$(M, g)$ = Riemannian manifold $\Rightarrow$ the Levi-Civita connection on $TM$

$\Rightarrow$ second derivatives $\dfrac{\nabla^2\gamma}{dt^2} = \dfrac{\nabla\left(\frac{d\gamma}{dt}\right)}{dt} : I \to TM$ for $\gamma : I \to M$.

$\Rightarrow$ the notion of geodesic : $\gamma$ s.t. $\dfrac{\nabla^2\gamma}{dt^2} = 0$.

geodesics

Noticed $(\gamma, \dot\gamma) : I \to TM$ are precisely the int. curves of a $\mathcal{V} \in \mathcal{X}(TM)$

$\Rightarrow (\forall)\ (\underset{\substack{\cap \\ M}}{p}, \underset{\substack{\cap \\ T_pM}}{v}) \in TM$ one finds a maximal geod $\gamma_{p,v} : I_{p,v} \longrightarrow M$. $\quad \gamma_{p,v}(t)$

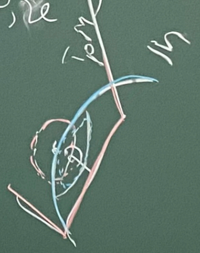
Noticed $\gamma(t, p, v) = \gamma(\frac{1}{c}t, p, cv) \Rightarrow$ all the info contained at $t = 1$.

$\mathcal{D}_{exp} = \{(p, v) / 1 \in I_{p,v}\}$, $\exp : \overset{\subseteq TM}{\mathcal{D}_{exp}} \longrightarrow M$,

$(p, v) \longmapsto \gamma_{p,v}(1)$

$0 : M \to TM,\ p \longmapsto (p, 0_p)$.

$\overset{\cup\!\!|}{\underset{0(M)}{}}$ $\quad i$



$0(M) \subseteq \mathcal{D}_{exp} \subseteq TM$

$$\boxed{\begin{array}{l} \exp(p, 0_p) = p \\[2ex] \left.\dfrac{d}{dt}\right|_{t=0} \exp(p, tv) = v \end{array}}$$

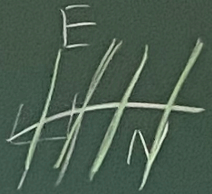
$\boxed{-2-}$

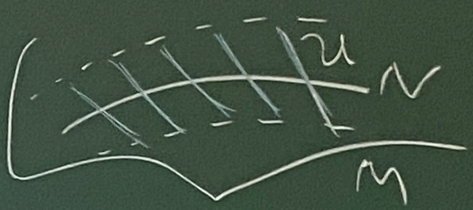
**Thm**: $N \subseteq M$ embedded submanifold. Then:

$$\begin{cases} \exists \text{ vector bundle } E \xrightarrow{} N \\ \exists\, \mathcal{U} \subseteq M \text{ open containing } N \end{cases} \quad \& \quad \begin{tikzcd} E \arrow[rr, "\sim"] & & \mathcal{U} \\ & N \arrow[ul, "0"] \arrow[ur, "\text{incl}"'] & \end{tikzcd} \ \phi : E \xrightarrow{\sim} \mathcal{U} \text{ diffeom.}$$

s.t $\phi(0_p) = p \quad (\forall)\, p \in N$

The vector bundle is actually (must be) the "normal bundle of $N$ in $M$":




**Descr 1**: "abstract" Over $N$ the vector bundles $\begin{matrix} TN \\ \cap \\ TM|_N \end{matrix}$ & form the quotient $\mathcal{N} = \dfrac{TM/_N}{TN}$

**Descr 2**: "concrete" but using a Riemannian metric on $M$ : $E := (TN)^{\perp} = \{ V_p \in T_pM : p \in N,\ g(V_p, W_p) = 0\ (\forall)\, W_p \in T_pN \}$

$\sim$

$V_p \bmod T_pN = \widehat{V_p}$

**Rk**: ...

band...

In the...

**Last** ...

(noth...

a le...

$\Rightarrow$ ...

$\boxed{-4-}$

~~pf~~: Choose $g$ on $M$ (when $N$ is compact)

$\longrightarrow E := (TN)^{\perp}$ rank $m - n$

dim $n + m - m = m$.

Take: $M^m$ $N^n$

$N = \{p\}$

$E = T_pM$

$\begin{matrix} T_pM \\ \cup \\ \exp|_{\mathcal{D}_p} : \mathcal{D}_p \longrightarrow M \end{matrix}$

Cla...

- 2 -

$\to \mathcal{U}$ diffeom.

incl

$p$ $(\forall)\, p \in N$

$\tilde{\mathcal{U}}$, $N$, $M$

the quotient

$= \dfrac{TM|_N}{TN}$

$\ni V_p \bmod T_pN = \widehat{V_p}$

$\ldots) = 0 \ (\forall)\, W_p \in T_pN \}$

---

- 3 -

<u>Rk</u>: The normal bundle of $N$ in $M$ is a vector bundle over $N$ which reflects the way $N$ sits inside $M$. In the same spirit: notice $TM|_N \cong TN \oplus \mathcal{N}$

---

<u>Last time</u>: If we are given a vector bdle $E \to N$ (nothing else) and look at $N \simeq \theta(N) \subseteq E$ as a (embedded) submanifold:

$$\mathcal{N}_p \overset{\sim}{\dashrightarrow} E_p$$

$$\begin{tikzcd} T_{0_p}E \arrow[rr, "\sim"] & & E_p \oplus T_pN & (*) \\ & T_pN \arrow[ul, "(d\theta)_p"] \arrow[ur, "(0,id)"'] & & \end{tikzcd}$$

$$\boxed{\left.\frac{d}{dt}\right|_{t=0} t v \longleftarrow (v, 0)}$$

$$(d\theta)_p(w) \longleftarrow (0, w)$$

$\Rightarrow \mathcal{N} \simeq E$ (as vector bdles over $N$).

---

$p \in N$

$U_p$

$B(p, \varepsilon$

$E =$

---

- 4 -

$T_pM$

- 5 -

<u>Claim</u>: $\exists\ \varepsilon > 0$ s.t. the $\varepsilon$-neighbhd. of $N$ in $E$

$\ldots = \{ (p, v) \in E \,/\, p \in N, \|v\| < \varepsilon \}$

$N$

cpt

pf: Choose $g$ on $M$ (when $N$ is compact.) Take: $M^m$ $N^n$

$N = \{p\}$

$E = T_pM$

$T_pM \supset \mathcal{D}_p \to M$ , $\exp|_{\mathcal{D}\cap T_pM}$

$\longrightarrow E := (TN)^{\perp}$ rank $m-n$, dim $n+m-m = m$.

$\longrightarrow$ the exponential map "restricted to $E$"

$$\begin{cases} \mathcal{D}_E := \mathcal{D}\cap E = \{ (p,v) \in E \,/\, \gamma_{p,v}(1) \text{ is defined} \} \\ \phi = \exp|_{\mathcal{D}_E} : \underset{m\text{-dim}}{\mathcal{D}_E} \longrightarrow M. \end{cases}$$

Aim: make $\phi$ into a diffeo.

Claim: $(d\phi)_{0_p} : T_{0_p}E \longrightarrow T_pM$ is an iso. $\frac{d}{dt}\Big|_{t=0} \exp(tv) = v$ , $w$

$\Vert$ orthog decomp. $\quad \{ \Vert (*)$

$$E_p \oplus T_pN \overset{?}{\dashrightarrow} E_p \oplus T_pN$$

$(v,0) \qquad (0,w)$

$(v,0)$

$(0,tw)$

$\frac{d}{dt}\Big|_{t=0}$

$(d\theta)_p(w)$

$(\phi\circ\theta)(p) =$
$= \phi(0_p) =$
$= \exp(0_p) = p$

Hence $? = Id$

So: by the IFT: $\forall p \in N$,
$\phi = \exp|_{\mathcal{D}_E} : \mathcal{D}_E \to M$
is a local diffeom. around $p$.

$\Rightarrow W \cong E$ (as vector bdles over $N$).

-4-

$M$

$w$

$(v,0)$ $(0,w)$

$(\forall)\, p\in N,$

$M$

om. around $p$.

-5-

Claim: $\exists\, \varepsilon>0$ s.t. the $\varepsilon$-neighbhd. of $N$ in $E$

$$B_\varepsilon(E)=\{(p,v)\in E \,/\, p\in N, \|v\|<\varepsilon\}$$

is inside $\mathcal{D}_E$ and $\underbrace{\phi|_{B_\varepsilon(E)} \text{ is injective}}_{\text{proceed by contradiction}}$

basically the tube lemma

Claim: After eventually shrinking $\varepsilon$: may assume

$\phi|_{B_\varepsilon(E)} = \exp|_{B_\varepsilon(E)}$ is a diffeo. into an open in $M$.

Pf: To make sure: $\boxed{(d\Phi)_v = \text{iso} \ (\forall)\, v\in B_\varepsilon(E)}$ (**)

For $p\in N \Rightarrow$ find $\varepsilon_p>0$, $U_p$ a neigh. of $p$ in $B_\varepsilon(E)$

s.t. $(d\phi)_v = \text{iso}$ $(\forall)\, \underline{q\in U_p}$, $(\forall)\, v\in B_{\varepsilon_p}(E)\cap T_qM$

cpct

extract finite $U_{p_1},\dots,U_{p_k}$ still covering $N$; set $\varepsilon' = \min\{\varepsilon_{p_1},\dots,\varepsilon_{p_k}\}$.

$\Rightarrow$ (**). ☐

$N = N\times\{0\} \subseteq$

cpct

$\Downarrow$

$\exists\, \varepsilon>0$ $N$

So for

$\phi|_{B_{\varepsilon_0}(E)}$

diffeom

Last step

$B_\varepsilon$

$\frac{\varphi(\|v\|)}{\|v\|}\cdot v$

$v$

$E$

where $\|v\|$

over N).

$$O(N) \subseteq \bigcup U_p \subseteq E$$

$\exists\, \varepsilon'_{\xi}$ s.t. $O(N) \subseteq \underline{B_{\varepsilon'}(E)} \subset \bigcup U_p$

ε – neighbd. of N in E

$\dots p \in N,\ \|v\| < \varepsilon \}$

is injective

$B_\varepsilon(E)$

proceed by contradiction

shrinking ε: may assume

a diffeo. into an open in M.

$\boxed{(d\phi)_v = \text{iso} \quad (\forall v \in B_\varepsilon(E))}$ (**)

$U_p$ a neigb. of p in $B_\varepsilon(E)$

$\dots, U_p$

$q \in U_p$, $(\forall u \in B_{\varepsilon_p}(E) \cap T_q M$

still covering N ; set $\varepsilon' = \min\{\varepsilon_{p_1}, \dots, \varepsilon_{p_k}\}$

$\Rightarrow$ (**). □

$\underset{\text{cpt}}{N} = N \times \{0\} \subseteq \underset{\text{open}}{\mathcal{U}} \subseteq N \times \mathbb{R}^k$

$\Downarrow$

$\exists\, \varepsilon > 0 \quad N \times (-\varepsilon, \varepsilon) \subseteq \mathcal{U}$

So far: found $\varepsilon_0$ s.t.

$\phi|_{B_{\varepsilon_0}(E)} : B_{\varepsilon_0}(E) \to M$

diffeom into its image $\mathcal{U} \subseteq M$

<u>Last step</u>: from $B_{\varepsilon_0}(E)$ to E

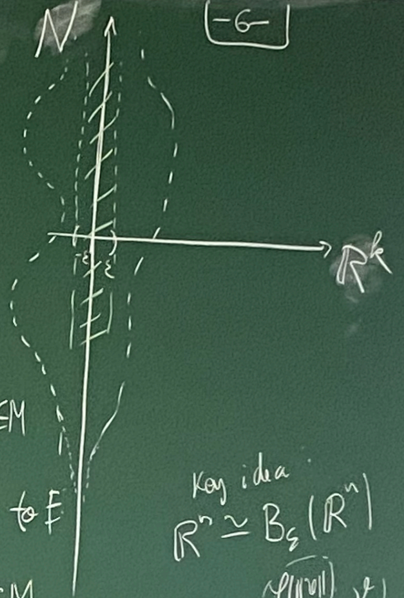
$$B_{\varepsilon_0}(E) \xrightarrow[\sim]{\ \phi\ } \mathcal{U} \subseteq M$$

$$\frac{\varphi(\|v\|)}{\|v\|} \cdot v \ \uparrow\ v \qquad \uparrow \wr \ E$$

HOERAAAA!

where $\|v\| = \sqrt{g(v,v)}$

-6-




Key idea:

$$\mathbb{R}^n \simeq B_\varepsilon(\mathbb{R}^n)$$

$$v \longmapsto \frac{\varphi(\|v\|)}{\|v\|} \cdot v$$

$\varphi : [0,\infty) \to [0,\varepsilon)$

Last time
$\Longrightarrow N \cong E$
After eventually
still covering $N$
Last step