DIFFERENTIAL GEOMETRY MASTERMATH RETAKE, FEBRUARY 8, 2024 (PLEASE EXPLAIN WHAT YOU DO, AND PLEASE GIVE DETAILS)

Exercise 1 (1pt)**.** Show that $P = SO(3)$ endowed with the projection $\pi : SO(3) \to S^2$ which associates to a matrix $A \in SO(3)$ its first column, and with the right action $\bullet$ of S^1 given by

$$A \bullet \lambda := A \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{for } A \in SO(3),\ \lambda = \cos(\alpha) + i\sin(\alpha) \in S^1.$$

is a principal S^1-bundle.

Exercise 2 (1+1+1pts)**.** On $M = S^2 \subset \mathbb{R}^3$ we consider the trivial vector bundle of rank 2, $E = S^2 \times \mathbb{R}^2$, and we denote by $\{e_1, e_2\}$ its canonical frame. Consider the connection ∇ on E whose connection matrix w.r.t. $\{e_1, e_2\}$ is

$$\omega = \begin{pmatrix} xdy & ydz \\ zdx & x^2dx + y^2dy + z^2dz \end{pmatrix},$$

where x, y, z are the coordinates in $\mathbb{R}^3$, and when writing xdy etc we mean their restriction to S^2. Please do the following:

(a) Compute $\nabla_{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}}(x^2e_1 + xyze_2)$ (more precisely, compute the coefficients of this expression w.r.t. the basis $\{e_1, e_2\}$).

(b) Compute the connection matrix of ∇ with respect to $\{e_1, e_2\}$. Should this matrix, or at least its trace, be zero, since the vector bundle is trivial? What is going on?

(c) Compute the parallel transport $T_\gamma^{0,2\pi} : \mathbb{R}^2 \to \mathbb{R}^2$ of ∇ along $\gamma : [0, 2\pi] \to S^2$, $\gamma(t) = (\cos t, \sin t, 0)$.

Exercise 3 (2+1pts)**.** This exercise is related to the question: can a manifold N be written as the boundary $N = \partial M$ of a it compact manifold with boundary M?

(1) Show that if a manifold N of dimension $n = 4k$ can be written $N = \partial M$ with M compact orientable manifold with boundary, then $p_k(TN) = 0$.

(2) Show that none of the manifolds $\mathbb{CP}^{2k}$ with $k \geq 1$ (viewed as real manifolds) cannot be written the boundary of a compact orientable manifold.

(Hint: ... a manifold N with boundary, with N-oriented and compact, is precisely what you need for Stokes theorem to hold: $\int_N d\omega = \int_{\partial N} \omega$ for all forms ω of degree $n - 1$. Please also be aware that much of what we discussed in the course (vector bundles, tangent bundles, connections, characteristic classes, geometric structures, etc, make sense, without any change, also for manifolds with boundary).

Exercise 4 (1+1+1pts+1 bouns pt)**.** Assume that $\pi : P \to M$ is a principal S^1-bundle, and let $L = P[\mathbb{C}]$ be the vector bundle obtained by attaching to P the fiber $\mathbb{C}$ using the standard representation $\mathbb{C}$ of S^1 ($\rho : S^1 \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ being just the inclusion).

(a) Construct a hermitian structure (fibrewise Hermitian inner product) h on L.

(b) Show that $D(L, h) := \{v \in L : h(v, v) \leq 1\}$ (the disk bundle of L w.r.t. h) is a manifold with boundary and describe its boundary directly in terms of P.

(c) What does that give you if applied to the bundle from the first exercise?

(d) (bonus) Try now to answer the question from the previous exercise for $\mathbb{RP}^{2k+1}$ and $\mathbb{CP}^{2k+1}$.