DIFFERENTIAL GEOMETRY MASTERMATH EXAM, JANUARY 11, 2024

Exercise 1 (2pt)**.** Please provide a brief overview on the pull-back operations

$$f^* : \mathrm{Vect}(N) \to \mathrm{Vect}(M)$$

along smooth maps $f : M \to N$ and explain how different types of "notions" that live/make sense on vector bundles $E \to N$ give rise to similar ones on f^*E. As "notions" think e.g. of "sections" of E, "local frames", "connections", connection and curvature matrices, maybe some "geometric structures", or maybe "differential forms with coefficients" (try to collect 3-4-5 different ones). Please also point out 2-3 properties of the pull-back operation that *you* find most remarkable (and maybe say why). However, please try to keep everything in under 2 pages (one page would be ok too).

Exercise 2 (2pt)**.** Show that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$.

Exercise 3. Consider $P = SO(3)$ endowed with the projection $\pi : SO(3) \to S^2$ which associates to a matrix $A \in SO(3)$ its first column, and with the right action $\bullet$ of S^1 given by

$$A \bullet \lambda := A \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \quad \text{for } A \in SO(3),\ \lambda = \cos(\alpha) + i\sin(\alpha) \in S^1.$$

(a) Prove that $\pi : SO(3) \to S^2$ becomes a principal S^1-bundle.

(b) Consider the vector bundle $E = E(SO(3), \mathbb{R}^2, r)$ obtained by fiber attachment using the representation of S^1 on $\mathbb{R}^2$ given by

$$r : S^1 \to GL_2(\mathbb{R}) = GL(\mathbb{R}^2),\ \cos(\alpha) + i\sin(\alpha) \mapsto \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Prove that E is isomorphic to TS^2.

(d) Can $SO(3)$ be seen as a subbundle of $Fr(S^2)$ defining a G-structure for some $G \subseteq GL_2(R)$? If so, what geometric interpretation (eg: complex structure, orientation, etc) does this structure have?

Exercise 4. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold, denote by $\widetilde{\nabla}$ the corresponding Levi-Civita connection and by $\frac{\widetilde{\nabla}}{dt}$ the corresponding derivatives of paths in $T\widetilde{M}$.

Assume now that $M \subset \widetilde{M}$ is a compact submanifold, we endow M with the Riemannian structure g obtained by restricting $\widetilde{g}$ to (the tangent spaces of) M and we consider the corresponding Levi-Civita connection ∇ on TM, and the operation $\frac{\nabla}{dt}$.

(a) show that ∇ coincides with the projection onto TM, via the orthogonal projection $\mathfrak{pr} : T\widetilde{M}|_M \to TM$ induced by g, of the restriction of $\widetilde{\nabla}$ to M (a connection on $T\widetilde{M}|_M$).

(b) show that for any path $u : I \to TM$, one has

$$\frac{\nabla u}{dt} = \mathfrak{pr}\left(\frac{\widetilde{\nabla} u}{dt}\right).$$

Assuming now that $\widetilde{M} = \mathbb{R}^3$ with the standard Riemannian structure and $M = S^2$,

(c) deduce that a path $u(t) = f(t)\frac{\partial}{\partial x} + g(t)\frac{\partial}{\partial y} + h(t)\frac{\partial}{\partial z}$ in TS^2 sitting above a path $\gamma = (x(t), y(t), z(t))$ in S^2 is horizontal w.r.t. ∇ if and only of

$$\dot{f} = -x(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{g} = -y(\dot{x}f + \dot{y}g + \dot{z}h), \quad \dot{h} = -z(\dot{x}f + \dot{y}g + \dot{z}h).$$

(d) describe by explicit formulas all the geodesics of (S^2, g) that start at the point $(1, 0, 0) \in S^2$.

NOTES:

- The order of the exercises is completely unrelated to their difficulty.
- Each of the subquestions inside Exercises 3 and 4 are worth one point. The mark for the exam is the minimum between 10 and the total number of points you collect.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! Especially in Exercise 4, where I would like to see all the careful details. Similarly if you do a computation (e.g. in part (4) of the same exercise) please do not just write down the final result (that may not count at all!) but explain how you got it/provide the details.
- In particular, please do not give "solutions" of the type: "the fact that $\mathbb{CP}^4$ cannot be embedded in $\mathbb{R}^{11}$ is something that I proved in the last question of the last homework" (if that is the case, just repeat the argument aiming at giving as much detail as possible).