v bdle over N)

is ... E

Explanation: $E \xrightarrow{\pi} M$

$$E_p \hookrightarrow T_{0_p}E \xrightarrow{d\pi} T_pM, \qquad (d0)_p : T_pM \dashrightarrow T_{0_p}E$$

$T_pN \to T_{0_p}E$, $\quad T_pN \to E_p \oplus T_pM$

$$T_{0_p}E \xleftrightarrow{\sim} E_p \oplus T_pM$$

$$(d0)(w) \longleftarrow w \in T_pM$$

$$\left.\frac{d}{dt}\right|_{t_0} tv \longleftarrow v \in E_p$$

$$\boxed{N_p \cong E_p}$$

[-4-]

... t the integral curves of a vector field on $\mathbb{R}^{2n}$ ... $(x(t), y(t))$

Prop For any $p \in M$, $\exists\, U_p \subseteq T_pM$ open containing 0 such that $\exp|_{U_p} : U_p \longrightarrow M$ is a local diffeom between $U_p \subseteq T_pM$ and an open neighborhood of p in M.

proof Use the inverse function theorem for $\exp_p = \exp|_{T_pM} : T_pM \longrightarrow M$

$(d\exp_p)_0 : T_0(T_pM) \longrightarrow T_pM$

$v = \frac{d}{dt}\Big|_{t=0} \exp(p, tv)$

Tubular neighborhoods

$N^n \subseteq M^m$, $k = m - n$

→ When $N = \{p\}$, M around p looks like a vector space

→ Arbitrary N

$N \times$ vector space? $N \times \mathbb{R}^k$

Answer M will look like a rank k vector bundle over N

→ which vector bundle over N? The normal bundle

- Abstract version: $T_NM = TM|_N \supseteq TN$ & take quotient: $\mathcal{N} = T_NM / TN$
- Concrete version: use g: $(TN)^\perp = \{(p, v) \in TM \,/\, p \in N,\ g(v, w) = 0 \ \forall w \in T_pN\}$

→ What if we forget about M & we start with N & a vector bundle $E \to N$ of rank r?

$E \longleftarrow N$

E is a manifold of dimension $n + r$

Claim: the resulting normal bundle of N in E is ... E

Explanation

$T_pE \longleftrightarrow E_p \oplus T_pM$

$\mathcal{N}_p \cong E_p$

Recall from the general theory of vector fields

→ $\forall (p, v) \in TM$ we have a maximal geodesic $\gamma_{p,v} : I_{p,v} \longrightarrow M$ with $\begin{cases} \gamma_{p,v}(0) = p \\ \dot\gamma_{p,v}(0) = v \end{cases}$

→ assembled all together in the "geodesic flow"

$\gamma : (t, p, v) \longmapsto \gamma_{p,v}(t)$

(t,p,v) in the domain of the geodesic flow; $\gamma_{p,v}(t) \in M$

$\{0\} \times TM \subseteq \mathcal{D}(\mathcal{V}) = \{(t, p, v) \,/\, (p, v) \in TM,\ t \in I_{p,v}\} \subseteq \mathbb{R} \times TM$ (open)

$\gamma : \mathcal{D}(\mathcal{V}) \longrightarrow M$

$\alpha(0) = p$, $\dot\alpha(0) = \frac{1}{c} \cdot cv = v$

→ **Exercise** $\gamma\left(\frac{1}{c}t, p, cv\right) = \gamma(t, p, v)$ $(\forall c)$ $\left(\& \ I_{p,cv} = \frac{1}{c} I_{p,v}\right)$

HENCE: all the info in γ is contained in what happens at $t = 1$

Def: $\exp(p, v) := \gamma(1, p, v)$

In more detail:

$$\begin{cases} \mathcal{D}_{\exp} = \{(p, v) \in TM \,/\, \gamma_{p,v} \text{ is defined up to } t = 1\} \subseteq TM \text{ open} \\ \exp : \mathcal{D}_{\exp} \longrightarrow M, \quad \exp(p, v) = \gamma_{p,v}(1) \quad \text{smooth map} \end{cases}$$

"sloppy notation $\exp : TM \longrightarrow M$"

$TM \supseteq \mathcal{D}_{\exp}$ (open) $\supseteq 0_M = \{(p, 0) \,|\, p \in M\}$

For general $(t, p, v) \in \mathcal{D}(\mathcal{V})$

$\gamma(t, p, v) = \exp(p, tv)$.

$\Rightarrow$ hence $\boxed{\frac{d}{dt}\Big|_{t=0} \exp(p, tv) = v}$ $\quad \boxed{\exp(p, 0) = p}$

THE EXPONENTIAL MAP OF (M, g)

This is about the integral curves of a vector field on $\mathbb{R}^{2n}$

$$\mathcal{V} = \sum_i y^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma^i_{jk}\, y^j y^k \frac{\partial}{\partial y^i}$$

... intrinsic on $\mathbb{R}^{2n} = T\mathbb{R}^n$ — THE GEODESIC VECTOR FIELD

Prop Given (M, g), $\exists!\ \mathcal{V} \in \mathfrak{X}(TM)$ s.t.

$$\left\{ \begin{matrix} \text{Integral curves} \\ u : I \longrightarrow TM \\ \text{of } \mathcal{V} \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{Geodesics} \\ \gamma : I \longrightarrow M \\ \text{of } (M, g) \end{matrix} \right\}$$

$(\gamma, \dot\gamma) \longleftarrow \gamma$

$u \longmapsto pr \circ u$

$u : I \to TM$, $pr : TM \to M$