Exercises for Group Theory
Week 48 (November 26)

Exercise 1 Exercises 16.1, 16.2 from the book.

Exercise 2 Prove (again) that SU_n is a normal subgroup of U_n, and the resulting quotient group is isomorphic to S^1.

Exercise 3 Consider the group S^1 and the subgroup $\{1, -1\}$. Prove (again) that $S^1/\{1, -1\}$ is isomorphic to S^1.

Exercise 4 Show that the commutator subgroup of the quaternion group Q is $\{-1, 1\}$.
(Hint: proceed as in the example in the class, and use theorem 15.2).

Exercise 5 Show that the, for $n \geq 3$, A_n is a normal subgroup of S_n, S_n/A_n is isomorphic to $\mathbb{Z}_2$, then show that the commutator subgroup of S_n is A_n.

Exercise 6 For each $n \geq 1$ integer, show that S^1 has a subgroup H isomorphic to $\mathbb{Z}_n$ such that S^1/H is isomorphic to S^1.

Exercise 7 Show that

$$H = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \equiv y \bmod 6\}$$

is a subgroup of $\mathbb{Z} \times \mathbb{Z}$ and compute the resulting quotient.

Exercise 8 For $G = D_4$, describe all its subgroups. Then

1. which ones are normal?
2. which one is the center of G?
3. which one is the commutator subgroup of G?
4. for each normal subgroup, compute the quotient group.

Do the same for $G = D_5, D_6, etc.$

Exercise 9 Exercise 16.3 and 15.9 from the book.

Exercise 10 Let G be a group. Show that

1. The set of $\mathcal{G}$ all automorphisms of G (i.e. group isomorphisms $\phi : G \longrightarrow G$) form a subgroup of S_G.
2. Show that, for any $x \in G$,

   $$\mathrm{Ad}_x : G \longrightarrow G,\ \mathrm{Ad}_x(g) = xgx^{-1}$$

   is an automorphism of G.

3. Denote by $\mathcal{I}$ the set of all automorphisms of type Ad_x (for some $x \in G$). Show that $\mathcal{I}$ is a normal subgroup of $\mathcal{G}$.

4. Show that $\mathcal{I}$ is isomoprphic to $G/Z(G)$.

Exercise 11 Exercise 16.9 from the book.