Final Exam, Group Theory, January 29, 2008
- solutions -

Solution Ex. 1: The condition is equivalent to $(xy)^{-1} = x^{-1}y^{-1}$. Taking the inverses of left and right hand side, this is equivalent to $xy = (x^{-1}y^{-1})^{-1}$. But the last expression is yx (in general, $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$). Hence our condition is equivalent to $xy = yx$ for all x and y.

Solution Ex. 2: The unit element is $(1, 0)$. Associativity can be checked directly. To find the inverse of (a, b), call it (x, y), we have to solve the equations (on x and y):

$$ax + by = 1, ay + bx = 0,$$

and we find

$$x = \frac{a}{a^2 - b^2}, y = \frac{b}{b^2 - a^2}.$$

We deduce that $(\mathbb{R}, \circ)$ is not a group (some elements, e.g. $(1, 1)$, do not have inverses), while $(G, \circ)$ is (the las formulas make sense if $a^2 \neq b^2$). For the second part, one remarks that

$$f : G \longrightarrow \mathbb{R}^*, f(a, b) = a^2 - b^2$$

is a surjective group hommomorphiosm, and then use the first isomorphism theorem. For the last part, you can, for instance, check that G has three elements of order two, while N has only one.

Other possible solution: can be based on the remark that

$$f : G \longrightarrow \mathbb{R}^* \times \mathbb{R}^*, f(x, y) = (x + y, x - y)$$

is a bijection and $f(u \circ v) = f(u)f(v)$ for all u and v in G (hence G is actually isomorhic to $\mathbb{R}^* \times \mathbb{R}^*$.

Solution Ex. 3: The permutation is, as a product of cycles,

$$\sigma = (1\,12\,2\,3)(4\,5\,6\,7)(8\,9\,10)$$

hence the order is $4 \cdot 3 = 12$ and its signature is $(-1) \cdot (-1) \cdot 1 = 1$. For the second part, one can reason in many ways. One possible way is the following. Any configuration determines a permutation in S_{12}- so that the initial configuration gives the identity permutation, while the other one in the picture gives the σ above (hence you assign the number 12 to the empty spot). Remark that a move into the empty spot corresponds to multiplication of the permutation by a transposition. So, if we want to reach σ, since σ is even, we have to perform an even number of moves.

On the other hand, we can evaluate the parity of the total number of moves by using the actual shape of the table. More precisely, we can divide all the moves into horizontal and vertical moves. How many vertical moves would there be needed to achieve the right hand side table? Answer: an odd number of moves. That is because any move along the way which is to the left should be compensated by one to the right, and, in total, we have to move the square three times to the right. Hence the number of horizontal moves should be congruent to 3 modulo 2, i.e. odd. Similarly, the number of vertical moves should be even. In total, an (odd + even), i.e. an odd number of moves. Contradiction.

Solution Ex. 4:

1. Element of order 5: yes because 5 is a prime divisor of $|D_{10}| = 20$. (or just point out r^2). Element of order 4: does not exist (the elements containinig s are of order 2, and the nontrivial elements containing only r have order divisible by 5, except for r^5 which has order two).

2. $\{1, r, s, rs\}$ does not contain $r \cdot r = r^2$, hence it is not a subgroup.

3. $\{1, r^5, s, r^5 s\}$ is a subgroup (direct check) but it is not normal because it does not contain $rsr^{-1} = r^2 s$.

4. $N = \{1, r^2, r^4, r^6, r^8\}$ is clearly a subgroup. From the Sylow theorem, D_{10} contain only one subgroup with 5 elements, hence N, as the only Sylow subgroup, must be normal (alternatively, but more unpleasant and longer, one can check directly that N is normal). In the quotient group, one has

$$D_{10}/N = \{\overline{e}, \overline{r}, \ldots, \overline{r}^9, \overline{s}, \overline{rs}, \ldots, \overline{r}^9\overline{s}\},$$

(where $\overline{x} = xN$) with $\overline{r}^2 = \overline{e}$ (since $r^2 \in N$), hence

$$D_{10}/N = \{\overline{e}, \overline{r}, \overline{s}, \overline{rs}\}$$

contains four elements, $\overline{r}^2 = \overline{s}^2$. We deduce that the quotient is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution Ex. 5: Assume G is a group with 2007 elements. Apply the Sylow theorem to deduce that there is exactely one subgroup H of G with 49 elements and exactely one subgroup N with 41 elements. Then we continue as in the lectures (or in the werkcolleges) to prove that

$$f : H \times K \longrightarrow G, f(h, k) = hk$$

is an isomorphism of groups: to see that it is a group homomorphism, we see that what we have to prove is that any element of H commutes with any element of K. To do this, first of all remark that H and K are normal subgroups (from the uniqueness from the Sylow theorems). Secondly, $K \cap H = \{e\}$ (since this intersection is a subgroup of H and K, its number of elements must divide both 49 and 41, hence it must be one). From thes we deduce that $hkh^{-1}k^{-1} \in H \cap K$ must be equal to e for all $h \in H$, $k \in K$ (hence $hk = kh$). In conclusion, f is a group homomorphism.

We now show that f is bijective. The injectivity is equivalent to the fact that the kernel of f is trivial, and this follows immediately from $H \cap K = \{e\}$. Since $H \times K$ and G have the same number of elements and f injective, we deduce that f is bijective as well.

In conclusion, G must be isomorphic to $H \times K$. Sincd $|K| = 41$- prime number, K must be isomorphc to $\mathbb{Z}_{41}$. Since $H = 7^2$ with 7-primes, H must be isomorphic to $\mathbb{Z}_7 \times \mathbb{Z}_7$, or to $\mathbb{Z}_{49}$. In conclusion, G is isomorphic to one of the two groups

$$\mathbb{Z}_{49} \times \mathbb{Z}_{41}, \ \mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{41}.$$

(Note also that the two groups above are not isomorphic- by an argument based on elements of order 49).