Exercises for Group Theory
Week 47 (November 19)

Exercise 1 Prove the following properties of left cosets: for a group G and a subgroup $H \subset G$:

1. For $x \in G$, one has $xH = H$ if and only if $x \in H$.
2. For $x, y \in G$, if $xHy = H$ then $H = x^{-1}Hy^{-1}$.
3. For $x, y \in G$, $xH = yH$ if and only if $Hx^{-1} = Hy^{-1}$.

Prove then that there is a bijection between the set G/H of all left cosets of H in G, and the set of all right cosets if H in G.

Exercise 2 In the group $D_5 = \{e, r, \ldots, r^4, s, rs \ldots, r^4 s\}$ take the subgroup $\langle r^2 \rangle$ generated by r^2.

1. Compute the index of $\langle r^2 \rangle$ in D_5.
2. Prove that $\langle r^2 \rangle$ is a normal subgroup.
3. Compute $D_5/\langle r^2 \rangle$.

Do the same for D_6. What happens for general n?

Exercise 3 Consider $(\mathbb{Z}, +)$ as a subgroup of $(\mathbb{R}, +)$ and prove that $\mathbb{R}/\mathbb{Z}$ is isomorphic to $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (with the multiplication of complex numbers).

Exercise 4 Prove that SO_n is a normal subgroup of O_n.

Exercise 5 In A_4, consider

$$H = \{e, (12)(34), (13)(24), (14)(23)\}.$$

1. Show that H is a normal subgroup of A_4.
2. Compute A_4/H.
3. Compute the index of H in S_4.
4. Is H a normal subgroup of S_4?

Exercise 6 Exercise 15.14, 15.7, 15.15 in the book.

Exercise 7 Consider the group S^1 and the subgroup $\{1, -1\}$. Show that $S^1/\{1, -1\}$ is isomorphic to S^1.

Exercise 8 Prove thst SU_n is a normal subgroup of U_n, and the resulting quotient group is isomorphic to S^1.