Extra- Exercises for Group Theory
Week 37 (September 10)

(the exercises from the book are: 2.3, 2.5, 2.8, 3.2, 3.3).

Recall that, given a group $(G, \bullet)$, and $x \in G$, the n-th power of x (in $(G, \bullet)$) was defined for any integer n by

$$x^n := \begin{cases} e & \text{if } n = 0 \\ \underbrace{x \bullet \ldots \bullet x}_{n \text{ times}} & \text{if } n > 0 \\ \underbrace{x^{-1} \bullet \cdots \bullet x^{-1}}_{-n \text{ times}} & \text{if } n < 0 \end{cases}$$

Exercise 1 Show that in any group $(G, \bullet)$, one has

$$x^n \bullet x^m = x^{n+m}, \ (x^n)^m = x^{nm}$$

for all $x \in G$ and all $n, m \in \mathbb{Z}$.

Exercise 2 On the set $\mathbb{R}$ of real numbers consider the operation $\cdot$ given by:

$$x \cdot y = x + y - xy.$$

Prove the axioms (G1) (associativity) and (G2) (existence of the identity) for $(R, \cdot)$. What about axiom (G3) (existence of inverses)? Then conclude that $(\mathbb{R} - \{1\}, \cdot)$ is a group.

Exercise 3 We have "presented" the group Z of rotational symmetries of a regular hexagon as

$$Z = \{e, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$$

where r and s satisfy the relations:

$$r^6 = e, s^2 = e, rsr = s.$$

The key point is that these relations allow us to "reconstruct" the product of Z: given two arbitrary elements of Z, their product is apriori not writtent in a form which appears in the previous list for Z. The relations listed above allow us to rewrite such products in the desired form. For instance, taking the product of r^2s and rs, we can write

$$(r^2s)(rs) = r(rsr)s = rss = rs^2 = re = r.$$

Do the same for the product of rs with r^2s and then for the product between r^3s and r^4s.