Exercises for Group Theory
Week 50 (December 10)

Exercise 1 Let X be a set. Show that giving an action of $\mathbb{Z}_2$ on X is the same thing as giving a function $\tau : X \longrightarrow X$ satisfying $\tau \circ \tau = \mathrm{Id}_X$. What if one replaces $\mathbb{Z}_2$ by $\mathbb{Z}_n$? What if one replaces $\mathbb{Z}_2$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$?

Exercise 2 Let G be a group acting on a set X. Show that, if $x, y \in X$ are in the same orbit, then G_x is isomorphic to G_y.

Exercise 3 Let G be a group acting on a set X. We define the fixed point set of the action as

$$X^G := \{x \in X : gx = x \ \forall \ g \in G.$$

Show that, for $x \in X$, x is in X^G if and only if $G_x = G$.

Exercise 4 Let $G = \mathbb{Z}_p$ be the cyclic group of order p (where p is a prime number) acting on a finite set X. Show that

$$|X| \equiv |X^G| \mod p.$$

Also, deduce from this that if n is an odd integer, then any permutation $\sigma \in S_n$ with $\sigma^2 = e$ has at least one fixed point (i.e. there exists $x \in \{1, \ldots, k\}$ such that $\sigma(x) = x$).

Exercise 5 If a group G with 27 elements acts on a set X with 2006 elements, show that X^G contains at least two elements.

Exercise 6 State and prove a more general result about actions of groups with p^k elements on finite sets, which imply the previous two exercises.

Exercise 7 Consider the action of $SO(n)$ on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ arbitrary.

1. what is the orbit of x?
2. what is the stabilizer of x?

(hint: assume first that $x = (1, 0, \ldots, 0)$).

Exercise 8 Indicate a non-trivial action of D_3 on $\mathbb{R}^3$. Describe the stabilizers of an arbitrary element $x \in \mathbb{R}^3$. (you can try to do the same for A_4 instead of D_3).

Exercise 9 Consider an action of D_3 on set X with 2003 elements, and assume that there are no fixed points. Show that there exist $x, y \in X$ such that G_x is isomorphic to $\mathbb{Z}_2$ and G_y is isomorphic to $\mathbb{Z}_3$.