Exercises for Group Theory
Week 43 (October 22)

Exercise 1 Prove that

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_{12} \times \mathbb{Z}_2.$$

Exercise 2 Prove that if G is a group with n elements, then it is cyclic if and only if there exists $x \in G$ such that $\mathrm{ord}(x) = n$.

Exercise 3 Let G be a group with $n = pq$ elements, where $p \neq q$ are two prime numbers. If G contains an element x of order p and an element y of order q such that $xy = yx$, show that that G is isomorphic to $\mathbb{Z}_{pq}$.

Exercise 4 Let G be a group with the following property: out of each four distinct elements of G, at least two of them commute. Show that G is abelian. Is the same true if one replaces "four" by "five"?

Exercise 5 Exercise 7.9 in the book.

Exercise 6 For $n \geq 1$ integer, we consider

$$R_n := \{k \in \mathbb{Z}_n : (k, n) = 1\}.$$

1. Show that, together with the multiplication modulo n, R_n becomes an abelian group.
2. Give a one-line proof of the fact that R_9 is isomoprphic to $\mathbb{Z}_6$.
3. What about R_{16}?

Extra- Exercises

Exercise 7 Exercises 11.12, 13.7, 13.8, 13.11 in the book.

Bonus exercise

Exercise 8 *(bonus exercise of* 0.50 *point)* In the class, we have shown that any group with 4 elements is isomorhic either to $\mathbb{Z}_4$ or two $\mathbb{Z}_2 \times \mathbb{Z}_2$. Also, that any group with six elements is isomorphic either to $\mathbb{Z}_6$ or to D_3. Using the same arguments, prove that, if p is a prime number, then any group with $2p$ elements is isomorphic either to $\mathbb{Z}_{2p}$ or to D_p.