Dirac structures, moment maps and quasi-Poisson manifolds

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Abstract

We extend the correspondence between Poisson maps and actions of symplectic groupoids, which generalizes the one between momentum maps and hamiltonian actions, to the realm of Dirac geometry. As an example, we show how hamiltonian quasi-Poisson manifolds fit into this framework by constructing an "inversion" procedure relating quasi-Poisson bivectors to twisted Dirac structures.

Dedicated to Alan Weinstein for his 60th birthday

Contents

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1 Introduction

This paper builds on three ideas pursued by Alan Weinstein in some of his many fundamental contributions to Poisson geometry: First, Lie algebroids play a prominent role in the study of Poisson manifolds [8, 30]; second, Poisson maps can be regarded as generalized momentum maps for actions of symplectic groupoids [25, 31]; third, Poisson structures on manifolds are particular examples of more general objects, called Dirac structures [12, 13, 28]. The main objective of this paper is to combine these three ideas in order to extend the notion of "moment map" to the realm of Dirac geometry. As an application, we obtain an alternative approach to hamiltonian quasi-Poisson manifolds [2] which answers many of the questions posed in [28, 31], shedding light on the relationship between various notions of generalized Poisson structures, hamiltonian actions and reduced spaces.

Let $\mathfrak{g}$ be a Lie algebra, and consider its dual $\mathfrak{g}^*$, equipped with its Lie-Poisson structure. The central ingredients in the formulation of classical hamiltonian $\mathfrak{g}$-actions are a Poisson manifold (Q, π_Q) and a Poisson map $J : Q \to \mathfrak{g}^*$, which we use to define an action of $\mathfrak{g}$ on Q by hamiltonian vector fields:

$$\mathfrak{g} \longrightarrow \mathcal{X}(Q), \quad v \mapsto X_{J_v} := i_{dJ_v}(\pi_Q), \tag{1.1}$$

where $J_v \in C^\infty(Q)$ is given by $J_v(x) = \langle J(x), v\rangle$. For the global picture, we assume that J is a *complete* Poisson map [8, Sec. 6.2], in which case the infinitesimal action (1.1) can be integrated to an action of the connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$, in such a way that J becomes G-equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$. The map J is called a **momentum map** for the G-action on Q, and we refer to the G-action as **hamiltonian**. A key observation, described in [25, 31], is that this construction of a hamiltonian action out of a Poisson map holds in much more generality: one may replace $\mathfrak{g}^*$ by any Poisson manifold, as long as Lie groups are replaced by symplectic groupoids [29]. In this sense, any Poisson map can be seen as a "Poisson-manifold valued moment map".

In this paper, we show that the correspondence between Poisson maps and hamiltonian actions by symplectic groupoids can be further extended to the context of Dirac geometry: in this setting, Poisson maps must be replaced by special types of Dirac maps, called Dirac realizations (see Def. 3.11); for the associated global actions, twisted presymplectic groupoids [6] (alternatively called quasi-symplectic groupoids [33]) play the role of symplectic groupoids. Our main results show that various important notions of generalized hamiltonian actions, such as the "quasi" objects of [2, 3], fit nicely into the Dirac geometry framework.

We organize our results as follows:

In Section 2, we discuss important connections between Lie algebroids and bivector fields. Our main result is that, just as ordinary Poisson structures give rise to Lie algebroid structures on their cotagent bundles, a quasi-Poisson manifold [2, Def. 2.1] (M, π) defines a Lie algebroid structure on $T^*M \oplus \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group acting on M. The leaves of this Lie algebroid coincide with the leaves of the "quasi-hamiltonian foliations" of [2, Sec. 9] in the hamiltonian case, though, in our framework, we make no assumption about the existence of group-valued moment maps.

In Section 3, we study hamiltonian actions in the context of Dirac geometry at the infinitesimal level. We observe that, just as Poisson maps, Dirac realizations are always associated with Lie algebroid actions. (This is, in fact, the guiding principle in our definition of Dirac realizations.) After discussing how classical notions of infinitesimal hamiltonian actions fit into this framework, we prove the main result of the section: Dirac realizations of Cartan-Dirac structures on Lie groups [6, 28] are equivalent to quasi-Poisson $\mathfrak{g}$-manifolds carrying group-valued moment maps. This equivalence involves an "inversion" procedure relating twisted Dirac structures and quasi-Poisson bivectors, revealing that these two objects are in a certain sense "mirror" to one another. The main ingredients in this discussion are the Lie algebroids of Section 2 and the bundle maps which appear in [6] as infinitesimal versions of multiplicative 2-forms. This result explains, in particular, the relationship between Cartan-Dirac and quasi-Poisson structures on Lie groups; on the other hand, it recovers the correspondence proven in [2, Thm. 10.3] between "non-degenerate" hamiltonian quasi-Poisson manifolds (i.e., those for which the Lie algebroids of Section 2 are *transitive*) and quasi-hamiltonian spaces [3].

In Section 4, we study moment maps in Dirac geometry from a global point of view. We show that *complete* Dirac realizations "integrate" to presymplectic groupoid actions, which are natural extensions of those studied in [33]. As our main example, we show that the "integration" of Dirac realizations of Cartan-Dirac structures on Lie groups results in hamiltonian quasi-Poisson G-manifolds. Finally, we show that the natural reduction procedure in the setting of Dirac geometry encompasses various classical reduction theorems [20, 24, 25] as well as their "quasi" counterparts [2, 3, 33].

We remark, following an observation of E. Meinrenken, that the results concerning quasi-Poisson manifolds in this paper only require the Lie algebras to be quadratic, in contrast with some of the constructions in [2], in which the positivity of the bilinear forms plays a key role. (In particular, our results hold for quasi-Poisson G-manifolds when G is a noncompact semisimple Lie group.) Most of our constructions can be carried out in the more general setting of [1], but this will be discussed in a separate paper.

A work which gave initial motivation and is closely related to the present paper is that of Xu [33], in which a Morita theory of quasi-symplectic groupoids is developed in order to compare "moment map theories". Our results show that twisted Dirac structures complement Xu's picture in two ways: on one hand, by providing the infinitesimal framework for Morita equivalence; on the other hand, by leading to more general "modules" (i.e., hamiltonian spaces).

It is a pleasure to dedicate this paper to Alan Weinstein, whose work and insightful ideas have been an unlimited source of inspiration to us.

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Notation: We use the following conventions for bundle maps: if π is a bivector field on M, then $\pi^\sharp : T^*M \to TM$, $\alpha \mapsto \pi(\alpha, \cdot)$; if ω is a 2-form, then $\omega^\sharp : TM \to T^*M$, $X \mapsto \omega(X, \cdot)$.

The space of k-multivector fields on M is denoted by $\mathcal{X}^k(M)$.

On a Lie group G, with lie algebra $\mathfrak{g}$, $(\cdot,\cdot)_{\mathfrak{g}}$ will denote a bi-invariant nondegenerate quadratic form; we write ϕ^G for the associated Cartan 3-form, and $\chi_G \in \Lambda^3\mathfrak{g}$ for the dual trivector.

2 Lie algebroids, bivector fields and Poisson geometry

2.1 Lie algebroids

A **Lie algebroid** over a manifold M is a vector bundle $A \to M$ together with a Lie algebra bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$, and a bundle map $\rho : A \to TM$, called the **anchor**, satisfying the Leibniz identity

$$[\xi, f\xi'] = f[\xi,\xi'] + \mathcal{L}_{\rho(\xi)}(f)\xi', \quad \text{for } \xi,\xi' \in \Gamma(A), \text{ and } f \in C^\infty(M). \tag{2.1}$$

Whenever there is no risk of confusion, we will write $\mathcal{L}_{\rho(\xi)}$ simply as $\mathcal{L}_\xi$.

If M is a point, then a Lie algebroid over M is a Lie algebra in the usual sense. An important feature of Lie algebroids $A \to M$ is that the image of the anchor, $\rho(A) \subseteq TM$, defines a generalized integrable distribution, determining a singular foliation of M. The leaves of this foliation are the **orbits** of the Lie algebroid. The following example plays a key role in the study of hamiltonian actions and moment maps.

Example 2.1 (*Transformation Lie algebroids*)

Consider an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M, given by a Lie algebra homomorphism $\bar\rho : \mathfrak{g} \to \mathcal{X}(M)$. The **transformation Lie algebroid** associated with this action is the trivial vector bundle $M \times \mathfrak{g}$, with anchor $(x, v) \mapsto \rho(x, v) := \bar\rho(v)(x)$ and Lie bracket on $\Gamma(M \times \mathfrak{g}) = C^\infty(M, \mathfrak{g})$ defined by

$$[u, v](x) := [v(x), w(x)]_{\mathfrak{g}} + (\bar\rho(u(x)) \cdot v)(x) - (\bar\rho(v(x)) \cdot u)(x). \tag{2.2}$$

We often denote a transformation Lie algebroid by $\mathfrak{g} \ltimes M$.

Note that $[\cdot,\cdot]$ is uniquely determined by the condition that it coincides with $[\cdot,\cdot]_{\mathfrak{g}}$ on constant functions and the Leibniz identity. The orbits of $\mathfrak{g} \ltimes M$ are the $\mathfrak{g}$-orbits on M.

The remaining of this section is devoted to examples of Lie algebroids closely related to Poisson manifolds.

2.2 Bivector fields and Poisson structures

If (M, π) is a Poisson manifold, then T^*M carries a Lie algebroid structure with anchor

$$\pi^\sharp : T^*M \to TM, \quad \beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta), \tag{2.3}$$

and bracket

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - d\pi(\alpha, \beta), \tag{2.4}$$

uniquely characterized by $[df, dg] = d\{f, g\}$ and the Leibniz identity (2.1). Here, as usual, $\{f, g\} = \pi(df, dg)$ is the Poisson bracket on $C^\infty(M)$. In this case, the orbits of T^*M are the symplectic leaves of M, i.e., the integral manifolds of the distribution defined by the hamiltonian vector fields $X_f = \pi^\sharp(df)$.

Example 2.2 *(Lie-Poisson structures)*

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra, and consider $\mathfrak{g}^*$ equipped with the associated Lie-Poisson structure

$$\{f,g\}(\mu) := \langle \mu, [df(\mu), dg(\mu)]\rangle, \ \ \mu \in \mathfrak{g}^*. \tag{2.5}$$

Under the identification $T^*\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}$, one can see that the Lie algebroid structure on $T^*\mathfrak{g}^*$ induced by (2.5) is that of a transformation Lie algebroid $\mathfrak{g} \ltimes \mathfrak{g}^*$, see Example 2.1, and a direct computation reveals that the action in question is the coadjoint action.

If $\pi \in \mathcal{X}^2(M)$ is an *arbitrary* bivector field, let us consider $\pi^\sharp$, $[\cdot,\cdot]$, $\{\cdot,\cdot\}$ and X_f as defined by the previous formulas, and let $\chi_\pi \in \mathcal{X}^3(M)$ be the trivector field defined by

$$\chi_\pi := [\pi, \pi], \tag{2.6}$$

i.e., χ_π satisfies

$$\frac{1}{2}\chi_\pi(df, dg, dh) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = \{\{f,g\},h\} + c.p.,$$

where we use *c.p.* to denote cyclic permutations.

Lemma 2.3 *For any bivector field π on M, one has*

$$\pi^\sharp([\alpha,\beta]) = [\pi^\sharp(\alpha), \pi^\sharp(\beta)] + \frac{1}{2} i_{\alpha\wedge\beta}(\chi_\pi), \tag{2.7}$$

$$[[\alpha,\beta],\gamma] + c.p. = \frac{1}{2}(\mathcal{L}_{i_{\alpha\wedge\beta}(\chi_\pi)}(\gamma) + c.p.) - d(\chi_\pi(\alpha,\beta,\gamma)), \tag{2.8}$$

for $\alpha, \beta, \gamma \in \Omega^1(M)$. As a result, the following are equivalent:

(i) π is a Poisson tensor;

(ii) $\pi^\sharp : \Omega^1(M) \to \mathcal{X}(M)$ preserves the brackets;

(iii) the bracket $[\cdot,\cdot]$ on $\Omega^1(M)$ satisfies the Jacobi identity;

(iv) $(T^*M, \pi^\sharp, [\cdot,\cdot])$ is a Lie algebroid.

Proof: The key remark is that the difference between the left and right hand sides of each of (2.7) and (2.8) is $C^\infty(M)$-multilinear in α, β and γ. So it is enough to prove the identities on exact forms, which is immediate. $\square$

Example 2.4 (*Twisted Poisson manifolds*)

Consider a closed 3-form $\phi \in \Omega^3(M)$. A **ϕ-twisted Poisson structure** on M [18, 27] consists of a bivector field $\pi \in \mathcal{X}^2(M)$ satisfying

$$\frac{1}{2}[\pi,\pi] = \pi^\sharp(\phi).$$

Here, we abuse notation and write $\pi^\sharp$ to denote the map induced by (2.3) on exterior algebras. We know from Lemma 2.3 that the bracket (2.4) induced by π is not preserved by $\pi^\sharp$ and does not satisfy the Jacobi identity. However,

$$\pi^\sharp([\alpha,\beta] + i_{\pi^\sharp(\alpha)\wedge\pi^\sharp(\beta)}(\phi)) = [\pi^\sharp(\alpha), \pi^\sharp(\beta)].$$

Hence, if we define a "twisted" version of the bracket (2.4),

$$[\alpha,\beta]_\phi := [\alpha,\beta] + i_{\pi^\sharp(\alpha)\wedge\pi^\sharp(\beta)}(\phi),$$

then $\pi^\sharp$ will preserve this new bracket, and $[\cdot,\cdot]_\phi$ satisfies the Jacobi identity. As a result, $(T^*M, \pi^\sharp, [\cdot,\cdot]_\phi)$ is a Lie algebroid. We leave it to the reader to prove a "twisted" version of Lemma 2.3.

2.3 The Lie algebroid of a quasi-Poisson manifold

Let G be a Lie group with Lie algebra $\mathfrak{g}$, equipped with a bi-invariant nondegenerate quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$. Let ϕ^G be the bi-invariant Cartan 3-form on G, and let $\chi_G \in \Lambda^3\mathfrak{g}$ be its dual trivector. On Lie algebra elements $u, v, w \in \mathfrak{g}$, we have

$$\phi^G(u,v,w) = \chi_G(u^\vee, v^\vee, w^\vee) = \frac{1}{2}(u,[v,w])_{\mathfrak{g}},$$

where $u^\vee, v^\vee, w^\vee$ are dual to u, v, w via $(\cdot,\cdot)_{\mathfrak{g}}$; when $(\cdot,\cdot)_{\mathfrak{g}}$ is a metric and e_a is an orthonormal basis of $\mathfrak{g}$, we can write

$$\chi_G = \frac{1}{12}\sum (e_a,[e_b,e_c])_{\mathfrak{g}} e_a \wedge e_b \wedge e_c.$$

A **quasi-Poisson G-manifold** [2] consists of a G-manifold M together with a G-invariant bivector field π satisfying

$$\chi_\pi = \rho_M(\chi_G), \tag{2.9}$$

where $\rho_M : \mathfrak{g} \longrightarrow \mathcal{X}(M)$ is the associated infinitesimal action, and we keep the same notation for the induced maps of exterior algebras. When M is just a $\mathfrak{g}$-manifold, we call the corresponding object a **quasi-Poisson $\mathfrak{g}$-manifold**. The two notions are related by the standard procedure of integration of infinitesimal actions; in particular, they coincide if M is compact and G is simply connected.

In analogy with ordinary or twisted Poisson manifolds, are quasi-Poisson structures also associated with Lie algebroids? As we now discuss, the answer is yes. Let us consider a more general set-up: let M be a $\mathfrak{g}$-manifold and let $\pi \in \mathcal{X}^2(M)$ be an arbitrary bivector field. Motivated by [2, Sec. 9], we consider on $T^*M \oplus \mathfrak{g}$ the "anchor" map

$$r : T^*M \oplus \mathfrak{g} \longrightarrow TM, \quad r(\alpha, v) = \pi^\sharp(\alpha) + \rho_M(v), \tag{2.10}$$

combining the bivector field and the action. On sections of $T^*M \oplus \mathfrak{g}$, we consider the bracket defined by

$$[(\alpha,0),(\beta,0)] = ([\alpha,\beta], -\frac{1}{2} i_{\rho_M^*(\alpha\wedge\beta)}(\chi_G)), \tag{2.11}$$

$$[(0,v),(0,v')] = (0,[v,v']), \tag{2.12}$$

$$[(0,v),(\alpha,0)] = (\mathcal{L}_{\rho_M(v)}(\alpha), 0), \tag{2.13}$$

for all 1-forms $\alpha, \beta \in \Omega^1(M)$ and all $v, v' \in \mathfrak{g}$ (thought of as constant sections in $C^\infty(M,\mathfrak{g})$). Of course, as in Example 2.1, the definition of the bracket on general elements in $\Gamma(T^*M \oplus \mathfrak{g}) = \Omega^1(M) \oplus C^\infty(M,\mathfrak{g})$ is obtained from the Leibniz formula (2.1). With these definitions, we obtain a quasi-Poisson analogue of Lemma (2.3):

Theorem 2.5 *Let M be a $\mathfrak{g}$-manifold equipped with a bivector field π. The following are equivalent:*

(i) (M,π) *is a quasi-Poisson $\mathfrak{g}$-manifold;*

(ii) $r : \Omega^1(M) \oplus C^\infty(M,\mathfrak{g}) \to \mathcal{X}(M)$ *preserves brackets;*

(iii) the bracket $[\cdot,\cdot]$ *on* $\Omega^1(M) \oplus C^\infty(M,\mathfrak{g})$ *satisfies the Jacobi identity;*

(iv) $(T^*M \oplus \mathfrak{g}, r, [\cdot,\cdot])$ *is a Lie algebroid.*

PROOF: Note that r preserves the bracket (2.12), since ρ_M is an action. From the identity (2.7) in Lemma 2.3, it follows that r preserves the bracket of type (2.11) if and only if $\chi_\pi = \rho_M(\chi_G)$. On the other hand, r preserves the bracket of type (2.13) if and only if $\pi^\sharp \mathcal{L}_{\rho_M(v)}(\xi) = \mathcal{L}_{\rho_M(v)}\pi^\sharp(\xi)$, which is equivalent to the $\mathfrak{g}$-invariance of π. This shows that (i) and (ii) are equivalent.

Let us prove that (i) implies (iii); from the proof, the converse will be clear. Assuming (i), we must show that $[\cdot,\cdot]$ on $\Omega^1(M) \oplus C^\infty(M,\mathfrak{g})$ satisfies the Jacobi identity. On elements of type $(0,v)$, this reduces to the Jacobi identity for $\mathfrak{g}$ (or, alternatively, for $\mathfrak{g} \ltimes M$). On elements $(0,v)$, $(0,w)$ and $(\alpha,0)$, the Jacobi identity of $[\cdot,\cdot]$ reduces to the fact that ρ_M is an action. Computing the "jacobiator" for elements of type $(0,v)$, $(\alpha,0)$, $(\beta,0)$, we see that the first component is

$$[\mathcal{L}_{\rho_M(v)}(\alpha),\beta] + [\alpha, \mathcal{L}_{\rho_M(v)}(\beta)] - \mathcal{L}_{\rho_M(v)}([\alpha,\beta]) \tag{2.14}$$

Using the Leibniz identity, we see that the $C^\infty(M)$-linearity of (2.14) with respect to β is equivalent to $\pi^\sharp\mathcal{L}_{\rho(v)}(\beta) = \mathcal{L}_{\rho(v)}\pi^\sharp(\beta)$, i.e., to the $\mathfrak{g}$-invariance of π. Hence, if π is invariant, (2.14) is $C^\infty(M)$-linear on α and β, and then one can check that it is zero by looking at the particular case when α and β are exact. The second component of the jacobiator of $(0,v)$, $(\alpha,0)$, $(\beta,0)$ can be computed similarly.

To complete the proof that (i) implies (iii), we must deal with the Jacobi identity for elements of type $(\alpha,0)$, $(\beta,0)$, $(\gamma,0)$. To this end, we first need to find the expression for the bracket between elements of type $(0,\tilde{v})$ and $(\alpha,0)$, with $\tilde{v} \in C^\infty(M,\mathfrak{g})$ not necessarily constant: pairing $d\tilde{v} \in \Omega^1(M;\mathfrak{g})$ with an element $\mu \in C^\infty(M,\mathfrak{g}^*)$ gives us a 1-form on M, denoted by $A_{\tilde{v}}(\mu)$, satisfying the following two properties:

$$A_{f\tilde{v}}(\mu) = fA_{\tilde{v}}(\mu) + \mu(\tilde{v})df, \quad \text{and} \quad A_{\tilde{v}}(f\mu) = fA_{\tilde{v}}(\mu),$$

for $f \in C^\infty(M)$. We claim that

$$[(0,\tilde{v}),(\alpha,0)] = (\mathcal{L}_{\rho_M(\tilde{v})}(\alpha) - A_{\tilde{v}}(\rho_M^*(\alpha)), -\mathcal{L}_{\pi^\sharp(\alpha)}(\tilde{v})). \tag{2.15}$$

To see that, note that (2.15) holds when $\tilde{v}$ is constant, and the difference between the left and right hand sides is $C^\infty(M)$-linear in $\tilde{v}$. We remark that

$$A_{i_{\mu\wedge\mu'}(\chi_G)}(\mu'') + c.p. = 2d(\chi_G(\mu,\mu',\mu'')). \tag{2.16}$$

Again, it is easy to check this identity when μ, μ' and μ'' are constant, so (2.16) follows from $C^\infty(M)$-linearity. Also, denoting $\chi_M := \rho_M(\chi_G)$, a direct computation shows that

$$\rho_M(i_{\rho_M^*(\alpha\wedge\beta)}(\chi_G)) = i_{\alpha\wedge\beta}(\chi_M).$$

We now turn to the computation of the jacobiator of the elements $(\alpha,0)$, $(\beta,0)$ and $(\gamma,0)$, that we denote by $\mathrm{Jac}(\alpha,\beta,\gamma)$. For the first component of $\mathrm{Jac}(\alpha,\beta,\gamma)$, we obtain

$$([[\alpha,\beta],\gamma]+c.p.)-\frac{1}{2}(\mathcal{L}_{i_{\alpha\wedge\beta}(\chi_M)}(\gamma)+c.p.)+\frac{1}{2}(A_{i_{\rho_M^*(\alpha)\wedge\rho_M^*(\beta)}(\chi_M)}(\rho_M^*(\gamma))+c.p.). \tag{2.17}$$

Combining the second identity of Lemma 2.3 with (2.16), we get that (2.17) equals

$$d((\rho_M(\chi_G)-\chi_\pi)(\alpha,\beta,\gamma)),$$

which vanishes by the condition $\chi_\pi=\rho_M(\chi_G)$. So we are left with proving that the second component of $\mathrm{Jac}(\alpha,\beta,\gamma)$ vanishes, which amounts to showing that

$$i_{\rho_M^*([\alpha,\beta]\wedge\gamma)}(\chi_G)+c.p.=\mathcal{L}_{\pi^\sharp(\gamma)}i_{\rho_M^*(\alpha\wedge\beta)}(\chi_G)+c.p.. \tag{2.18}$$

In order to do that, consider the operators $i_{\rho_M^*([\alpha,\beta])}$ and $\mathcal{L}_{\pi^\sharp(\alpha)}i_{\rho_M^*(\beta)}-\mathcal{L}_{\pi^\sharp(\beta)}i_{\rho_M^*(\alpha)}$ acting on $C^\infty(M,\Lambda\mathfrak{g})$, for $\alpha,\beta\in\Omega^1(M)$.

Claim 2.6 *On $\Lambda\mathfrak{g}$, seen as constant functions in $C^\infty(M,\Lambda\mathfrak{g})$, we have*

$$i_{\rho_M^*([\alpha,\beta])}=\mathcal{L}_{\pi^\sharp(\alpha)}i_{\rho_M^*(\beta)}-\mathcal{L}_{\pi^\sharp(\beta)}i_{\rho_M^*(\alpha)}. \tag{2.19}$$

Proof: Both operators are derivations of degree -1 on $\Lambda\mathfrak{g}$, hence it suffices to show (2.19) for elements $v\in\mathfrak{g}$. As we now check, this follows from the definition of the bracket induced by π and the invariance of π: on one hand,

$$i_{\rho_M^*([\alpha,\beta])}(v)=[\alpha,\beta](\rho_M(v))=i_{\rho_M(v)}\mathcal{L}_{\pi^\sharp(\alpha)}(\beta)-i_{\rho_M(v)}\mathcal{L}_{\pi^\sharp(\beta)}(\alpha)-i_{\rho_M(v)}d\pi(\alpha,\beta). \tag{2.20}$$

Using that $i_{[X,Y]}=\mathcal{L}_X i_Y-i_Y\mathcal{L}_X$ for vector fields X,Y, we have

$$\begin{aligned} i_{\rho_M(v)}\mathcal{L}_{\pi^\sharp(\alpha)}(\beta) &= \mathcal{L}_{\pi^\sharp(\alpha)}(\beta(\rho_M(v)))-\beta([\pi^\sharp(\alpha),\rho_M(v)]) \\ &= \mathcal{L}_{\pi^\sharp(\alpha)}i_{\rho_M^*(\beta)}(v)-\pi(\mathcal{L}_{\rho_M(v)}(\alpha),\beta), \end{aligned} \tag{2.21}$$

where the last equality follows from the $\mathfrak{g}$-invariance of π. Using the identity (2.21) (and its analogue for α and β interchanged) in (2.20), (2.19) follows. □

Using the claim, we see that

$$i_{\rho_M^*([\alpha,\beta]\wedge\gamma)}+c.p.=i_{\rho_M^*(\gamma)}i_{\rho_M^*([\alpha,\beta])}+c.p.=(i_{\rho_M^*(\gamma)}\mathcal{L}_{\pi^\sharp(\alpha)}i_{\rho_M^*(\beta)}-i_{\rho_M^*(\gamma)}\mathcal{L}_{\pi^\sharp(\beta)}i_{\rho_M^*(\alpha)})+c.p. \tag{2.22}$$

when restricted to constant elements in $C^\infty(M,\Lambda\mathfrak{g})$. On the other hand, it follows from (2.19) that, on $\Lambda\mathfrak{g}$, we can write

$$i_{\rho_M^*([\alpha,\beta])}=[\mathcal{L}_{\pi^\sharp(\alpha)},i_{\rho_M^*(\beta)}]-[\mathcal{L}_{\pi^\sharp(\beta)},i_{\rho_M^*(\alpha)}] \tag{2.23}$$

since the Lie derivatives are zero on constant functions. But both sides of (2.23) are $C^\infty(M)$-linear, so this equality is valid for all $C^\infty(M,\mathfrak{g})$. So we can write

$$\begin{aligned} i_{\rho_M^*([\alpha,\beta]\wedge\gamma)}+c.p. &= -i_{\rho_M^*([\alpha,\beta])}i_{\rho_M^*(\gamma)}+c.p. \\ &= -([\mathcal{L}_{\pi^\sharp(\alpha)},i_{\rho_M^*(\beta)}]-[\mathcal{L}_{\pi^\sharp(\beta)},i_{\rho_M^*(\alpha)}])i_{\rho_M^*(\gamma)}+c.p., \end{aligned}$$

from where we deduce that

$$i_{\rho_M^*([\alpha,\beta]\wedge\gamma)} + c.p. = 2(\mathcal{L}_{\pi^\sharp(\alpha)} i_{\rho_M^*(\beta\wedge\gamma)} + c.p.) - (i_{\rho_M^*(\gamma)}\mathcal{L}_{\pi^\sharp(\alpha)} i_{\rho_M^*(\beta)} - i_{\rho_M^*(\beta)}\mathcal{L}_{\pi^\sharp(\alpha)} i_{\rho_M^*(\gamma)} + c.p.).$$

On constant functions, we can use (2.22) to conclude that

$$(i_{\rho_M^*([\alpha,\beta]\wedge\gamma)} + c.p.) = 2(\mathcal{L}_{\pi^\sharp(\alpha)} i_{\rho_M^*(\beta\wedge\gamma)} + c.p.) - (i_{\rho_M^*([\alpha,\beta]\wedge\gamma)} + c.p.),$$

i.e., $i_{\rho_M^*([\alpha,\beta]\wedge\gamma)} + c.p. = \mathcal{L}_{\pi^\sharp(\alpha)} i_{\rho_M^*(\beta\wedge\gamma)}$. Evaluating this identity at χ_G proves (2.18), showing that (i) implies (iii). Looking back into the proof, one can check that the same formulas show the converse, so that (i) and (iii) are equivalent.

Since (ii) and (iii) together are equivalent to (iv), it follows that (i) – (iv) are equivalent to each other. $\square$

Corollary 2.7 *If (M, π) is a quasi-Poisson $\mathfrak{g}$-manifold, then the generalized distribution*

$$\pi^\sharp(\alpha) + \rho_M(v) \subseteq TM, \quad \textit{for} \;\; \alpha \in T^*M, \;\; v \in \mathfrak{g},$$

is integrable.

This result shows that the singular distribution discussed in [2, Thm. 9.2] in the context of *hamiltonian* quasi-Poisson manifolds is integrable even without the presence of a moment map (and without the positivity of $(\cdot,\cdot)_\mathfrak{g}$). As in the case of ordinary Poisson manifolds, we call a quasi-Poisson manifold **nondegenerate** if its associated Lie algebroid is transitive (i.e., its anchor map is onto).

Example 2.8 *(Quasi-Poisson structures on Lie groups)*

Let G be a Lie group with Lie algebra $\mathfrak{g}$, which we assume to be equipped with an invariant nondegenerate quadratic form $(\cdot,\cdot)_\mathfrak{g}$. We consider G acting on itself by conjugation. As shown in [2, Sec. 3], the bivector field π_G, defined on left invariant 1-forms by

$$\pi_G(dl^*_{g^{-1}}(\mu), dl^*_{g^{-1}}(\nu)) := \frac{1}{2}\big((\mathrm{Ad}_{g^{-1}} - \mathrm{Ad}_g)(\mu^\vee), \nu^\vee\big)_\mathfrak{g}, \tag{2.24}$$

where l_g denotes left multiplication by $g \in G$, $\mu, \nu \in \mathfrak{g}^*$, and $\mu^\vee$ is the element in $\mathfrak{g}$ dual to μ via $(\cdot,\cdot)_\mathfrak{g}$, makes G into a quasi-Poisson G-manifold. If $(\cdot,\cdot)_\mathfrak{g}$ is a metric, then we can write

$$\pi_G = \frac{1}{2}\sum e_a^r \wedge e_a^l, \tag{2.25}$$

where e_a is an orthonormal basis of $\mathfrak{g}$ and e_a^r (resp. e_a^l) are the corresponding right (resp. left) translations.

In this example, the image of $\pi_G^\sharp$ is tangent to the G-orbits, so the leaves of the corresponding foliation are the conjugacy classes. The formula for the Lie algebroid bracket on $T^*G \oplus \mathfrak{g}$ bears close resemblance with the one for the bracket in the "double" of the Lie quasi-bialgebra of G, as in [4]. We will discuss this connection in a separate work.

3 Moment maps in Dirac geometry: the infinitesimal picture

3.1 Dirac manifolds

Let ϕ be a closed 3-form on a manifold M. A **ϕ-twisted Dirac structure** on M [28] is a subbundle $L \subset E = TM \oplus T^*M$ satisfying the following two conditions:

1. L is maximally isotropic with respect to the symmetric pairing $\langle\cdot,\cdot\rangle_+ : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$,
$$\langle (X,\alpha),(Y,\beta)\rangle_+ := \beta(X) + \alpha(Y); \tag{3.1}$$
2. The space of sections $\Gamma(L)$ is closed under the bracket $[\![\cdot,\cdot]\!]_\phi : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,
$$[\![(X,\alpha),(Y,\beta)]\!]_\phi := ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha + i_{X\wedge Y}\phi). \tag{3.2}$$

Since the pairing (3.1) has zero signature, condition 1. is equivalent to requiring that L has rank equal to $\dim(M)$ and that $\langle\cdot,\cdot\rangle_+|_L = 0$. The bracket (3.2) is the **ϕ-twisted Courant bracket** considered in [28]. When ϕ=0, this bracket is a non-skew-symmetric version of Courant's original bracket introduced in [12].

Twisted Dirac structures are always associated with Lie algebroids. Indeed, the restriction of the Courant bracket $[\![\cdot,\cdot]\!]_\phi$ to a Dirac subbundle $L \subset TM \oplus T^*M$ defines a Lie algebra bracket on the space of sections $\Gamma(L)$, making $L \to M$ into a Lie algebroid with anchor
$$\rho = \mathrm{pr}_1|_L : L \to TM,$$
where pr_1 is the first projection. The orbits of this algebroid are also called the **leaves** of L.

Example 3.1 (*Twisted Poisson structures*)
If π is a bivector field on M, then
$$L_\pi := \mathrm{graph}(\pi^\sharp) \subset TM \oplus T^*M$$
satisfies condition 1., and L_π is a ϕ-twisted Dirac structure if and only if π is a ϕ-twisted Poisson structure in the sense of Example 2.4. In this case, the second projection
$$\mathrm{pr}_2|_L : L \to T^*M$$
establishes an isomorphism of Lie algebroids, where T^*M is equipped with the Lie algebroid structure described in Example 2.4. Setting $\phi = 0$, we obtain a one-to-one correspondence between ordinary Poisson structures on M and Dirac structures satisfying the extra condition $L \cap TM = \{0\}$.

Example 3.2 (*Twisted presymplectic forms*)
Similarly, the graph associated with a 2-form $\omega \in \Omega^2(M)$, $L_\omega = \mathrm{graph}(\omega^\sharp)$, is a ϕ-twisted Dirac structure if and only if
$$d\omega + \phi = 0,$$
and we refer to ω as a **ϕ-twisted presymplectic form**. In this case, setting $\phi = 0$, we have an identification of closed 2-forms on M with Dirac structures satisfying $L \cap T^*M = \{0\}$.

In general, the leaves of a ϕ-twisted Dirac structure L carry twisted presymplectic forms defined as follows: at each $x \in M$, we define a skew symmetric bilinear form θ_x on $\rho(L)_x = \mathrm{pr}_1(L)_x$ by

$$\theta_x(X_1, X_2) = \alpha(X_2), \tag{3.3}$$

where α is any element in T_x^*M satisfying $(X_1, \alpha) \in L_x$. The fact that L is maximally isotropic with respect to (3.1) guarantees that (3.3) is independent of the choice of α, and these forms fit into a smooth leafwise 2-form θ. Using that $\Gamma(L)$ is closed with respect to (3.2), one can show that, on each leaf $\iota : \mathcal{O} \hookrightarrow M$, the 2-form θ satisfies

$$d\theta + \iota^*\phi = 0.$$

At each point $x \in M$, the kernel of θ coincides with $L_x \cap T_xM$, which shows that the leafwise presymplectic forms are nondegenerate if and only if L comes from a ϕ-twisted Poisson structure. We will denote the distribution $L \cap TM$ on M by $\ker(L)$.

Since Dirac structures are always associated with Lie algebroids, it is natural to consider how to obtain Dirac structures from them. The following is a useful construction, see [6]: for a Lie algebroid A over M with anchor $\rho : A \longrightarrow TM$, we define a **$\phi$-IM form of** A to be any bundle map[1]

$$\sigma : A \longrightarrow T^*M$$

satisfying the following properties:

$$\begin{aligned} \langle \sigma(\xi), \rho(\xi') \rangle &= -\langle \sigma(\xi'), \rho(\xi) \rangle; & (3.4)\\ \sigma([\xi, \xi']) &= \mathcal{L}_\xi(\sigma(\xi')) - \mathcal{L}_{\xi'}(\sigma(\xi)) + d\langle \sigma(\xi), \rho(\xi') \rangle + i_{\rho(\xi)\wedge\rho(\xi')}(\phi), & (3.5) \end{aligned}$$

for $\xi, \xi' \in \Gamma(A)$ (here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between a vector space and its dual). Let $L_\sigma \subset TM \oplus T^*M$ be the image of the map $(\rho, \sigma) : A \longrightarrow TM \oplus T^*M$. Then the following is immediate.

Lemma 3.3 *If σ is a ϕ-IM form of A and $\mathrm{rank}(L_\sigma) = \dim(M)$, then L_σ is a ϕ-twisted Dirac structure on M.*

Of course, any Dirac structure can be realized as the image of an IM form by taking $A = L$, viewed as an algebroid with $\rho = \mathrm{pr}_1|L$, and $\sigma = \mathrm{pr}_2|L$.

The following is a key example.

Example 3.4 (*Cartan-Dirac structures on Lie groups*)

Cartan-Dirac structures on Lie groups play a role in Dirac geometry analogous to the one played by Lie-Poisson structures (Example 2.2) in Poisson geometry. Just as Lie-Poisson structures on dual of Lie algebras are completely determined by the Kostant-Kirillov-Souriau (KKS) symplectic forms along coadjoint orbits, Cartan-Dirac structures on Lie groups "assemble" certain 2-forms defined on conjugacy classes defined as follows.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $(\cdot, \cdot)_\mathfrak{g}$ be a bi-invariant nondegenerate quadratic form, which we use to identify TG and T^*G. For $v \in \mathfrak{g}$, let $v_G = v_r - v_l$ be the

[1] These bundle maps are infinitesimal versions of multiplicative 2-forms on groupoids, see [6]; the terminology "IM" stands for "infinitesimal multiplicative".

infinitesimal generator of the action of G on itself by conjugation. We define, on each conjugacy class $\iota : \mathcal{C} \hookrightarrow G$, a 2-form θ by

$$\theta_g(u_G, v_G) := (\frac{1}{2}(\mathrm{Ad}_{g^{-1}} - \mathrm{Ad}_g)u, v)_{\mathfrak{g}}, \quad g \in \mathcal{C}. \tag{3.6}$$

Direct computations show that $d\theta + \iota^*\phi^G = 0$, where ϕ^G is the bi-invariant Cartan 3-form on G, and that θ_g is nondegenerate at a point g if and only if $(\mathrm{Ad}_g + 1)$ is invertible. The 2-forms (3.6) appear in [17] in the study of symplectic structures of moduli spaces, and we will refer to them as **GHJW 2-forms**.

Since GHJW 2-forms are not symplectic, but *twisted presymplectic*, they should correspond to a ϕ^G-twisted Dirac structure L_G on G rather than a Poisson structure. A simple computation shows that

$$L_G = \{(v_r - v_l, \frac{1}{2}(v_r + v_l)), \;\; v \in \mathfrak{g}\} \subset TG \oplus TG. \tag{3.7}$$

We call L_G the **Cartan-Dirac structure** on G associated with $(\cdot,\cdot)_{\mathfrak{g}}$ [28, 6].

Note that $\rho(v) = v_r - v_l$ is the anchor of the action Lie algebroid (Example 2.1) $\mathfrak{g} \ltimes G$ with respect to the action by conjugation, and the map

$$\sigma : G \times \mathfrak{g} \longrightarrow TG, \;\; \sigma(v) = \frac{1}{2}(v_r + v_l) \tag{3.8}$$

satisfies the conditions of Lemma 3.3 (recall that we are identifying TG with T^*G via $(\cdot,\cdot)_{\mathfrak{g}}$). So σ is a ϕ^G-IM form of $\mathfrak{g} \ltimes G$, and the Cartan-Dirac structure L_G arises as the image of (ρ, σ). In this case, (ρ, σ) actually establishes an isomorphism between $\mathfrak{g} \ltimes G$ and L_G. (Note the analogy with Example 2.2, which shows that Lie algebroids of Lie-Poisson structures are isomorphic to action Lie algebroids for the coadjoint action!)

Let us finally recall an important operation involving Dirac structures: if L is a ϕ-twisted Dirac structure on M and $B \in \Omega^2(M)$, then

$$\tau_B(L) := \{(X, \alpha + B^\sharp(X)) \mid (X, \alpha) \in L\} \tag{3.9}$$

defines a $(\phi - dB)$-twisted Dirac structure on M [28]. The operation τ_B is called a **gauge transformation associated with** B, and it has the effect of modifying L by adding the pull-back of B to the presymplectic form on each leaf.

3.2 Dirac maps

Since Dirac structures generalize both Poisson and presymplectic structures, we have two possible definitions of Dirac maps, see [7].

Let (M, L_M) and (N, L_N) be twisted Dirac manifolds. A smooth map $f : N \to M$ is a **forward Dirac map**, or **f-Dirac** in short, if L_N and L_M are related as follows:

$$L_M = \{(df(Y), \alpha) \mid Y \in TN, \, \alpha \in T^*M \text{ and } (Y, df^*\alpha) \in L_N\}. \tag{3.10}$$

If L_M and L_N are associated with twisted Poisson structures, then an f-Dirac map is equivalent to a Poisson map. The terminology "forward" is due to the fact that, at each point, (3.10) extends the usual notion of "push-forward" of a linear bivector. For this reason, we may write

$$L_M = f_* L_N$$

instead of (3.10), in analogy with the notation for "f-related" bivector fields on a manifold.

Similarly, $f : N \to M$ is a **backward Dirac map**, or simply **b-Dirac**, if

$$L_N = \{(Y, df^*\alpha) \mid Y \in TN,\ \alpha \in T^*M \text{ and } (df(Y), \alpha) \in L_M\}. \tag{3.11}$$

If L_M and L_N are associated with twisted presymplectic structures ω_M and ω_N, then a b-Dirac map is just a map satisfying $f^*\omega_M = \omega_N$. As before, we will write

$$L_N = f^*L_M$$

to denote that f is a b-Dirac map. Note that f^*L_M is always a well-defined, though not necessarily smooth, subbundle of TN, in contrast with f_*L_N, which may not be well-defined at all. In fact, f^*L_M defines a Dirac structure on N provided it is smooth, which is the case, e.g., when f is a submersion. However, as illustrated in the next example, f^*L_M may define a Dirac structure even when f is not a submersion.

Example 3.5 (*Inclusion of presymplectic leaves*)

Let L be a twisted Dirac structure on M, and consider a presymplectic leaf $\mathcal{O}$, equipped with Dirac structure L_θ associated with the twisted presymplectic form θ. Denoting by $\iota : \mathcal{O} \hookrightarrow M$ the inclusion map, it follows from the definition of θ that

$$L_\theta = \{(X, i_X\theta) \mid X \in T\mathcal{O}\} = \{(X, (d\iota)^*\alpha) \mid (d\iota(X), \alpha) \in L\} = \iota^*L. \tag{3.12}$$

So $\iota : (\mathcal{O}, L_\theta) \hookrightarrow (M, L)$ is a b-Dirac map. On the other hand, at each point of M, we have

$$\iota_*L_\theta = \{(d\iota(X), \alpha) \mid (X, (d\iota)^*\alpha) \in L_\theta\}.$$

By the second equality in (3.12), it follows that $\iota_*L_\theta \subseteq L$, but since they have the same dimension, we get

$$\iota_*L_\theta = L, \tag{3.13}$$

so ι is f-Dirac as well.

Note that the fact that the inclusion of presymplectic leaves into a Dirac manifold is an f-Dirac map is a direct generalization of the fact that the inclusion of symplectic leaves into a Poisson manifold is a Poisson map. As a simple consequence, we have

Corollary 3.6 *Let (N, L_N) and (M, L_M) be twisted Dirac manifolds. A map $J : N \to M$ is f-Dirac if and only if its restriction to each presymplectic leaf of N is f-Dirac.*

We remark that Example 3.5 is very special in that the inclusion map of presymplectic leaves is both forward and backward Dirac (see also Remark 4.12). In general, f-Dirac maps need not be b-Dirac, nor the other way around.

3.3 Poisson maps as infinitesimal hamiltonian actions

The usual notion of Lie algebra action can be extended to the realm of Lie algebroids, the main difference being that algebroids, rather than acting on manifolds, act on maps from manifolds into their base: An **action of a Lie algebroid** $A \to M$ **on a map** $J : N \to M$ consists of a Lie algebra homomorphism $\rho_N : \Gamma(A) \to \mathcal{X}(N)$ satisfying

$$dJ \circ \rho_N(\xi) = \rho(\xi), \quad \text{for all } \xi \in \Gamma(A), \tag{3.14}$$

and such that, for $f \in C^\infty(M)$ and $\xi \in \Gamma(A)$, $\rho_N(f\xi) = J^* f \rho_N(\xi)$ (i.e., the induced map $\Gamma(J^*A) \to \mathcal{X}(N)$ comes from a vector bundle morphism $J^*A \to TN$, where $J^*A = A \times_M N$ is the pull-back of the vector bundle A by J).

Example 3.7 (*Actions of transformations Lie algebroids*)
Consider an infinitesimal action ρ of $\mathfrak{g}$ on a manifold M. Then an action ρ_N of the transformation Lie algebroid $A = \mathfrak{g} \ltimes M$ on a map $J : N \to M$ is equivalent to an infinitesimal action $\overline{\rho_N}$ of $\mathfrak{g}$ on N for which J is $\mathfrak{g}$-equivariant. Indeed, ρ_N and $\overline{\rho_N}$ are related by the formula

$$\rho_N(v)_y = \overline{\rho_N}(v(J(y)))_y, \quad \text{where } v \in C^\infty(M, \mathfrak{g}),\, y \in N, \tag{3.15}$$

and the $\mathfrak{g}$-equivariance of J corresponds to (3.14).

In Poisson geometry, Poisson maps are *always* associated with Lie algebroid actions: If (Q, π_Q) and (P, π_P) are Poisson manifolds, then any Poisson map $J : Q \to P$ induces a Lie algebroid action of T^*P on Q by

$$\Omega^1(P) \longrightarrow \mathcal{X}(Q), \quad \alpha \mapsto {\pi_Q}^\sharp(J^*\alpha). \tag{3.16}$$

When the target P is the dual of a Lie algebra, we recover a familiar example:

Example 3.8 (*Infinitesimal hamiltonian actions*)
Consider $\mathfrak{g}^*$ equipped with its Lie-Poisson structure. As remarked in Example 2.2, the Lie algebroid structure on $T^*\mathfrak{g}^*$ induced by (2.5) is that of a transformation Lie algebroid $\mathfrak{g} \ltimes \mathfrak{g}^*$ with respect to the coadjoint action. If $J : Q \to \mathfrak{g}^*$ is a Poisson map, then it induces an action of $T^*\mathfrak{g}^*$ via (3.16), which, by Example 3.7, is equivalent to an ordinary $\mathfrak{g}$-action on Q for which J is equivariant. A simple computation shows that the $\mathfrak{g}$-action arising in this way is just a hamiltonian action in the usual sense, making Q into a hamiltonian Poisson $\mathfrak{g}$-manifold having J as a momentum map.

Recall that a Poisson map $J : Q \to P$ is called a **symplectic realization** if Q is symplectic. The following is an immediate consequence:

Proposition 3.9 *There is a one-to-one correspondence between Poisson maps into* $\mathfrak{g}^*$ *and hamiltonian Poisson* $\mathfrak{g}$*-manifolds, and this correspondence restricts to a one-to-one correspondence between symplectic realizations of* $\mathfrak{g}^*$ *and hamiltonian symplectic* $\mathfrak{g}$*-manifolds.*

Remark 3.10 An analogue of Prop. 3.9 holds more generally in the context of Poisson-Lie groups [19, 21]. Let (G, π) be a simply-connected Poisson-Lie group, and let G^* be its dual. The Lie algebroid structure on $T^*G^* \cong G^* \times \mathfrak{g}$ induced from the dual Poisson structure is a transformation Lie algebroid, now associated with the infinitesimal dressing action of $\mathfrak{g}$ on G^*. For a Poisson map $J : Q \to G^*$, the general Lie algebroid action described by (3.16) reduces to a Poisson $\mathfrak{g}$-action for which J is an equivariant momentum map in the sense of Lu [20].

In order to extend this discussion to Dirac geometry, let us consider $L_{\pi_P} = \mathrm{graph}(\pi_P{}^\sharp)$, the associated Dirac structure on (P, π_P). Using the Lie algebroid isomorphism $T^*P \cong L_{\pi_P}$, we can rewrite the infinitesimal action (3.16) as

$$\Gamma(L_{\pi_P}) \to \mathcal{X}(Q), \quad (X, \alpha) \mapsto Y, \tag{3.17}$$

where $Y \in \mathcal{X}(Q)$ is uniquely determined by the condition $(Y, J^*\alpha) \in L_{\pi_Q}$. Also note that Y is related to X by $dJ(Y) = X$, since J is a Poisson map. The question of whether this procedure can be carried out for f-Dirac maps leads us to the notion of Dirac realization.

3.4 Dirac realizations

If (N, L_N) and (M, L_M) are twisted Dirac manifolds, then, by definition, a smooth $J : N \to M$ is an f-Dirac map if and only if, given $(X, \alpha) \in (L_M)_{J(y)}$, there exists a $Y \in T_yQ$ with the property that

$$(Y, dJ^*\alpha) \in (L_N)_y \quad \text{and} \quad X = (dJ)_y(Y). \tag{3.18}$$

It is natural to try to define an action of L_M on N just as in the case of Poisson maps, see (3.17), except that (3.18) does *not* determine Y uniquely in general. In fact, this is the case if and only if the following extra "nondegeneracy" condition holds:

$$\ker(J) \cap \ker(L_N) = \{0\}. \tag{3.19}$$

A similar argument as in [6, Section 7.1] shows that (3.19) is equivalent to $J : \ker(L_N) \to \ker(L_M)$ being an isomorphism.

Definition 3.11 *A* ***Dirac realization*** *of a ϕ-twisted Dirac manifold (M, L_M) is an f-Dirac map $J : (N, L_N) \to (M, L_M)$, where L_N is a $J^*\phi$-twisted Dirac structure on N, satisfying* (3.19)*.*

As a consequence of Definition 3.11, we have

Corollary 3.12 *Let $J : N \to M$ be a Dirac realization. Then the map $\Gamma(L_M) \to \mathcal{X}(N)$, $(X, \alpha) \mapsto Y$, where Y is determined by the conditions in* (3.18)*, is a Lie algebroid action.*

Dirac realizations $J : N \to M$ for which N is presymplectic were studied in [6, Sec. 7.1] under the name of **presymplectic realizations**. As a result of Corollary 3.6, we have

Corollary 3.13 *A map $J : N \to M$ is a Dirac realization if and only if its restriction to each presymplectic leaf of N is a presymplectic realization.*

Similarly to Poisson geometry, the connection between Dirac realizations and "hamiltonian actions" is established by a suitable choice of "target" M. Following the analogy between Lie-Poisson structures and Cartan-Dirac structures, it is natural to study the "hamiltonian spaces" associated with Dirac realizations of Cartan-Dirac structures. The particular case of presymplectic realizations is discussed in [6, Sec. 7.2]:

Example 3.14 (*Presymplectic realizations of Cartan-Dirac structures*)
Let G be a Lie group with Lie algebra $\mathfrak{g}$, equipped with a bi-invariant nondegenerate quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$. Let L_G be the associated Cartan-Dirac structure on G. If (M, ω_M) is a twisted presymplectic manifold, then the conditions for $J : M \to G$ being a presymplectic realization can be expressed as follows:

1. ω_M is $\mathfrak{g}$-invariant and satisfies $d\omega_M = -J^*\phi^G$;
2. at each $x \in M$, $\mathrm{Ker}(\omega_M)_x = \{(\rho_M)_x(v) : v \in \mathrm{Ker}(\mathrm{Ad}_{J(p)} + 1)\}$;
3. the map J satisfies the moment map condition

$$\omega^\sharp \rho_M = J^*\sigma. \tag{3.20}$$

The invariance of ω_M in 1. is with respect to the $\mathfrak{g}$-action ρ_M induced by J (recall that $L_G \cong \mathfrak{g} \ltimes G$, see Example 3.4, so an L_G-action defines an ordinary $\mathfrak{g}$-action), for which J is equivariant; in 3., σ is the IM-form of the Cartan-Dirac structure (3.8),

$$\sigma : \mathfrak{g} \longrightarrow T^*G, \quad \sigma(v) = \frac{1}{2}(v_r + v_l)^\vee, \tag{3.21}$$

where $v \longrightarrow v^\vee$ denotes the isomorphism $TG \longrightarrow T^*G$ induced by the quadratic form. The "relative closedness" of ω_M in 1. expresses that the associated Dirac structure is $J^*\phi^G$-twisted, while condition 2. is the "non-degeneracy" condition (3.19) applied to this particular case; finally, condition 3. follows from J being an f-Dirac map.

Conditions 1., 2. and 3. are exactly the defining axioms of a **quasi-hamiltonian $\mathfrak{g}$-space**, in the sense of [3], for which J is the group-valued moment map. Conversely, any group-valued moment map of a quasi-hamiltonian $\mathfrak{g}$-space is a presymplectic realization of (G, L_G).

We summarize Example 3.14 in the next result, analogous to Proposition 3.9, see [6, Thm. 7.6].

Theorem 3.15 *There is a one-two-one correspondence between presymplectic realizations of G endowed with the Cartan-Dirac structure, and quasi-hamiltonian $\mathfrak{g}$-spaces.*

Combining Corollary 3.13 with Theorem 3.15, we conclude that general Dirac realizations of Cartan-Dirac structures must be "foliated" by quasi-hamiltonian $\mathfrak{g}$-spaces. Since hamiltonian quasi-Poisson manifolds, in the sense of [2], also have this property [2, Sec. 10], we are led to investigate the relationship between these objects.

3.5 Dirac realizations and hamiltonian quasi-Poisson $\mathfrak{g}$-manifolds

3.5.1 The equivalence theorem

For a quasi-Poisson $\mathfrak{g}$-manifold (M, π), a **moment map** is a $\mathfrak{g}$-equivariant map $J : M \longrightarrow G$ (with respect to the infinitesimal action by conjugation on G) satisfying the condition [2, Lem. 2.3]

$$\pi^\sharp J^* = \rho_M \sigma^\vee, \tag{3.22}$$

where

$$\sigma^\vee : T^*G \longrightarrow \mathfrak{g}, \quad \sigma^\vee(\xi_g) = \frac{1}{2}(dr_{g^{-1}}(\xi_g^\vee) + dl_{g^{-1}}(\xi_g^\vee)) \tag{3.23}$$

is the adjoint of σ (3.21) with respect to the form $(\cdot,\cdot)_\mathfrak{g}$, and $\rho_M : \mathfrak{g} \to TM$ is the $\mathfrak{g}$-action. Here l_g and r_g denote the left and right translations by g, respectively, and, as in Example 3.14, the symbol $^\vee$ on elements of T^*G is used to denote the corresponding element in TG via the identification induced by $(\cdot,\cdot)_\mathfrak{g}$ (and vice-versa).

The following is our main result in this section.

Theorem 3.16 *There is a one-to-one correspondence between Dirac realizations of G, endowed with the Cartan-Dirac structure, and hamiltonian quasi-Poisson $\mathfrak{g}$-manifolds.*

Before proving Theorem 3.16, let us collect some useful formulas relating the maps σ, $\sigma^\vee$, $\rho : \mathfrak{g} \to TG$, $\rho(v) = v_r - v_l$, and, for symmetry, the dual of ρ with respect to $(\cdot,\cdot)_{\mathfrak{g}}$,

$$\rho^\vee : TG \longrightarrow \mathfrak{g}, \quad \rho^\vee(V_g) = dr_{g^{-1}}(V_g) - dl_{g^{-1}}(V_g). \tag{3.24}$$

The following lemma follows from a straightforward computation.

Lemma 3.17 *The following formulas hold true:*

$$\begin{aligned}
4\sigma^\vee\sigma + \rho^\vee\rho &= 4\mathrm{Id}_{\mathfrak{g}}, & (3.25)\\
4\sigma\sigma^\vee + (\rho\rho^\vee)^* &= 4\mathrm{Id}_{T^*G}, & (3.26)\\
\sigma^*\rho &= \rho^*\sigma, & (3.27)\\
\sigma\rho^\vee &= -(\rho^\vee)^*\sigma^*, & (3.28)\\
\rho^\vee(\sigma^\vee)^* &= -\sigma^\vee(\rho^\vee)^*. & (3.29)
\end{aligned}$$

Motivated by the equivalence between quasi-hamiltonian manifolds and nondegenerate hamiltonian quasi-Poisson manifolds [2, Thm. 10.3], which will also follow from Theorem 3.16, it is natural to combine the two moment-map conditions (3.20) and (3.22). By applying $\pi^\sharp$ to (3.20) and using (3.22), we obtain, in particular, the relation

$$\rho_M\sigma^\vee\sigma = \pi^\sharp\omega^\sharp\rho_M.$$

This suggests the importance of writing $\rho_M\sigma^\vee\sigma$ as the composition of some operator $C : TM \to TM$ with ρ_M in general. Using (3.25) and the equivariance of J, written as $\rho = dJ \circ \rho_M$, it is easy to find an expression for C (which already appears in [2, Lem. 10.2]):

Lemma 3.18 *For any manifold M equipped with an infinitesimal action $\rho_M : \mathfrak{g} \longrightarrow TM$, and any $\mathfrak{g}$-equivariant map $J : M \longrightarrow G$, the operator*

$$C = 1 - \frac{1}{4}\rho_M\rho^\vee(dJ) : TM \longrightarrow TM,$$

and its dual $C^* : T^*M \longrightarrow T^*M$, satisfy the formulas

$$\rho_M\sigma^\vee\sigma = C\rho_M, \quad \text{and} \quad J^*\sigma\sigma^\vee = C^*J^*. \tag{3.30}$$

Theorem 3.16 follows from the next two propositions, each one of them describing explicitly one direction of the asserted one-to-one correspondence.

Proposition 3.19 *Let M be a $\mathfrak{g}$-quasi-Poisson space, and let $A = T^*M \oplus \mathfrak{g}$ be its associated Lie algebroid, with anchor r. Then any moment map $J : M \longrightarrow G$ induces a ϕ-IM form of A by*

$$s : A \longrightarrow T^*M, \quad s(\alpha, v) = C^*(\alpha) + J^*\sigma(v), \tag{3.31}$$

so that the image L of the map $(r, s) : A \longrightarrow TM \oplus T^*M$ is a $J^*\phi^G$-twisted Dirac structure on M, and $J : (M, L) \longrightarrow (G, L_G)$ is a Dirac realization of the Cartan-Dirac structure on G.

This proposition also suggest the converse construction.

Proposition 3.20 *Let $J : (M, L) \longrightarrow (G, L_G)$ be a Dirac realization of the Cartan-Dirac structure on G. Then*

(i) for any $v \in \mathfrak{g}$ there is an unique vector $V \in TM$ satisfying

$$dJ(V) = v, \quad \text{and} \quad (V, J^*\sigma(v)) \in L. \tag{3.32}$$

(ii) for any $\alpha \in T^*M$, there is an unique vector $X \in TM$ satisfying

$$dJ(X) = -(\rho_M \sigma^\vee)^*\alpha, \tag{3.33}$$
$$(X, C^*(\alpha)) \in L. \tag{3.34}$$

Moreover, $v \mapsto \rho_M(v) := V$ defines a $\mathfrak{g}$-action on M, and $\alpha \mapsto \pi^\sharp(\alpha) := X$ defines a quasi-Poisson tensor π on M so that (M, π) is a hamiltonian quasi-Poisson $\mathfrak{g}$-manifold with moment map J.

Note that the $\mathfrak{g}$-action defined by (3.32) is just the one induced by the infinitesimal $L_G = \mathfrak{g} \ltimes G$-action with "moment" $J : M \to G$.

It is simple to check that the constructions in Propositions 3.19 and 3.20 are inverses to one another. For example, if L is obtained from π as in Prop. 3.19, then

$$L = \{(\pi^\sharp(\alpha) + \rho_M(v), C^*(\alpha) + J^*\sigma(v)) \mid \alpha \in T^*M, v \in \mathfrak{g}\},$$

and it is clear that, given $\alpha \in T^*M$, $X = \pi^\sharp(\alpha)$ satisfies conditions (3.33) (which is just the dual of the moment map condition (3.22)) and (3.34), so Prop. 3.20 constructs π back.

Since the proofs of Propositions 3.19 and 3.20 involve long computations, we will postpone them to the next subsection; we will discuss some examples and implications of the results first.

Example 3.21 (*Nondegenerate quasi-Poisson and quasi-hamiltonian $\mathfrak{g}$-manifolds*)

It is clear from the correspondence constructed in Prop. 3.19 that the singular foliation associated with π, tangent to $\mathrm{Im}(\pi^\sharp) + \mathrm{Im}(\rho_M) \subseteq TM$, coincides with the singular foliation of the Dirac structure L, tangent to $\mathrm{pr}_1(L) \subseteq TM$. In other words, the Lie algebroids associated with π and L have the same leaves, so one is transitive if and only if the other one is. Note that, for Dirac structures, $\mathrm{pr}_1(L) = TM$ means exactly that L is defined by a 2-form. As a result, it follows that the correspondence established by Theorem 3.16 restricts to a one-to-one correspondence between *nondegenerate* hamiltonian quasi-Poisson manifolds and *presymplectic* realizations Cartan-Dirac structures on Lie groups.

Combining Example 3.21 with Theorem 3.15, we obtain

Corollary 3.22 *There is a one-to-one correspondence between*

(i) non-degenerate quasi-Poisson hamiltonian $\mathfrak{g}$-spaces.

(ii) quasi-hamiltonian $\mathfrak{g}$-spaces.

(iii) presymplectic realizations of G endowed with the Cartan-Dirac structure.

Of course, in general, the leaves of a hamiltonian quasi-Poisson $\mathfrak{g}$-space are quasi-hamiltonian $\mathfrak{g}$-spaces, which can now be seen as a particular case of Dirac structures having presymplectic foliations (see also Corollary 3.13).

The next example answers a question posed in [28, Ex. 5.2].

Example 3.23 (*Cartan-Dirac and quasi-Poisson structures on Lie groups*)

If G is a Lie group with Lie algebra $\mathfrak{g}$ equipped with a bi-invariant nondegenerate quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$, then one can regard it as a ϕ^G-twisted Dirac manifold with respect to the Cartan-Dirac structure L_G, or as a hamiltonian quasi-Poisson $\mathfrak{g}$-manifold. In the latter case, we consider G acting on itself by conjugation, the quasi-Poisson tensor is π_G, defined Example 2.8, and the moment map $J: G \to G$ is the identity map, see [2, Sec. 3].

We claim that these two structures are "dual" to each other in the sense of Theorem 3.16. Indeed, starting with π_G and using Prop. 3.19, we see that the corresponding Dirac structure is

$$L = \{(\pi_G^\sharp(\alpha) + \rho(v), C^*(\alpha) + \sigma(v)),\, |\, (\alpha, v) \in T^*M \oplus \mathfrak{g}\}.$$

Since L_G is the image of (ρ, σ), it is clear that $L_G \subseteq L$, which implies that $L_G = L$ since they have the same dimension.

To complete the "duality" picture between Dirac realizations of Cartan-Dirac structures and hamiltonian quasi-Poisson $\mathfrak{g}$-manifolds, we note that the the correspondence established in Theorem 3.16 preserves maps.

If (M_i, π_i) is a hamiltonian quasi-Poisson $\mathfrak{g}$-manifold with moment map $J_i : M_i \to G$, $i = 1, 2$, then a map $f: M_1 \to M_2$ is a **hamiltonian quasi-Poisson map** if $f_*\pi_1 = \pi_2$, f is $\mathfrak{g}$-equivariant, and $J_2 \circ f = J_1$. Suppose that L_i is the Dirac structure on M_i corresponding to π_i, $i = 1, 2$, via Theorem 3.16.

Proposition 3.24 *A map $f : (M_1, \pi_1) \to (M_2, \pi_2)$ is a hamiltonian quasi-Poisson map if and only if $f : (M_1, L_1) \to (M_2, L_2)$ is f-Dirac and commutes with the realization maps, $J_2 \circ f = J_1$.*

Proof: Suppose $f : M_1 \to M_2$ is a hamiltonian quasi-Poisson map. In order to check that f is f-Dirac, we have to compare, at each point of M_2, L_2 with

$$f_*L_1 = \{(df(X), \beta) \mid (X, df^*(\beta)) \in L_1\}. \tag{3.35}$$

To simplify the notation, we will denote the infinitesimal actions of $\mathfrak{g}$ on M_i by ρ_i, and $C_i = 1 - (1/4)\rho_i\rho^\vee dJ$, $i = 1, 2$. Since L_2 corresponds to π_2, we have

$$L_2 = \{(\pi_2^\sharp(\beta) + \rho_2(v), C_2^*(\beta) + J_2^*\sigma(v)) \,|\, \beta \in T^*M_2, v \in \mathfrak{g}\}.$$

Using that $df\pi_1^\sharp df^* = \pi_2^\sharp$ (which is another way of writing $f_*\pi_1 = \pi_2$) and $df\rho_1 = \rho_2$ (which is f $\mathfrak{g}$-equivariance), we deduce that

$$\pi_2^\sharp(\beta) + \rho_2(v) = df(\pi_1^\sharp(df^*(\beta)) + \rho_1(v)). \tag{3.36}$$

On the other hand, using the $\mathfrak{g}$-equivariance of f and $J_2 \circ f = J_1$, it follows that $dfC_1 = C_2 df$, and we obtain

$$df^*(C_2^*(\beta) + J_2^*\sigma(v)) = C_1^*(df^*(\beta)) + J_1^*\sigma(v). \tag{3.37}$$

Since
$$(\pi_1^\sharp(df^*(\beta))+\rho_1(v), C_1^*(df^*(\beta))+J_1^*\sigma(v))\in L_1,$$
for L_1 corresponds to π_1, it follows that, at each point, $L_2\subseteq f_*L_1$. But since they have equal dimension, we conclude that $L_2=f_*L_1$, so f is forward Dirac.

For the converse, suppose that $f_*L_1=L_2$ and $J_2\circ f=J_1$. It is easy to check that, in this case, f is automatically $\mathfrak{g}$-equivariant with respect to (3.32). From that, it follows that $dfC_1=C_2df$. In order to prove that f is a hamiltonian quasi-Poisson map, we must still check that $f_*\pi_1=\pi_2$, or, equivalently, that $df\pi_1^\sharp df^*=\pi_2^\sharp$. By Prop. 3.20, it suffices to prove that, for $\beta\in T^*M_2$, $Y=df\pi_1^\sharp df^*(\beta)\in TM_2$ satisfies
$$(Y, C_2^*(\beta))\in L_2 \quad \text{and} \quad dJ_2(Y)=-(\rho_2\sigma^\vee)^*\beta. \tag{3.38}$$
Since f is an f-Dirac map, the first condition in (3.38) holds since
$$(\pi_1^\sharp df^*(\beta), df^*(C_2^*(\beta)))=(\pi_1^\sharp df^*(\beta), C_1^*(df^*(\beta)))\in L_1,$$
for L_1 corresponds to π_1. The second condition holds since
$$dJ_2(df\pi_1^\sharp df^*(\beta))=dJ_1(\pi_1^\sharp df^*(\beta))=-(\rho_1\sigma^\vee)^*df^*(\beta)=-(\rho_2\sigma^\vee)^*(\beta).$$
□

We now proceed to the proofs of Propositions 3.19 and 3.20.

3.5.2 The proofs

Proof of Proposition 3.19:
To simplify our formulas, we set
$$T=\rho_M\rho^\vee dJ,$$
and we denote by $\langle\cdot,\cdot\rangle$ the pairing between vector spaces and their duals.

First, we have to show that $\langle s(\xi), s(\xi')\rangle$ is antisymmetric in $\xi,\xi'\in A=T^*M\oplus\mathfrak{g}$. We check this on elements of type $(\alpha,0)$, $(\beta,0)$, and we leave the other cases to the reader. Since π is antisymmetric, we only have to show that $\langle T^*\beta, \pi^\sharp(\alpha)\rangle$ is antisymmetric in α and β. Using that $dJ\pi^\sharp=-(\sigma^\vee)^*(\rho_M)^*$, which is the adjoint of the moment map condition (3.22), we see that
$$\pi(\alpha, T^*\beta)=\Big\langle T^*\beta, \pi^\sharp(\alpha)\Big\rangle=-\big\langle\rho_M^*(\beta), \rho^\vee(\sigma^\vee)^*\rho_M^*(\alpha)\big\rangle, \tag{3.39}$$
which is antisymmetric by (3.29).

We now turn to proving that s satifies (3.5). For sections of A of type $\xi=(0,u)$, $\xi'=(0,v)$, with $u,v\in\mathfrak{g}$, (3.5) follows from the next lemma.

Lemma 3.25 *Given a $\mathfrak{g}$-manifold M, and an equivariant map $J: M\longrightarrow G$, then, for any $u,v\in\mathfrak{g}$,*
$$J^*\sigma([u,v])=\mathcal{L}_{\rho_M(u)}(J^*\sigma(v))-\mathcal{L}_{\rho_M(v)}(J^*\sigma(u))+d\langle J^*\sigma(v),\rho_M(u)\rangle+i_{\rho_M(u)\wedge\rho_M(v)}(J^*\phi^G).$$

Proof: Using the equivariance of J, $dJ\rho_M=\rho$, we immediately see that this equation is the pull-back by J of (3.5) for σ (for instance, the last term in the equation equals to $J^*i_{u\wedge v}(\phi^G)$). □

Next we consider the case $\xi=(0,u)$ and $\xi'=(\alpha,0)$, which is handled by the next result.

Lemma 3.26 *Given a $\mathfrak{g}$-manifold M, an equivariant map $J : M \longrightarrow G$, and a bivector π on M satisfying the moment map condition (3.22), then, for all $u \in \mathfrak{g}$ and $\alpha \in T^*M$,*

$$C^*\mathcal{L}_{\rho_M(v)}(\alpha) = \mathcal{L}_{\rho_M(v)}(C^*(\alpha)) - \mathcal{L}_{\pi^\sharp(\alpha)}(J^*\sigma(v)) + d\Big\langle J^*\sigma(v), \pi^\sharp(\alpha)\Big\rangle + i_{\rho_M(v)\wedge\pi^\sharp(\alpha)}(J^*\phi^G).$$

Although Lemma 3.26 is more difficult than Lemma 3.25, it is still simpler than the next case, treated in Lemma 3.27 below. Since a formula that holds under the same assumptions and is proven by the same method is proven in detail in Claim 3.37 below, we will omit its proof.

Let us now consider $\xi = (\alpha, 0)$ and $\xi' = (\beta, 0)$. Comparing with Lemmas 3.25 and 3.26, the greater technical difficulty of this case comes from the fact that now the formulas involve both χ_G and ϕ^G, the bracket $[\cdot,\cdot]$ induced by π on 1-forms, and require more than just the moment map condition.

Lemma 3.27 *Given a $\mathfrak{g}$-manifold M, an equivariant map $J : M \longrightarrow G$, and an invariant bivector π on M satisfying the moment map condition (3.22), then, for all $\alpha, \beta \in T^*M$,*

$$C^*([\alpha,\beta]) - \frac{1}{2}J^*\sigma(i_{(\rho_M)^*(\alpha\wedge\beta)}\chi_G) = \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp\beta)}((\alpha) - d\Big\langle C^*(\beta), \pi^\sharp(\alpha)\Big\rangle + i_{\pi^\sharp(\alpha)\wedge\pi^\sharp(\beta)}(J^*\phi^G)).$$

Proof: Since $C = 1 - \frac{1}{4}T$, using the definition of s and of the bracket in $\Gamma(A)$, we see that we can rewrite the equation in the lemma as

$$\begin{aligned} T^*([\alpha,\beta]) - \mathcal{L}_{\pi^\sharp(\alpha)}(T^*(\beta)) + \mathcal{L}_{\pi^\sharp(\beta)}(T^*(\alpha)) + d\pi(\alpha, T^*(\beta)) = \quad & -2J^*\sigma i_{\rho_M^*(\alpha)\rho_M^*(\beta)}\chi_G \\ & -4i_{\pi^\sharp(\alpha)\wedge\pi^\sharp(\beta)}J^*(\phi^G). \end{aligned} \tag{3.40}$$

Let us evaluate all the terms of (3.40) on an arbitrary vector field $X \in \mathcal{X}(M)$. To simplify the formulas, we set

$$a = \rho_M^*(\alpha), \quad b = \rho_M^*(\beta), \quad V = J(X) \tag{3.41}$$

and we consider the $Hom(\mathfrak{g}^*, \mathfrak{g})$-valued function on G given by

$$D = \rho^\vee(\sigma^\vee)^*.$$

Claim 3.28 *The following formula holds:*

$$\langle d\pi(\alpha, T^*(\beta)), X\rangle = \langle a, \mathcal{L}_V(Db)\rangle - \langle b, \mathcal{L}_V(Da)\rangle - \langle a, \mathcal{L}_V(D)b\rangle. \tag{3.42}$$

Proof: The left hand side of (3.42) is $\mathcal{L}_X\pi(\alpha, T^*\beta)$. Hence, using (3.39), it equals

$$-\big\langle \mathcal{L}_{J(X)}\rho_M^*(\beta), \rho^\vee(\sigma^\vee)^*\rho_M^*(\alpha)\big\rangle - \big\langle \rho_M^*(\beta), \mathcal{L}_{dJ(X)}(\rho^\vee(\sigma^\vee)^*)\big\rangle\rho_M^*(\alpha). \tag{3.43}$$

With the notation of (3.41), and using $D^* = -D$ (i.e. (3.29)) to rewrite the first term, we see that (3.43) equals

$$\langle D\mathcal{L}_V(b), a\rangle - \langle b, \mathcal{L}_V(Da)\rangle.$$

To obtain (3.42), we write $D\mathcal{L}_V(b) = \mathcal{L}_V(Db) - \mathcal{L}_V(D)(b)$. □

From the definition of $[\alpha,\beta]$ (2.4), we have

$$T^*([\alpha,\beta]) = T^*\mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - T^*\mathcal{L}_{\pi^\sharp(\beta)}(\alpha) - T^*d\pi(\alpha,\beta). \tag{3.44}$$

Claim 3.29 *The following formula holds:*

$$\langle T^* d\pi(\alpha, \beta), X \rangle = \pi(\mathcal{L}_{T(X)}(\alpha), \beta) + \pi(\alpha, \mathcal{L}_{T(X)}(\beta)). \tag{3.45}$$

PROOF: This follows from the invariance of π and the fact that the image of T sits inside that of ρ_M. $\square$

Using (3.44) and (3.45), we can split the left hand side of (3.40) as a difference of two terms which are symmetric to each other. The next claim deals with such a term.

Claim 3.30 *The following formula holds:*

$$\begin{aligned} \Big\langle T^* \mathcal{L}_{\pi^\sharp(\alpha)}(\beta) - \mathcal{L}_{\pi^\sharp(\alpha)}(T^*(\beta)), X \Big\rangle &= \pi(\mathcal{L}_{T(X)}(\alpha), \beta) + \langle b, \mathcal{L}_V(Da) \rangle \\ &\quad + \big\langle b, d\rho^\vee((\sigma^\vee)^*(a), V) \big\rangle \end{aligned} \tag{3.46}$$

PROOF: The left hand side of (3.46) equals

$$-\Big\langle \beta, [\pi^\sharp(\alpha), T(X)] + T([\pi^\sharp(\alpha), X]) \Big\rangle. \tag{3.47}$$

To rewrite $[\pi^\sharp(\alpha), T(X)]$, we note that

$$[\pi^\sharp(\alpha), \rho_M(\tilde{v})] = -\pi^\sharp \mathcal{L}_{\rho_M(\tilde{v})}(\alpha) + \rho_M \mathcal{L}_{\pi^\sharp(\alpha)}(\tilde{v}), \tag{3.48}$$

for all $\tilde{v} \in C^\infty(M, \mathfrak{g})$: Indeed, due to $C^\infty(M)$-linearity with respect to $\tilde{v}$, it suffices to check (3.48) for $\tilde{v}$ constant; in this case, the equation is just the invariance of π. We now use (3.48) for $\tilde{v} = \rho^\vee J(X)$ to get

$$[\pi^\sharp(\alpha), T(X)] = -\pi^\sharp \mathcal{L}_{T(X)}(\alpha) + \rho_M \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee dJ(X)).$$

We deduce that (3.47) equals to

$$\begin{aligned} &\Big\langle \beta, \pi^\sharp \mathcal{L}_{T(X)}(\alpha) \Big\rangle - \Big\langle \rho_M^*(\beta), \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee dJ(X)) \Big\rangle + \Big\langle \rho_M^*(\beta), \rho^\vee J[\pi^\sharp(\alpha), X] \Big\rangle = \\ &\pi(\mathcal{L}_{T(X)}(\alpha), \beta) - \Big\langle \rho_M^*(\beta), \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee J)(X) \Big\rangle. \end{aligned} \tag{3.49}$$

On the other hand, for all vector fields Y on M and $\mathfrak{g}$-valued 1-forms ν on G, we have

$$\mathcal{L}_Y(J^*\nu) = \mathcal{L}_{dJ(X)}(\theta(dJ(Y))) + (d\theta)(dJ(Y), dJ(X)).$$

Using this identity for $Y = \pi^\sharp\alpha$ and $\nu = \rho^\vee$ in (3.49), together with the dual of the moment map condition, $dJ\pi^\sharp = -((\sigma)^\vee)^*(\rho_M)^*$, we obtain (3.46) and prove the claim. $\square$

Combining the formulas of claims 3.28, 3.29 and 3.30, we conclude that the left hand side of (3.40) evaluated at a vector field X is

$$\big\langle b, d\rho^\vee((\sigma^\vee)^* a, V) \big\rangle - \big\langle a, d\rho^\vee((\sigma^\vee)^* b, V) \big\rangle - \langle a, \mathcal{L}_V(D) b \rangle. \tag{3.50}$$

On the other hand, the right hand side of (3.40) applied to X equals to

$$\begin{aligned} &-2(\chi_G(\rho_M^*(\alpha), \rho_M^*(\beta), \sigma^* dJ(X)) + 2\phi^G(dJ\pi^\sharp(\alpha), dJ\pi^\sharp(\beta), dJ(X))) = \\ &-2(\chi_G(a, b, \sigma^* V) + 2\phi^G((\sigma^\vee)^* a, (\sigma^\vee)^* b, V)), \end{aligned} \tag{3.51}$$

where we have used again that $dJ\pi^\sharp = -((\sigma)^\vee)^*(\rho_M)^*$.

To conclude the proof of the lemma, it suffices to show that χ_G and ϕ^G are related as follows.

Claim 3.31 *For all $a, b \in \mathfrak{g}^*$ and all vector fields V on G, one has*

$$\begin{aligned} \frac{1}{2}\chi_G(a, b, \sigma^* V) + \phi^G((\sigma^\vee)^* a, (\sigma^\vee)^* b, V) &= \frac{1}{4}(\langle a, d\rho^\vee((\sigma^\vee)^* b, V)\rangle - \langle b, d\rho^\vee((\sigma^\vee)^* a, V)\rangle \\ &\quad +\langle a, \mathcal{L}_V(D)b\rangle). \end{aligned} \tag{3.52}$$

PROOF: It suffices to prove (3.52) on elements of type

$$a = u^\vee, b = v^\vee, V = w_r,$$

where $u, v, w \in \mathfrak{g}$, and we recall that $u^\vee \in \mathfrak{g}^*$ denotes the dual of u with respect to the quadratic form, and w_r is the vector field on G obtained from w by right translations. We will also denote by $\mathrm{Ad}(u) \in C^\infty(G, \mathfrak{g})$ the function $g \mapsto \mathrm{Ad}_g(u)$, and we define $\mathrm{Ad}^{-1}(u)$ similarly. We will need the explicit formulas for σ^* and $(\sigma^\vee)^*$:

$$\sigma^*(w_r) = \frac{1}{2}(w + \mathrm{Ad}^{-1}(w))^\vee, \qquad (\sigma^\vee)^*(u^\vee) = \frac{1}{2}(u_r + u_l).$$

Using these formulas, combined with the invariance of χ_G and ϕ^G, the formula $u_l = \mathrm{Ad}(u_r)$, and the explicit formulas for χ_G and ϕ^G on elements of $\mathfrak{g}$, one can check that the left hand side of (3.52) is

$$\frac{1}{8}\big(([u, v], w + \mathrm{Ad}^{-1}(w))_{\mathfrak{g}} + ([u + \mathrm{Ad}(u), v + \mathrm{Ad}(v)], w)_{\mathfrak{g}}\big). \tag{3.53}$$

Since $\rho^\vee$ is the difference between the right and left Maurer-Cartan forms on G, we have

$$(d\rho^\vee)(u_r, v_r) = [u, v] + \mathrm{Ad}^{-1}[u, v].$$

By the invariance of $(\cdot, \cdot)_{\mathfrak{g}}$ with respect to Ad, we get that

$$\begin{aligned} \langle a, d\rho^\vee((\sigma^\vee)^* b, V)\rangle &= \frac{1}{2}\big((u, [v, w] + \mathrm{Ad}^{-1}([v, w]))_{\mathfrak{g}} + (u, [\mathrm{Ad}(v), w] + [v, \mathrm{Ad}^{-1}(w)])_{\mathfrak{g}}\big) \\ &= \frac{1}{2}([u + \mathrm{Ad}(u), v + \mathrm{Ad}(v)], w)_{\mathfrak{g}}. \end{aligned} \tag{3.54}$$

Since $D = \frac{1}{2}(\mathrm{Ad} - \mathrm{Ad}^{-1})$, and $\mathcal{L}_{w_r}(D)(v) = [w, \mathrm{Ad}(v)] + \mathrm{Ad}^{-1}([w, v])$, it follows that

$$\langle a, \mathcal{L}_V(D)b\rangle = -\frac{1}{2}(([u, \mathrm{Ad}(v)], w)_{\mathfrak{g}} + ([\mathrm{Ad}(u), v], w)_{\mathfrak{g}}).$$

Hence the right hand side of (3.52) equals to

$$\frac{1}{8}(2([u + \mathrm{Ad}(u), v + \mathrm{Ad}(v)], w)_{\mathfrak{g}} - ([u, \mathrm{Ad}(v)], w)_{\mathfrak{g}} - ([\mathrm{Ad}(u), v], w)_{\mathfrak{g}}),$$

which is easily seen to coincide with (3.53). This concludes the proof of the claim. □

Using Claim 3.31, we conclude that (3.50) and (3.51) coincide, and this proves Lemma 3.27 □

From Lemmas 3.25, 3.26 and 3.27, it follows that s is a ϕ^G-IM form for A. To conclude that $L = \mathrm{Im}(r, s)$ is a Dirac structure, we must still prove that L has rank $n = \dim(M)$. This follows from the next lemma (which also serves as inspiration for the proof of Prop. 3.20).

Lemma 3.32 *The sequence*

$$0 \longrightarrow T^*G \xrightarrow{j} A \xrightarrow{(r,s)} L \longrightarrow 0$$

is exact, where $j(a) = (-J^*a, \sigma^\vee(a))$, $a \in T^*G$.

PROOF: The fact that $(r,s) \circ j = 0$ is equivalent to (3.22) and the second formula in (3.30). We define the maps

$$U : A \longrightarrow T^*G, \qquad U(\alpha, v) = -\frac{1}{4}(\rho^\vee)^*\rho_M^*(\alpha) + \sigma(v), \tag{3.55}$$

$$i : L \longrightarrow A, \qquad i(X, \alpha) = (\alpha, \frac{1}{4}\rho^\vee dJ(X)). \tag{3.56}$$

We claim that

$$U \circ j = \mathrm{Id}, \quad (r,s) \circ i = \mathrm{Id}, \quad \text{and} \quad j \circ U + i \circ (r,s) = \mathrm{Id}, \tag{3.57}$$

and these identities imply that the sequence is exact. For the first identity in (3.57), write

$$U(j(a)) = \frac{1}{4}\rho^\vee\rho_M^*J^*a + \sigma\sigma^\vee a,$$

and then, using that $dJ\rho_M = \rho$ and (3.26), we see that $U \circ j = \mathrm{Id}$. The second identity is immediate from the first and the last ones. To prove the last identity, we evaluate $j(U(\alpha, v))$: The first component gives

$$\begin{aligned} -J^*(-\frac{1}{4}(\rho^\vee)^*\rho_M^*(\alpha) + \sigma(v)) &= -J^*\sigma(v) + \alpha - (1 - \frac{1}{4}(\rho_M\rho^\vee J)^*)\alpha \\ &= -(J^*\sigma(v) + C^*(\alpha)) + \alpha \end{aligned} \tag{3.58}$$

The second component is

$$\sigma^\vee(-\frac{1}{4}(\rho^\vee)^*\rho_M^*\alpha + \sigma(v)). \tag{3.59}$$

But $\sigma^\vee(\rho^\vee)^*\rho_M^* = -\rho^\vee(\sigma^\vee)^*\rho_M^* = \rho^\vee dJ\pi^\sharp$, where we have used (3.29), and the moment map condition (3.22). Expressing $\sigma^\vee\sigma$ using (3.25), we see that (3.59) is

$$v - \frac{1}{4}\rho^\vee dJ(\rho_M(v) + \pi^\sharp(\alpha)).$$

Hence

$$j(U(\alpha, v)) = (\alpha, v) - (s(\alpha, v), \frac{1}{4}\rho^\vee dJr(\alpha, v)),$$

i.e. $j \circ U + i \circ (r,s) = \mathrm{Id}$. □

This concludes the proof of Proposition 3.19.

Proof of Proposition 3.20:

Let $J : (M, L) \to (G, L_G)$ be a Dirac realization. Identifying L_G with $\mathfrak{g} \ltimes G$, we know that there is an induced action of $\mathfrak{g}$ on M, denoted by ρ_M. Spelling out the definition, for $v \in \mathfrak{g}$, $\rho_M(v)$ is the unique vector field satisfying the equations in (3.32).

From the second condition in (3.32) and the fact that L is isotropic, we immediately deduce

Lemma 3.33 *For all* $(X, \alpha) \in L$,

$$\rho_M^*(\alpha) + \sigma^* dJ(X) = 0. \tag{3.60}$$

Inspired by Lemma 3.32, we prove:

Lemma 3.34 *There is an exact sequence*

$$0 \longrightarrow L \stackrel{i}{\longrightarrow} T^*M \oplus \mathfrak{g} \stackrel{U}{\longrightarrow} T^*G \longrightarrow 0,$$

where U and i are given by (3.55) and (3.56), respectively.

PROOF: First of all, U is surjective since, as in Lemma 3.32 (and keeping the same notion), $U \circ j = \mathrm{Id}$. Next, $U \circ i = 0$ is an immediate consequence of (3.28) and (3.60). Finally, using the nondegeneracy condition (3.19) for a Dirac realization, it follows that i is injective. By a dimension argument, it follows that the sequence is exact. □

We now concentrate on constructing the quasi-Poisson bivector field π. Following (ii) of Prop. 3.20, we have

Claim 3.35 $\pi^\sharp$ *is well defined.*

PROOF: We first show that (3.33) and (3.34) have a solution X, for any given α: the point is that the element

$$(-C^*(\alpha), \frac{1}{4}\rho^\vee(\sigma^\vee)^*\rho_M^*(\alpha)))$$

is in the kernel of U; this is a simple computation using (3.26) and (3.29). Hence it must be in the image of i. More explicitely, we find that there exists an X such that

$$(X, C^*(\alpha)) \in L, \quad \rho^\vee(dJ(X) + (\sigma^\vee)^*\rho_M^*(\alpha)) = 0. \tag{3.61}$$

On the other hand, applying Lemma 3.33 to $(X, C^*(\alpha))$, and then using the first equation in (3.30) to replace $C\rho_M$, we find that

$$\sigma^*(dJ(X) + (\sigma^\vee)^*\rho_M^*(\alpha)) = 0. \tag{3.62}$$

Since $\mathrm{Ker}(\sigma^*) \cap \mathrm{Ker}(\rho^\vee) = 0$, equations (3.61) and (3.62) imply (3.33). The uniqueness of X follows from the nondegeneracy condition (3.19). □

Claim 3.36 $\pi^\sharp$ *defines a bivector field* π *which satisfies the moment map condition (3.22).*

PROOF: We have to show that

$$\alpha(\pi^\sharp(\beta)) + \beta(\pi^\sharp(\alpha)) = 0$$

for all 1-forms α and β. Let $X = \pi^\sharp(\alpha)$ and $Y = \pi^\sharp(\beta)$. Using (3.34) for (α, X) and (β, Y), the fact that L is isotropic, and the definition of C, we find that

$$4(\alpha(Y) + \beta(X)) = \alpha(\rho_M\rho^\vee dJ(Y)) + \beta(\rho_M\rho^\vee dJ(X)). \tag{3.63}$$

Let us show that the right hand side of (3.63) is zero: using (3.33), (3.63) becomes:

$$\alpha(\rho_M\rho^\vee(\rho_M\sigma^\vee)^*(\beta)) + \beta(\rho_M\rho^\vee(\rho_M\sigma^\vee)^*(\alpha)),$$

and this is zero due to (3.29). On the other hand, (3.33) shows that $dJ\pi^\sharp = -(\rho_M\sigma^\vee)^*$; dualizing it (and using $(\pi^\sharp)^* = -\pi^\sharp$, which holds by the first part of the lemma), we obtain the moment map condition. $\square$

Claim 3.37 *The bivector field π is $\mathfrak{g}$-invariant.*

PROOF: We have to show that $\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)) = \pi^\sharp(\mathcal{L}_{\rho_M(v)}(\alpha))$ for $v\in\mathfrak{g}$, and 1-forms α. For that, it suffices to show that $\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha))$ satisfies (3.33) and (3.34).

$$dJ(\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha))) = -(\rho_M\sigma^\vee)^*\mathcal{L}_{\rho_M(v)}(\alpha), \tag{3.64}$$
$$(\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)), C^*\mathcal{L}_{\rho_M(v)}(\alpha)) \in L \tag{3.65}$$

These conditions are related to Lemma 3.26. Let us first prove (3.65). Using (3.34), (3.32), and the fact that L is isotropic, we conclude that

$$([\rho_M(v),\pi^\sharp(\alpha)], \mathcal{L}_{\rho_M(v)}(C^*\alpha) - \mathcal{L}_{\pi^\sharp(\alpha)}(J^*\sigma(v)) + d\Big\langle J^*\sigma(v),\pi^\sharp(\alpha)\Big\rangle + i_{\rho_M(v)\wedge\pi^\sharp(\alpha)}(J^*\phi^G)) \in L.$$

Using Lemma 3.26, we see that this expression is precisely $(\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)), C^*\mathcal{L}_{\rho_M(v)}(\alpha))$.

Formula (3.64) is closely related to the one in Lemma (3.26): the proofs are similar and hold under the same hypothesis (which might be a bit surprising since (3.64) says that, although the invariance condition on π is not assumed, it must be satisfied modulo the kernel of J). Since we have omited the proof of Lemma 3.26, we will give the details for (3.64). We evaluate both sides of (3.64) on an arbitrary 1-form $\mu \in \Omega^1(G)$. The left hand side gives

$$\begin{aligned}\Big\langle J^*\mu, [\rho_M(v),\pi^\sharp(\alpha)])\Big\rangle &= d(J^*\mu)(\rho_M(v),\pi^\sharp(\alpha)]) + \mathcal{L}_{\rho_M(v)}\Big\langle J^*\mu,\pi^\sharp(\alpha)]\Big\rangle - \mathcal{L}_{\pi^\sharp(\alpha)}\langle J^*\mu,\rho_M(v)\rangle\\ &= -(d\mu)(\rho(v),(\sigma^\vee)^*\rho_M^*(\alpha)) - \mathcal{L}_{\rho_M(v)}\big\langle\mu,(\sigma^\vee)^*\rho_M^*(\alpha)\big\rangle +\\ &\quad \mathcal{L}_{(\sigma^\vee)^*\rho_M^*(\alpha)}\langle\mu,\rho(v)\rangle\end{aligned} \tag{3.66}$$

Evaluating μ on the right hand side, we get

$$-\big\langle\mathcal{L}_{\rho_M(v)}(\alpha),\rho_M\sigma^\vee\mu\big\rangle = -\mathcal{L}_{\rho_M(v)}\big\langle\alpha,\rho_M\sigma^\vee\mu\big\rangle + \big\langle\alpha,[\rho_M(v),\rho_M\sigma^\vee\mu]\big\rangle.$$

Now, using $[\rho_M(v),\rho_M(\tilde v)] = \rho_M([v,\tilde v]) + \rho_M\mathcal{L}_{\rho_M(v)}(\tilde v)$ for $\tilde v = \sigma^\vee\mu \in C^\infty(M,\mathfrak{g})$, we get

$$-\mathcal{L}_{\rho_M(v)}\big\langle\rho_M^*(\alpha),\sigma^\vee\mu\big\rangle + \big\langle\rho_M^*(\alpha),[v,\sigma^\vee\mu]\big\rangle + \big\langle\rho_M^*(\alpha),\mathcal{L}_{\rho_M(v)}(\sigma^\vee\mu)\big\rangle. \tag{3.67}$$

We have to show that this coincides with r.h.s. of (3.66). Comparing the two formulas, we see that the resulting equation makes sense for $\rho_M^*\alpha$ replaced by any element in $C^\infty(M,\mathfrak{g}^*)$. On the other hand, since the equation is $C^\infty(M)$-linear with respect to this element, we may assume that the element is a constant $a\in\mathfrak{g}^*$ (and the remaining apearances of ρ_M will become ρ). The identity to be proven, relating the r.h.s of (3.66) and (3.67), becomes

$$-(d\mu)(\rho(v),(\sigma^\vee)^*a) - \mathcal{L}_{\rho(v)}\big\langle\mu,(\sigma^\vee)^*a\big\rangle + \mathcal{L}_{(\sigma^\vee)^*a}\langle\mu,\rho(v)\rangle = \big\langle a,[v,\sigma^\vee\mu]\big\rangle,$$

or, equivalently,

$$-\mu([\rho(v),(\sigma^\vee)^*a]) = \langle a,[v,\sigma^\vee\mu]\rangle. \tag{3.68}$$

We may assume that μ is the dual (with respect to the quadratic form) of the vector field w_r for some $w \in \mathfrak{g}$, and that a is the dual of an element $u \in \mathfrak{g}$. Equation (3.68) becomes (after multiplying by 2):

$$-(w_r,[v_r - v_l, u_r + u_l]) = (u,[v, w + \mathrm{Ad}^{-1}(w)]),$$

and this can be proven to hold from the invariance of the quadratic form and the identities $[v_l,u_l] = -[v,u]_l$, $[v_r,u_l] = [v_l,u_r]$. □

Claim 3.38 *The bivector field π is a quasi-Poisson tensor.*

PROOF: We must show that $\pi^\sharp([\alpha,\beta]) = [\pi^\sharp(\alpha),\pi^\sharp(\beta)] + \frac{1}{2}i_{\alpha\wedge\beta}(\rho_M(\chi_G))$. Using the definition of $\pi^\sharp$ ((ii) of Prop. 3.20) evaluated at $[\alpha,\beta]$, we have to show that

$$J([\pi^\sharp(\alpha),\pi^\sharp(\beta)] + \frac{1}{2}i_{\alpha\wedge\beta}(\rho_M(\chi_G)) = -(\rho_M\sigma^\vee)^*[\alpha,\beta], \tag{3.69}$$

$$([\pi^\sharp(\alpha),\pi^\sharp(\beta)] + \frac{1}{2}i_{\alpha\wedge\beta}(\rho_M(\chi_G)), C^*([\alpha,\beta])) \in L. \tag{3.70}$$

. Similarly to the discussion in the previous claim, these conditions are related to Lemma 3.27. The first equation holds under the same assumptions, and it is proven by the same method, so it will be left to the reader (similar to the discussion in the previous proof, the equation tells us that, although the quasi-Poisson condition is not assumed, it must be satisfied modulo the kernel of J).

We now prove (3.70). First, we use that $(\pi^\sharp(\alpha),C^*(\alpha)) \in L$, $(\pi^\sharp(\beta),C^*(\beta)) \in L$, the fact that L is isotropic, and then apply the formula in Lemma 3.27, to conclude that

$$([\pi^\sharp(\alpha),\pi^\sharp(\beta)], C^*([\alpha,\beta]) - \frac{1}{2}J^*\sigma i_{\rho_M^*(\alpha\wedge\beta)}(\chi_G)) \in L. \tag{3.71}$$

On the other hand, applying the action in (3.32) to $v = i_{\rho_M^*(\alpha\wedge\beta)}(\chi_G)$ and observing that $\rho_M(v) = i_{\alpha\wedge\beta}(\rho_M(\chi_G))$, we find that

$$(i_{\alpha\wedge\beta}(\rho_M(\chi_G)), J^*\sigma i_{\rho_M^*(\alpha\wedge\beta)}(\chi_G)) \in L. \tag{3.72}$$

Since L, at each point, is a vector space, (3.71) and (3.72) imply (3.70). □

4 Moment maps in Dirac geometry: the global picture

4.1 Integrating Lie algebroids and infinitesimal actions

Lie groupoids are the global counterparts of Lie algebroids. In order to fix our notation, we recall that a Lie groupoid over a manifold M consists of a manifold $\mathcal{G}$ together with surjective submersions $\mathsf{t},\mathsf{s} : \mathcal{G} \to M$, called **target** and **source**, a partially defined multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G}\times\mathcal{G} \,|\, \mathsf{s}(g) = \mathsf{t}(h)\}$, a **unit section** $\varepsilon : M \to \mathcal{G}$ and an

inversion $i : \mathcal{G} \to \mathcal{G}$, all related by the appropriate axioms, see e.g. [8]. To simplify our notation, we will often identify an element $x \in M$ with its image $\varepsilon(x) \in \mathcal{G}$.

For a Lie groupoid $\mathcal{G}$, the associated Lie algebroid $A(\mathcal{G})$ consists of the vector bundle

$$\ker(d\mathsf{s})|_M \to M,$$

with anchor $\rho = d\mathsf{t} : \ker(d\mathsf{s})|_M \to TM$ and bracket induced from the Lie bracket on $\mathcal{X}(\mathcal{G})$ via the identification of sections $\Gamma(\ker(d\mathsf{s})|_M)$ with right-invariant vector fields on $\mathcal{G}$ tangent to the s-fibers.

An **integration** of a Lie algebroid A is a Lie groupoid $\mathcal{G}$ together with an isomorphism $A \cong A(\mathcal{G})$. Unlike Lie algebras, not every Lie algebroid admits an integration, see [14] for a description of the obstructions. On the other hand, if a Lie algebroid is integrable, then there exists a canonical source-simply-connected integration $\mathcal{G}(A)$, see [14].

If M is a point, then a Lie groupoid over M is a Lie group, and the associated Lie algebroid is its Lie algebra.

Example 4.1 (*Transformation Lie groupoids*)

Let G be a Lie group acting from the left on a manifold M. The associated **transformation Lie groupoid**, denoted by $G \ltimes M$, is a Lie groupoid over M with underlying manifold $G \times M$, source map $\mathsf{s}(g,x) = x$, target map $\mathsf{t}(g,x) = g \cdot x$, and multiplication

$$(g,x) \cdot (g',x') = (gg',x').$$

In this case, $A(G \ltimes M) = \mathfrak{g} \ltimes M$, the transformation Lie algebroid associated with the the infinitesimal action of $\mathfrak{g}$ on M corresponding to the given G-action. (However, even if a $\mathfrak{g}$-action does not come from a global action of a Lie group, one can always find a Lie groupoid integrating the transformation Lie algebroid $\mathfrak{g} \ltimes M$, see [16, 26].)

Similarly to infinitesimal actions, Lie groupoids act on maps into their identity sections: if $\mathcal{G}$ is a Lie groupoid over M, then a (left) **action of $\mathcal{G}$ on a map** $J : N \to M$ is a map $m_N : \mathcal{G} \times_M N \to N$, $(g,y) \mapsto g \cdot y$, satisfying

1. $J(g \cdot y) = \mathsf{t}(g)$,
2. $(gg')y = g(g'y)$,
3. $J(y) \cdot y = y$.

Here $\mathcal{G} \times_M N := \{(g,y) \in \mathcal{G} \times N \,|\, \mathsf{s}(g) = J(y)\}$. For reasons that will be clear in the next two subsections, the map $J : N \to M$ is often referred to as the **moment map** of the action m_N [25].

Example 4.2 (*Actions of transformation Lie algebroids*)

Analogously to Example 3.7, an action m_N of a transformation Lie groupoid $\mathcal{G} = G \ltimes M$ on a map $J : N \to M$ is equivalent to an ordinary action $\overline{m_N}$ of the Lie group G on N for which J is G-equivariant. Indeed, m_N and $\overline{m_N}$ are related by

$$m_N((g,J(y)),y) = \overline{m_N}(g,y), \quad \text{where } g \in G \text{ and } y \in N. \tag{4.1}$$

The link between infinitesimal and global actions is based on the following notion: An infinitesimal action ρ_N of a Lie algebroid A is called **complete** if $\rho_N(\xi) \in \mathcal{X}(N)$ is a complete vector field whenever $\xi \in \Gamma(A)$ has compact support. As in the case of Lie algebras, a complete action of a Lie algebroid A can be integrated to an action of its canonical source-simply-connected integration $\mathcal{G}(A)$, see e.g. [26].

4.2 Poisson maps as moment maps for symplectic groupoid actions

A 2-form ω on a Lie groupoid $\mathcal{G}$ is called **multiplicative** if the graph of the groupoid multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is an isotropic submanifold of $(\mathcal{G}, \omega) \times (\mathcal{G}, \omega) \times (\mathcal{G}, -\omega)$. Equivalently, the multiplicativity condition for ω can be written as

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega, \tag{4.2}$$

where $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1, 2$, are the canonical projections. A **symplectic groupoid** [29] is a Lie groupoid together with a multiplicative symplectic form.

Symplectic groupoids are the global counterparts of Poisson manifolds in the following sense: If π is a Poisson structure on a manifold P inducing an integrable Lie algebroid structure on $A = T^*P$ (as in Section 2.2), then the associated source-simply-connected groupoid $\mathcal{G}(P) := \mathcal{G}(A)$ carries a natural multiplicative symplectic structure [9, 15, 23]; on the other hand, on any symplectic groupoid $(\mathcal{G}, \omega)$ over a manifold P, condition (4.2) automatically implies that P has an induced Poisson structure uniquely determined by the condition that the target map $\mathsf{t} : \mathcal{G} \to P$ (resp. source map $\mathsf{s} : \mathcal{G} \to P$) is a Poisson map (resp. anti-Poisson map) [11].

An **integration** of a Poisson manifold (P, π) is a symplectic groupoid $(\mathcal{G}, \omega)$ over P for which the induced Poisson structure coincides with π. Note that the symplectic form ω defines a vector bundle map

$$\ker(d\mathsf{s})|_P \longrightarrow T^*P, \;\; \xi \mapsto i_\xi \omega|_{TP} \tag{4.3}$$

inducing an isomorphism of Lie algebroids $A(\mathcal{G}) \cong T^*P$ [11]. This immediately implies that $\dim(\mathcal{G}) = 2\dim(P)$.

Example 4.3 (*Integrating Lie-Poisson structures*)
Let us consider $\mathfrak{g}^*$, equipped with its Lie-Poisson structure. If G is a Lie group with Lie algebra $\mathfrak{g}$, then the transformation groupoid $\mathcal{G} = G \ltimes \mathfrak{g}^*$, with respect to the coadjoint action, integrates $T^*\mathfrak{g}^* = \mathfrak{g} \ltimes \mathfrak{g}^*$. The identification $G \times \mathfrak{g}^* \cong T^*G$ by right translations induces a multiplicative symplectic form ω on $\mathcal{G}$, in such a way that $(\mathcal{G}, \omega)$ is a symplectic groupoid integrating $\mathfrak{g}^*$.

Remark 4.4 The construction of the symplectic groupoid in the previous example can be extended to the context of Poisson-Lie groups, see Remark 3.10: If (G, π) is a simply-connected Poisson-Lie group and G^* is its dual, then, assuming that the dressing action is complete, the transformation groupoid $G \ltimes G^*$ carries a symplectic structure making it into a symplectic groupoid integrating G^*. (This symplectic structure is basically the one associated with the semi-direct product Poisson structure on $G \times G^*$ induced from the action of G on itself by right multiplication.) For a more general construction when the actions are not complete, see [21].

Let us assume that P is an integrable Poisson manifold. We have seen that any Poisson map $J : Q \to P$ induces a Lie algebroid action of T^*P on Q. Analogous to the case of Lie algebras, when this action is complete, it can be "integrated" to an action of $\mathcal{G}(P)$, the canonical source-simply-connected symplectic groupoid of P. We remark that the completeness of the T^*P action in the Lie algebroid sense coincides with the notion of $J : Q \to P$ being complete as a Poisson map, i.e., if $f \in C^\infty(P)$ has compact support (or if X_f is complete), then $X_{J^*(f)}$ is complete.

The global action $m_N : \mathcal{G}(P) \times_P Q \to Q$ arising in this way is compatible with the Poisson structure on Q in the sense that $\mathrm{graph}(m_N)$ is a *lagrangian* submanifold of $(\mathcal{G}(P), \pi) \times (Q, \pi_Q) \times$

$(Q, -\pi_Q)^2$, where π is the Poisson structure associated with the symplectic form ω on $\mathcal{G}(P)$. Since inclusions of symplectic leaves of Poisson manifolds are Poisson maps, an equivalent way to express this compatibility is that the restricted action $m_N : \mathcal{G}(P) \times_P S \to S$ to each symplectic leaf $(S, \omega_S) \hookrightarrow (Q, \pi_Q)$ satisfies

$$m_N^* \omega_S = \mathrm{pr}_{\mathcal{G}}^* \omega + \mathrm{pr}_S^* \omega_S, \tag{4.4}$$

where $\mathrm{pr}_{\mathcal{G}} : \mathcal{G}(P) \times_P S \to \mathcal{G}(P)$ and $\mathrm{pr}_S : \mathcal{G}(P) \times_P S \to S$ are the natural projections, see [25, 32]. On the other hand, if (Q, π_Q) is a Poisson manifold and m_N is an action of a symplectic groupoid $\mathcal{G}$ on $J : Q \to P$ compatible with π_Q in the sense just described, then J is automatically a Poisson map (this is just a leafwise version of [25, Thm. 3.8]).

The next example is the global version of Example 3.8.

Example 4.5 (*Global hamiltonian actions*)

Consider $\mathfrak{g}^*$ with its Lie-Poisson structure, and let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. As in Example 3.8, the starting point is a Poisson map $J : Q \to \mathfrak{g}^*$. Note that J is complete as a Poisson map if and only if the associated infinitesimal $\mathfrak{g}$-action is by complete vector fields. In this case, the global action of the symplectic groupoid $T^*G \cong G \ltimes \mathfrak{g}^*$ is equivalent, in the sense of Example 3.7, to the hamiltonian G-action obtained by integrating the infinitesimal hamiltonian $\mathfrak{g}$-action on Q.

So, in the previous example, the "moment" $J : Q \to \mathfrak{g}^*$ for the symplectic groupoid action of T^*G^* is just a momentum map for a hamiltonian G-action in the ordinary sense.

Remark 4.6 Analogously to the previous example and following Remarks 3.10 and 4.4, a Poisson map $J : Q \to G^*$, where G^* is the dual group to a complete simply-connected Poisson Lie group, can be "integrated" to an action of the symplectic groupoid $G \ltimes G^*$, which is equivalent to a G-action on Q for which J is equivariant (with respect to the dressing action on G^*). The "moment" J in this case coincides with Lu's momentum map [20] for a Poisson action of a Poisson-Lie group on a Poisson manifold.

4.3 Dirac realizations as moment maps for presymplectic groupoid actions

In order to describe the global actions "integrating" Dirac realizations, we should first identify the global objects integrating Dirac manifolds, generalizing symplectic groupoids. This was done in [6]: if ϕ is a closed 3-form on M, then a **ϕ-twisted presymplectic groupoid** over M is a Lie groupoid $\mathcal{G}$ over M equipped with a multiplicative 2-form ω such that

1. $d\omega = \mathsf{s}^*\phi - \mathsf{t}^*\phi$,
2. $\dim(\mathcal{G}) = 2\dim(M)$,
3. $\ker(\omega_x) \cap \ker(d_x\mathsf{s}) \cap \ker(d_x\mathsf{t}) = \{0\}$, for all $x \in M$.

(Twisted presymplectic groupoids are called **quasi-symplectic groupoids** in [33].) Condition 1. in this definition guarantees that the map

$$\sigma_\omega : A \to T^*M, \quad \xi \mapsto i_\xi \omega|_{TM} \tag{4.5}$$

[2] A submanifold C of a Poisson manifold (P, π) is **lagrangian** if, at each $x \in P$, the intersection of T_xC with $\widetilde{\pi}(T_x^*P)$, the tangent space to the symplectic leaf at x, is a lagrangian subspace of $\widetilde{\pi}(T_x^*P)$.

is a ϕ-IM form for A, while 2. and 3. are the extra-conditions needed in Lemma 3.3 to insure that the image L of (ρ, σ_ω) is a ϕ-twisted Dirac structure. When (G, ω) is a symplectic groupoid, such L is precisely the Dirac structure associated with the induced Poisson structure on M. As proven in [6], L is uniquely determined by the condition that t is an f-Dirac map (resp., s is an anti-f-Dirac map). Conversely, the canonical groupoid $\mathcal{G}(L)$ integrating the Lie algebroid associated with a ϕ-twisted Dirac structure (assuming it is integrable) is naturally a ϕ-twisted presymplectic groupoid [6, Sec. 5]. This correspondence generalizes the one between Poisson manifolds and symplectic groupoids [9, 15, 23] (see also [10] for the integration of twisted Poisson structures).

We now have all the ingredients to generalize the "integration" procedure of Poisson maps to symplectic groupoid actions, explained in Section 4.2, to the context of Dirac geometry. Let L_M be a ϕ-twisted Dirac structure on M associated with an integrable Lie algebroid. We call a Dirac realization $J : N \to M$ **complete** if the induced Lie algebroid action of L_M on N is complete, in which case it integrates to an action $m_N : \mathcal{G}(L_M) \times_M N \to N$, where $(\mathcal{G}(L_M), \omega)$ is the canonical twisted presymplectic groupoid associated with L_M. In this situation, we will simply say that the action m_N **integrates** the realization J.

Theorem 4.7 *Let (M, L_M) be a ϕ-twisted Dirac manifold and assume that L_M is integrable. A complete Dirac realization $J : N \to M$ integrates to an action $m_N : \mathcal{G}(L_M) \times_M N \to N$ satisfying*

$$m_N^* L_N = \tau_{\mathrm{pr}_{\mathcal{G}}^* \omega}(\mathrm{pr}_N^* L_N), \tag{4.6}$$

where $\mathrm{pr}_{\mathcal{G}}$ and pr_N are the projections from $\mathcal{G}(L_N) \times_M N$ onto $\mathcal{G}(L_N)$ and N, respectively, and $\tau_{\mathrm{pr}_{\mathcal{G}}^* \omega}$ denotes a gauge transformation.

Conversely, if m_N is an action of $\mathcal{G}(L_M)$ on $J : N \to M$ satisfying (4.6)*, then J is f-Dirac; if J also satisfies* (3.19)*, then it is a Dirac realization whose integration is m_N.*

In order to prove the theorem, we need the following result.

Lemma 4.8 *Let (M, L_M) be a ϕ-twisted Dirac manifold and assume that L_M is integrable. Let $m_N : \mathcal{G}(L_M) \times_M N \to N$ be an action of $\mathcal{G}(L_M)$ on $J : N \to M$, and assume that N is equipped with a $J^*\phi$-twisted presymplectic form ω_N. Then J is an f-Dirac map if and only if*

$$m_N^* \omega_N = \mathrm{pr}_N^* \omega_N + \mathrm{pr}_{\mathcal{G}}^* \omega. \tag{4.7}$$

Proof: To simplify the notation, let $\mathcal{G} = \mathcal{G}(L_M)$, and let us denote by A the corresponding Lie algebroid (which is just L_M). The source and target maps in $\mathcal{G}$ are denoted by s and t. Also, let $\omega_1 = m_N^* \omega_N - \mathrm{pr}_N^* \omega_N$ and $\omega_2 = \mathrm{pr}_{\mathcal{G}}^* \omega$. With these definitions, our goal is to show that J is f-Dirac if and only if $\omega_1 = \omega_2$.

The key observation is that if we regard $\mathcal{G} \times_M N$ as a transformation Lie groupoid over N, with source pr_N and target m_N, a direct computation shows that both ω_1 and ω_2 are multiplicative. Hence, by [6, Thm. 2.5], $\omega_1 = \omega_2$ if and only if the corresponding bundle maps

$$\sigma_{\omega_i} : A \times_M N \to T^*N, \;\; \xi_y \mapsto \sigma_{\omega_i} = (i_{\xi_y} \omega_i)|_{TN}$$

$i = 1, 2$, see (4.5), coincide. For $\xi_y \in A \times_M N$ ($\xi \in A_x$ and $x = J(y)$) and $Y \in TN$ (as usual, we identity TN with $T\varepsilon(N)$, where $\varepsilon : N \to \mathcal{G} \times_M N$ is the identity section), we have

$$\sigma_{\omega_1}(\xi_y, Y) = \omega_N(dm_N(\xi_y), Y) \quad \text{and} \quad \sigma_{\omega_2}(\xi_y, Y) = \omega(\xi, dJ(Y)). \tag{4.8}$$

For the first identity in (4.8), we used that $i_{\xi_y}\mathrm{pr}_N^*\omega_N = 0$ for $\xi_y \in A \times_M N$, since pr_N is the source map in $\mathcal{G} \times_M N$, and $A \times_M N$ is its Lie algebroid, which is tangent to the source fibres along the identity section.

Since $L_M = \{(d\mathsf{t}(\xi), i_\xi\omega|_{TM}) \mid \xi \in A\}$, $J : N \to M$ being f-Dirac means that

$$\{(d\mathsf{t}(\xi), i_\xi\omega|_{TM}) \mid \xi \in A\} = \{(dJ(Y), \alpha) \mid i_Y\omega_N = J^*\alpha\}. \tag{4.9}$$

But, for $\xi_y \in A \times_M N$, we have $dJ(dm_N(\xi_y)) = d\mathsf{t}(\xi)$. It then follows from (4.9) that

$$\omega(\xi, dJ(Y)) = \omega_N(dm_N(\xi_y), Y)$$

for all $Y \in TN$, which implies that $\sigma_{\omega_1} = \sigma_{\omega_2}$, i.e., $\omega_1 = \omega_2$.

The converse follows from the same arguments, reversing the steps. $\square$

We can now prove Theorem 4.7:

PROOF: We keep writing $\mathcal{G}$ for $\mathcal{G}(L_M)$. Suppose that m_N integrates a Dirac realization $J : N \to M$. The bundles $m_N^*L_N$ and $\mathrm{pr}_N^*L_N$, seen as subbundles of $T(\mathcal{G}\times_M N)\oplus T^*(\mathcal{G}\times_M N)$, have the same projection onto the first factor: at a point (g, y), they both coincide with $T_g\mathcal{G} \times T_y\mathcal{O}$, where $\mathcal{O}$ is the leaf of L_N through y. Note that, since pr_N is a submersion, $\mathrm{pr}_N^*L_N$ is a smooth subbundle, so it is an honest Dirac structure.

By Corollary 3.13, since J is a Dirac realization of M, its restriction to any leaf of L_N, $(\mathcal{O}, \theta)$, is a presymplectic realization, and m_N is tangent to the leaves. By Lemma 4.8,

$$m_N^*\theta = \mathrm{pr}_N^*\theta + \mathrm{pr}_{\mathcal{G}}^*\omega,$$

which implies the compatibility (4.6).

Conversely, (4.6) implies that J is tangent to the leaves of L_N. Restricting m_N to these leaves, (4.6) amounts to (4.7). So, by Lemma 4.8, J is an f-Dirac map when restricted to each leaf, which implies that J is f-Dirac by Corollary 3.6. The last statement follows from Corollary 3.13 and a direct check. $\square$

Remark 4.9 The presymplectic groupoid actions resulting from *presymplectic* realizations are exactly the "modules" considered in the Morita theory developed in [33] to compare various notions of moment maps. More general Dirac realizations give rise to more general "hamiltonian spaces" which still fit with the constructions in [33].

We will discuss examples of the "integration" in Theorem 4.7 related to "quasi" hamiltonian actions in Section 4.5.

4.4 Reduction in Dirac geometry

Just as in Poisson geometry, one can also carry out reduction in the context of Dirac manifolds. The general construction described in this section recovers reduction procedures in various settings, including [2, 3, 25, 33].

The set-up is as follows. Let $J : N \to M$ be a Dirac realization of a ϕ-twisted Dirac manifold (M, L_M). Let $x \in M$ be a regular value of J, and consider the submanifold $\iota : \mathcal{C} = J^{-1}(x) \hookrightarrow N$. Following [25, 33], let $\mathfrak{l}_x = \ker(\rho)_x$ be the isotropy Lie algebra of L_M at x. Since the anchor ρ is the projection $\mathrm{pr}_1|_{L_M}$, it follows that

$$\mathfrak{l}_x = (L_M \cap T^*M)_x. \tag{4.10}$$

The induced Lie algebroid action of L_M on $J : N \to M$ defines a vector bundle morphism $L_M \times_M N \to TN$, and a simple computation shows that this morphism gives rise to an action of the Lie algebra $\mathfrak{l}_x$ on $\mathcal{C}$. Our object of interest is the orbit space $\mathcal{C}/\mathfrak{l}_x$.

Lemma 4.10 *If the stabilizer algebras of the $\mathfrak{l}_x$-action on $\mathcal{C}$ have constant dimension, then $\iota^* L_N$ is a (untwisted) Dirac structure on $\mathcal{C}$.*

Proof: As mentioned in Section 3.2, the conclusion in the lemma holds as long as we show that $\iota^* L_N$ is a *smooth* subbundle of $T\mathcal{C} \oplus T^*\mathcal{C}$.

As a vector bundle, $\iota^* L_N$ is naturally identified with

$$\frac{L_N \cap (T\mathcal{C} \oplus T^*N)}{(L_N \cap T\mathcal{C}^\circ)}, \tag{4.11}$$

see [12], and $\iota^* L_M$ will be smooth if we show that both bundles in (4.11) are smooth. For that, it suffices to show that each one has constant dimension. But since their quotient $\iota^* L_N$ has constant dimension, it suffices to show that either $L_N \cap (T\mathcal{C} \oplus T^*N)$ or $L_N \cap T\mathcal{C}^\circ$ has constant dimension. We will prove that for $L_N \cap T\mathcal{C}^\circ$.

On one hand,

$$L_N \cap T\mathcal{C}^\circ = \{(0, \beta) \in L_N \mid \iota^*\beta = 0\} = \{(0, dJ^*\alpha) \in L_N \mid \alpha \in T^*M\}.$$

It follows from J being an f-Dirac map that if $(0, dJ^*\alpha) \in L_N$, then $(0, \alpha) \in L_M$. So, if ρ_N is the infinitesimal action of L_M on J, we can write

$$L_N \cap T\mathcal{C}^\circ = \{(0, dJ^*\alpha) \in L_N \mid \alpha \in T^*M\} \cong \frac{\ker(\rho_N) \cap L_M \cap T^*M}{\ker(dJ^*)}.$$

But $\ker(\rho_N) \cap L_M \cap T^*M$ is the stabilizer of the $\mathfrak{l}_x$-action on $\mathcal{C}$, which is assumed to have constant dimension. Since x is a regular value, dJ has maximal rank on $\mathcal{C}$, so $\ker(dJ^*)$ also has constant dimension. As a result, the dimension of (4.11) is constant, and $\iota^* L_N$ is a smooth bundle.

Finally, note that $\iota^* L_N$ is a $(\iota^* J^* \phi)$-Dirac structure on $\mathcal{C}$, but $\iota^* J^* \phi = 0$. So $\iota^* L_N$ is an ordinary Dirac structure. $\square$

We now show that the quotient $\mathcal{C}/\mathfrak{l}_x$ carries a natural Poisson structure.

Theorem 4.11 *Suppose that the orbit space $\mathcal{C}/\mathfrak{l}_x$ is a smooth manifold so that projection $\mathcal{C} \longrightarrow \mathcal{C}/\mathfrak{l}_x$ is a submersion. Then there is a unique Poisson structure π_{red} on $\mathcal{C}/\mathfrak{l}_x$ for which the projection $(\mathcal{C}, \iota^* L_N) \to (\mathcal{C}/\mathfrak{l}_x, \pi_{red})$ is an f-Dirac map.*

Remark 4.12 The projection $(\mathcal{C}, \iota^* L_N) \to (\mathcal{C}/\mathfrak{l}_x, \pi_{red})$ is also a b-Dirac map, and this property characterizes π_{red} uniquely as well.

Proof: It follows from our assumptions that the $\mathfrak{l}_x$-orbits on $\mathcal{C}$ have constant dimension, so the same holds for the stabilizer algebras. By Lemma 4.10, $(\mathcal{C}, \iota^* L_N)$ is a Dirac manifold.

The *admissible functions* on $(\mathcal{C}, \iota^* L_N)$, i.e., the set of functions on $\mathcal{C}$ whose differential vanish on $\ker(\iota^* L_N) = \iota^* L_N \cap T\mathcal{C}$ form a Poisson algebra, see [12, Sec. 2.5], under the bracket

$$\{f, g\} := \mathcal{L}_{X_f} g,$$

where X_f is a local vector field such that $(X_f, df) \in \iota^* L_N$. We will show that this Poisson algebra induces a Poisson structure on $\mathcal{C}/\mathfrak{l}_x$ by showing that the kernel of $\iota^* L_N$ coincides with the $\mathfrak{l}_x$-orbits, i.e.,

$$\ker(\iota^* L_N) = \rho_N(\mathfrak{l}_x). \tag{4.12}$$

On one hand,

$$\begin{aligned} \iota^* L_N \cap TQ &= \{Y \in TQ \mid \exists \beta \in T^*N \text{ with } (Y, \beta) \in L_N,\ \iota^*\beta = 0\} \\ &= \{Y \in TQ \mid \exists \alpha \in T^*M \text{ with } (Y, dJ^*\alpha) \in L_N\}. \end{aligned}$$

But since J is f-Dirac and $dJ(Y) = 0$, we can write

$$\iota^* L_N \cap TQ = \{Y \in TQ \mid \exists \alpha \in L_M \cap T^*M \text{ with } (Y, dJ^*\alpha) \in L_N\}.$$

On the other hand,

$$\begin{aligned} \rho_N(\mathfrak{l}_x) &= \{Y \in TN \mid \exists \alpha \in L_M \cap T^*M \text{ with } (Y, dJ^*\alpha) \in L_N,\ dJ(Y) = 0\} \\ &= \{Y \in TQ \mid \exists \alpha \in L_M \cap T^*M \text{ with } (Y, dJ^*\alpha) \in L_N\}. \end{aligned}$$

So (4.12) follows.

The fact that the projection $(\mathcal{C}, \iota^* L_N) \to (\mathcal{C}/\mathfrak{l}_x, \pi_{red})$ is an f-Dirac map and the claim in Remark 4.12 follow from a direct computation, see e.g. [7]. □

Of course, if the Dirac realization $J : N \to M$ is complete, one can state Theorem 4.11 in terms of the action of the isotropy group of $\mathcal{G}(L_M)$ at $x \in M$ on $\mathcal{C} = J^{-1}(x)$. A version of Theorem 4.11 can also be derived when this action is locally free and the quotient is an orbifold.

Remark 4.13 (*Other reductions*)

The following are important particular cases of the reduction in Theorem 4.11:

- If M is Poisson and $J : N \to M$ is a symplectic realization, we recover [25, Thm.3.12]; in particular, when $M = \mathfrak{g}^*$, this reduces to Marsden-Weinstein classical theorem [24], and when $M = G^*$, the dual of a Poisson-Lie group, we get Lu's reduction [20]. If $J : N \to M$ is a Poisson map, we get the "Poisson-version" of these results.
- If M is ϕ-twisted Dirac and $J : N \to M$ is a presymplectic realization, then we obtain Xu's reduction [33, Thm. 3.17]; in particular, when M is a Lie group equipped with Cartan-Dirac structure, one recovers the quasi-hamiltonian reduction of [3].
- If $J : N \to G$ is a general Dirac realization of a Lie group with Cartan-Dirac structure, then we recover the reduction of quasi-Poisson manifolds of [2] via the identification established in Theorem 3.16, see Remark 4.16 below.

4.5 AMM-groupoids and hamiltonian quasi-Poisson G-manifolds

We now discuss global actions, in the sense of Theorem 4.7, associated with complete Dirac realizations of Cartan-Dirac structures.

Let G be a Lie groups equipped with a Cartan-Dirac structure L_G with respect to a bi-invariant nondegenerate quadratic form $(\cdot,\cdot)_{\mathfrak{g}}$. The first step is to identify $\mathcal{G}(L_G)$, the canonical presymplectic groupoid integrating L_G.

As shown in [6, Sec. 7], $\mathcal{G}(L_G)$ is closely related to the AMM-groupoids of [5]: if $\mathcal{G} = G \ltimes G$ is the transformation groupoid with respect to the conjugation action, then the 2-form [3]

$$\omega_{(g,x)} = \frac{1}{2}\big(\big(\mathrm{Ad}_x p_g^*\lambda, p_g^*\lambda\big)_{\mathfrak{g}} + \big(p_g^*\lambda, p_x^*(\lambda + \overline{\lambda})\big)_{\mathfrak{g}}\big),$$

where $p_g, p_x : G \times G \to G$ are the first and second projections, and λ and $\overline{\lambda}$ are the left and right Maurer-Cartan forms, makes $\mathcal{G}$ into a ϕ^G-twisted presymplectic groupoid. If G is simply-connected, then $(\mathcal{G}, \omega)$ is isomorphic to $\mathcal{G}(L_G)$, the canonical source-simply-connected integration of L_G. In general, $\mathcal{G}(L_G)$ is obtained from the AMM groupoid by pulling back ω to $\widetilde{G} \ltimes G$, where $\widetilde{G}$ is the universal cover of G [6, Thm. 7.6]. As a result, just as Lie-Poisson structures "integrate" to cotangent bundles of Lie groups, see Example 4.3, Cartan-Dirac structures "integrate" to the "double" $(G \times G, \omega)$ in the sense of [3].

For simplicity, let G be simply connected. A complete Dirac realization $J : M \to G$ induces a presymplectic groupoid action of $(\mathcal{G}, \omega)$, as in Theorem 4.7, which is equivalent to a G-action on M for which J is G-equivariant, see Example 4.2; this G-action is just an integration of the infinitesimal $\mathfrak{g}$-action which makes M into a quasi-Poisson $\mathfrak{g}$-manifold, as constructed in Proposition 3.20. So M becomes a hamiltonian quasi-Poisson G-manifold for which $J : M \to G$ is the group valued moment map [2]. This construction yields the following global version of Theorem 3.16.

Theorem 4.14 *There is a one-to-one correspondence between complete Dirac realizations of* (G, L_G) *and hamiltonian quasi-Poisson* G*-manifolds.*

Corollary 4.15 *There is a one-to-one correspondence between compact Dirac realizations of* (G, L_G) *and compact hamiltonian quasi-Poisson* G*-manifolds.*

Of course, a global version of Prop. 3.24 also holds.

Remark 4.16 (*Reduction*)
Given a Dirac realization of (G, L_G), $J : M \to G$, the Dirac reduction of Theorem 4.11 produces Poisson spaces $J^{-1}(g)/G_g$, where G_g is the centralizer of $g \in G$. Using Remark 4.13 and [2, Prop. 10.6], one can check that these are the same Poisson spaces obtained by quasi-Poisson reduction [2, Thm.6.1] if we regard M as a hamiltonian quasi-Poisson G-manifold instead.

References

[1] Alekseev, A., Kosmann-Schwarzbach, Y.: *Manin pairs and moment maps* J. Differential Geom. **56** (2000), 133-165.

[2] Alekseev, A., Kosmann-Schwarzbach, Y., Meinrenken, E.: *Quasi-Poisson manifolds* Canadian J. Math. **54** (2002), 3–29.

[3] Alekseev, A., Malkin, A., Meinrenken, E.: *Lie group valued moment maps.* J. Differential Geom. **48** (1998), 445–495.

[4] Bangoura, M., Kosmann-Schwarzbach, Y.: *The double of a Jacobian quasi-bialgebra.* Lett. Math. Phys. **28** (1993), 13–29.

[5] Behrend. K., Xu, P., Zhang, B.: *Equivariant gerbes over compact simple Lie groups.* C. R. Acad. Sci. Paris **336** (2003), 251–256.

[6] Bursztyn, H., Crainic, M., Weinstein, A., Zhu, C.: *Integration of twisted Dirac brackets.* Duke Math. J., to appear. (Math.DG/0303180.)

[7] Bursztyn, H., Radko, O.: *Gauge equivalence of Dirac structures and symplectic groupoids.* Ann. Inst. Fourier (Grenoble) **53** (2003), 309–337.

[8] Cannas da Silva, A., Weinstein, A.: *Geometric models for noncommutative algebras.* American Mathematical Society, Providence, RI, 1999.

[9] Cattaneo, A., Felder, G.: *Poisson sigma models and symplectic groupoids.* In: *Quantization of singular symplectic quotients*, Progr. Math. 198, 61–93, Birkhäuser, Basel, 2001.

[10] Cattaneo, A., Xu, P.: *Integration of twisted Poisson structures.* J. Geom. Physics, to appear. (Math.SG/0302268)

[11] Coste, A., Dazord, P., Weinstein, A.: *Groupoïdes symplectiques.* In: *Publications du Département de Mathématiques. Nouvelle Série. A, Vol. 2*, i–ii, 1–62. Univ. Claude-Bernard, Lyon, 1987.

[12] Courant, T.: *Dirac manifolds.* Trans. Amer. Math. Soc. **319** (1990), 631–661.

[13] Courant, T., Weinstein, A.: *Beyond Poisson structures.* Séminaire sud-rhodanien de g/'eométrie VIII. Travaux en Cours **27**, Hermann, Paris (1988), 39-49.

[14] Crainic, M., Fernandes, R.: *Integrability of Lie brackets.* Ann. of Math. **157** (2003), 575–620.

[15] Crainic, M., Fernandes, R.: *Integrability of Poisson brackets.* Math.DG/0210152.

[16] Dazord, P.: *Groupoïde d'holonomie et géomeétrie globale.* C.R. Acad. Sci. Paris **324** (1997), 77–80.

[17] Guruprasad, K., Huebschmann, J., Jeffrey, L., Weinstein, A.: *Groups systems, groupoids, and moduli spaces of parabolic bundles* Duke Math. J. **89** (1997), 377–412.

[18] Klimcik, C., Ströbl, T.: *WZW-Poisson manifolds*, J. Geom. Physics **4** (2002), 341–344.

[19] Lu, J.-H.: *Multiplicative and affine Poisson structures on Lie groups.* Ph.D. Thesis, University of California, Berkeley.

[20] Lu, J.-H.: *Momentum mappings and reduction of Poisson actions.* In: *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, 291–311. Springer, New York, 1991.

[21] Lu, J.-H., Weinstein, A.: *Groupoïdes symplectiques double des groupes de Lie-Poisson.* C.R. Acad. Sci. Paris, **309** (1989), 951–954.

[22] Mackenzie, K., Higgins, P.: *Algebraic constructions in the category of Lie algebroids.* J. Algebra **129** (1990), 194–230.

[23] Mackenzie, K., Xu , P.: *Integration of Lie bialgebroids.* Topology **39** (2000), 445–467.

[24] Marsden, J., Weinstein , A.: *Reduction of symplectic manifolds with symmetry.* Rep. Mathematical Phys. **5** (1974), 121–130.

[25] Mikami, K., Weinstein, A.: *Moments and reduction for symplectic groupoid actions.* Publ. RIMS, Kyoto Univ. **24** (1988), 121–140.

[26] Moerdijk, I., Mrčun., J.: *On integrability of infinitesimal actions.* Amer. J. Math. **124** (2002), 567–593.

[27] Park, J.-S.: *Topological open p-branes.* Symplectic geometry and Mirror symmetry (Seoul 2000), 311–384, World Sci. Publishing, River Edge, NJ, 2001.

[28] Ševera, P., Weinstein, A.: *Poisson geometry with a 3-form background.* Prog. Theo. Phys. Suppl. **144** (2001), 145–154.

[29] Weinstein, A.: *Symplectic groupoids and Poisson manifolds.* Bull. Amer. Math. Soc. (N.S.) **16** (1987), 101–104.

[30] Weinstein, A.: *Poisson geometry. Symplectic geometry.* Differential Geom. Appl. **9** (1998), 213–238.

[31] Weinstein, A.: *The geometry of momentum.* Proceedings of conference on "Geometry in the 20th Century: 1930 - 2000", to appear. (Math.SG/0208108)

[32] Xu, P.: *Morita equivalent symplectic groupoids.* In: *Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989)*, 291–311. Springer, New York, 1991.

[33] Xu, P.: *Morita equivalence and momentum maps.* Preprint Math.SG/0307319.