

Exam Manifolds (November 8th, 2017)

Exercise 1. (1 pt) Show that, for a vector field X on a manifold M and $f \in C^\infty(M)$, one has $L_X(f) = 0$ if and only if f is constant on the integral curves of X .

Exercise 2. Consider the sphere $S^2 \subset \mathbb{R}^3$ and we use (x, y, z) to denote the standard coordinates in \mathbb{R}^3 . We consider the following vector field tangent to the sphere

$$X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(S^2)$$

as well as the volume form on the sphere:

$$\theta = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy \in \Omega^2(S^2)$$

(as before, while the previous formula defines a 2-form on \mathbb{R}^3 , θ is the restriction to S^2).

- (a) (0.5 pt) Compute $i_X(\theta)$ and $d(i_X(\theta))$.
- (b) (0.5 pt) Compute $d\theta$ and $i_X(d\theta)$.
- (c) (0.5 pt) Compute $L_X(\theta)$ in two ways: one using the Cartan formula, and one using the properties of L_X (being a derivation, and commuting with d).
- (d) (0.5 pt) Compute the flow ϕ^t of X .
- (e) (0.5 pt) Show that $(\phi^t)^*\theta = \theta$ for all $t \in \mathbb{R}$.

Exercise 3. Consider

$$M := \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 - 5)^2 + 16z^2 = 16\} \subset \mathbb{R}^3.$$

- (a) (1 pt) Show that M is a submanifold of \mathbb{R}^3 .
- (b) (1 pt) Compute the tangent space of M at the point $p = (3, 0, 0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial y} \right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z} \right)_p \in T_p M.$$

Similarly, compute the tangent space at the point $q = (2, 0, 1)$.

- (c) (0.5 pt) Draw a picture of M (hint: if you do not see how M looks like, maybe compute the tangent space at one more point).
- (d) (0.5 pt) Prove that M is diffeomorphic to $S^1 \times S^1$.

Exercise 4. Assume that M is a connected 3-dimensional manifold, and $V^1, V^2, V^3 \in \mathfrak{X}(M)$ are vector fields on M with the property that V_p^1, V_p^2, V_p^3 form a basis of $T_p M$ for all $p \in M$ and let $\theta_1, \theta_2, \theta_3 \in \Omega^1(M)$ be the 1-forms that are dual to V^1, V^2, V^3 , i.e. satisfying

$$\theta_i(V^j) = \delta_j^i \text{ (1 if } i = j \text{ and 0 otherwise)}.$$

(a) (0.5 pt) Show that, for any $f \in C^\infty(M)$ one has

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V^3}(f) \cdot \theta_3.$$

(b) (1 pt) Show that the vector fields V^i satisfy

$$[V^1, V^2] = 2V^3, \quad [V^2, V^3] = 2V^1, \quad [V^3, V^1] = 2V^2.$$

if and only if the 1-forms θ_i satisfy

$$d\theta_1 = -2\theta_2 \wedge \theta_3, \quad d\theta_2 = -2\theta_3 \wedge \theta_1, \quad d\theta_3 = -2\theta_1 \wedge \theta_2.$$

From now on we assume that all these are satisfied. Assume furthermore that the 1-forms are invariant with respect to a vector field $V \in \mathfrak{X}(M)$ in the sense that

$$L_V(\theta_1) = L_V(\theta_2) = L_V(\theta_3) = 0.$$

Introduce the following real-valued functions on M :

$$h_1 = i_V(\theta_1), \quad h_2 = i_V(\theta_2), \quad h_3 = i_V(\theta_3).$$

(c) (0.5 pt) Prove that

$$\begin{aligned} dh_1 &= 2h_2 \cdot \theta_3 - 2h_3 \cdot \theta_2, \\ dh_2 &= 2h_3 \cdot \theta_1 - 2h_1 \cdot \theta_3, \\ dh_3 &= 2h_1 \cdot \theta_2 - 2h_2 \cdot \theta_1 \end{aligned}$$

(d) (0.5 pt) Deduce that

$$h = (h_1, h_2, h_3) : M \rightarrow \mathbb{R}^3$$

takes values in a sphere S_r^2 (of some radius $r \geq 0$).

From now on we assume that $r = 1$ (i.e. h takes values in S^2).

(e) (0.5 pt) Show that h is constant on each integral curve of V .

(f) (0.5 pt) Show that V^1 is h -projectable to the following tangent vector on S^2 :

$$E^1 = 2 \left(z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \right) \in \mathfrak{X}(S^2),$$

i.e. $(dh)_p(V_p^1) = E_{h(p)}^1$ for all $p \in M$. And similarly for V^2 and V^3 .

(g) (0.5 pt) Show that if M is compact then h is surjective submersion onto S^2 and any fiber $h^{-1}(q)$ (with $q \in S^2$) which is connected is diffeomorphic to a circle.

(h) (0.5 pt) The homeworks show that the previous scenario can arise on $M = S^3$. Find another example, with M still connected, but for which the fibers of h are not connected.

Notes:

1. you are allowed to use the lecture notes and the homeworks. But no other material and/or other devices.
2. yes, I know, there are many questions, but please do not panic: I think it is easier when a hard question is split into four (say) easier ones!
3. also, please be aware that the points above add up to a total of 11 (i.e. you do not have to do all of them correctly in order to obtain a 10)!
4. in a sequence of items, if you are not able to do one of them, then move to the next one (and you are allowed to use the item that you skipped but did not do). But, hopefully, this will not be necessary.
5. in Exercise 3, item (c), if you do not see how the picture looks like, do not spend too much time with it (e.g. more than 15min)- just move on and return to it later.
6. in Exercise 4: please do not answer the questions/give the proofs in the particular case when $M = S^3$ like in the homeworks, but work with a general M (and then the proofs should be easier than in the homework's, or at least less computational).
7. probably the most difficult questions are items (g) and (h) from the last Exercise.
8. as an overall advice: do not hurry up too much- i.e. think a little bit before any question and before jumping to do a computation (just think of what you know, what should be done, etc, so that you do not waste your time because of an unfortunate choice of strategy).

GOOD LUCK!