

Smooth manifolds

1. Charts and smooth atlases

Recall from Definition 1.3 that, given a topological space M , an m -dimensional chart is a homeomorphism χ between an open U in M and an open subset $\chi(U)$ of \mathbb{R}^m ,

$$\chi : U \rightarrow \chi(U) \subset \mathbb{R}^m.$$

Given such a chart, each point $p \in U$ is determined/parameterized by its coordinates w.r.t. χ :

$$(\chi_1(p), \dots, \chi_m(p)) \in \mathbb{R}^m$$

(a more intuitive notation would be: $(x_\chi^1(p), \dots, x_\chi^m(p))$).

Given a second chart

$$\chi' : U' \rightarrow \chi'(U') \subset \mathbb{R}^m,$$

the map

$$c_{\chi, \chi'} := \chi' \circ \chi^{-1} : \chi(U \cap U') \rightarrow \chi'(U \cap U')$$

is a diffeomorphism between two opens in \mathbb{R}^m . It will be called **the change of coordinates map** from the chart χ to the chart χ' . The terminology is motivated by the fact that, denoting

$$c_{\chi, \chi'} = (c_1, \dots, c_m),$$

the coordinates of a point $p \in U \cap U'$ w.r.t. χ' can be expressed in terms of those w.r.t. χ by:

$$\chi'_i(p) = c_i(\chi_1(p), \dots, \chi_m(p)).$$

DEFINITION 3.1. *We say that two charts (U, χ) and (U', χ') are **smoothly compatible** if the change of coordinates map $c_{\chi, \chi'}$ (a map between two opens in \mathbb{R}^d) is a diffeomorphism.*

DEFINITION 3.2. *A (m -dimensional) **smooth atlas** on a topological space M is a collection \mathcal{A} of (m -dimensional) charts of M with the following properties:*

- any point in M belongs to the domain of at least one chart from \mathcal{A} .
- each two charts from \mathcal{A} are smoothly compatible.

The charts of an atlas are used to transfer the smoothness properties that make sense on (opens inside) \mathbb{R}^m to M ; the compatibility of the charts ensures that the resulting notions (now on M) do not depend on the choice of the charts from the atlas. Hence one may say that an atlas on M allows us to put a "smooth structure" on M . For instance, given a smooth m -dimensional atlas \mathcal{A} on the space M , a function $f : M \rightarrow \mathbb{R}$ is called **smooth w.r.t. the atlas \mathcal{A}** if for any chart (U, χ) that belongs to \mathcal{A} ,

$$f \circ \chi^{-1} : \chi(U) \rightarrow \mathbb{R}$$

is smooth in the usual sense ($\chi(U)$ is an open in \mathbb{R}^m !). We will temporarily denote by

$$\mathcal{C}^\infty(M, \mathcal{A})$$

the set of such smooth functions.

There is one aspect that requires a bit of attention: the fact that two different atlases may give rise to the same "smooth structure". Therefore the following definition.

DEFINITION 3.3. We say that two smooth atlases \mathcal{A}_1 and \mathcal{A}_2 are **smoothly equivalent** if any chart in \mathcal{A}_1 is smoothly compatible with any chart in \mathcal{A}_2 (or, shorter: if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a smooth atlas).

EXERCISE 3.1. Show that, indeed, for any two atlases \mathcal{A}_1 and \mathcal{A}_2 on M , one has:

$$\mathcal{C}^\infty(M, \mathcal{A}_1) = \mathcal{C}^\infty(M, \mathcal{A}_2) \iff \mathcal{A}_1 \text{ is smoothly equivalent to } \mathcal{A}_2.$$

If we want to avoid working with equivalence classes of atlases, one may restrict the attention only to atlases that are "largest".

DEFINITION 3.4. An atlas \mathcal{A} is called **maximal** if there is no larger smooth atlas containing it. Given an arbitrary (smooth) atlas \mathcal{A} , one defines the associated maximal atlas \mathcal{A}^{\max} by adding to \mathcal{A} all the charts that are smoothly compatible with all the charts of \mathcal{A} .

EXERCISE 3.2. Show that, indeed, \mathcal{A}^{\max} is an atlas, and that it is maximal. Also deduce that

$$\mathcal{C}^\infty(M, \mathcal{A}) = \mathcal{C}^\infty(M, \mathcal{A}^{\max}).$$

The following simple exercise shows that any equivalence class of atlases can be represented by an unique maximal atlas.

EXERCISE 3.3. Show that, for any atlas \mathcal{A} , \mathcal{A} is (smoothly) equivalent to \mathcal{A}^{\max} ; then show that for two atlases \mathcal{A}_1 and \mathcal{A}_2 , one has

$$\mathcal{A}_1 \text{ is smoothly equivalent to } \mathcal{A}_2 \iff \mathcal{A}_1^{\max} = \mathcal{A}_2^{\max}.$$

EXAMPLE 3.5. On \mathbb{R}^m one can consider a very small atlas \mathcal{A}_0 - the one consisting only of the identity map

$$\chi = \text{Id} : U = \mathbb{R}^m \rightarrow \mathbb{R}^m.$$

Then \mathcal{A}_0^{\max} consists of all diffeomorphisms between opens in \mathbb{R}^m . Another natural atlas would be

$$\mathcal{A}_1 := \{\text{Id}_U : U \subset \mathbb{R}^m \text{ open}\},$$

or \mathcal{A}_2 defined similarly, but using only open balls $B \subset \mathbb{R}^m$. Then \mathcal{A}_0 and \mathcal{A}_2 are disjoint, they are both inside \mathcal{A}_1 , all three of them are equivalent, and

$$\mathcal{A}_0^{\max} = \mathcal{A}_1^{\max} = \mathcal{A}_2^{\max}.$$

EXERCISE 3.4. Show that the unit circle S^1 (with the topology induced from \mathbb{R}^2) admits an atlas made of two charts, but does not admit an atlas made of a single chart.

2. Smooth structures; manifolds

The main point of the equivalence of atlases is that two atlases that are equivalent give rise to the same notion of smoothness on M . Therefore:

DEFINITION 3.6. An m -dimensional smooth structure on a topological space M is an equivalence class of m -dimensional smooth atlases on M .

The main point of mentioning maximal atlases is the fact that in each equivalence class of an atlas there is a unique maximal one. Therefore one may say that:

a smooth structure on M is a maximal atlas on M .

However, one should keep in mind that very often a smooth structure is exhibited by describing a smaller atlas (ideally "the smallest possible one"). E.g., already for \mathbb{R}^m , the simplest choice is the single chart $\text{Id} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ (see the previous example). In any case, it is clear that:

any atlas \mathcal{A} on M induces a smooth structure on M ;

it may be interpreted as the equivalence class $[\mathcal{A}]$ (with respect to the notion of smooth equivalence of atlases), or as the maximal atlas \mathcal{A}^{\max} associated to \mathcal{A} , respectively.

With these:

DEFINITION 3.7. A **smooth m -dimensional manifold** is a Hausdorff, second countable topological space M together with an m -dimensional smooth structure on M .

The charts belonging to the maximal atlas corresponding to the smooth structure are called **the charts of the smooth manifold M** .

EXERCISE 3.5. In the definition of smooth manifolds show that the condition that M is second countable is equivalent to the fact that the smooth structure on M can be defined by an atlas that is at most countable.

3. Smooth maps

Having introduced the main objects of Differential Geometry, we now move to the maps between them.

DEFINITION 3.8. Let $f : M \rightarrow N$ be a map between two manifold M and N of dimensions m and n , respectively. Given charts (U, χ) and (U', χ') of M and N , respectively, the representation of f with respect to these two charts is defined as

$$f_{\chi, \chi'} := \chi' \circ f \circ \chi^{-1}.$$

This map makes sense when applied to a point of type $\chi(p) \in \mathbb{R}^d$ with $p \in U$ with the property that $f(p) \in U'$, i.e. $p \in U$ and $p \in f^{-1}(U')$. Therefore, it is a map

$$f_{\chi, \chi'} : \text{Domain}(f_{\chi, \chi'}) \rightarrow \mathbb{R}^n.$$

whose domain is the following open subset of \mathbb{R}^m :

$$\text{Domain}(f_{\chi, \chi'}) = \chi(U \cap f^{-1}(U')) \subset \mathbb{R}^m.$$

DEFINITION 3.9. Let M and N be two manifolds and

$$f : M \rightarrow N$$

a map between them. We say that f is **smooth** if its representation $f_{\chi, \chi'}$ with respect to any chart χ of M and χ' of N , is smooth (in the usual sense of Analysis).

We say that f is a **diffeomorphism** if f is bijective and both f and f^{-1} are smooth.

EXERCISE 3.6. With the notation from the previous definitions, let \mathcal{A}_M and \mathcal{A}_N be two arbitrary (i.e. not necessarily maximal) atlases inducing the smooth structure on M and N , respectively. Show that, to check that f is smooth, it suffices to check that $f_{\chi, \chi'}$ is smooth for $\chi \in \mathcal{A}_M$ and $\chi' \in \mathcal{A}_N$.

Like in the case of \mathbb{R}^m , there are certain types of smooth maps that deserve separate names.

DEFINITION 3.10. Let $f : M \rightarrow N$ be a smooth map between two manifolds. We say that f is an **immersion at p** if its local representations $f_{\chi, \chi'}$ w.r.t. any chart χ of M around p and χ' of N around $f(p)$ is an immersion at $\chi(p)$ (as a map from an open inside \mathbb{R}^m to one inside \mathbb{R}^n - see Theorem 2.12).

We say that f is an **immersion** if it is an immersion at all points $p \in M$.

Similarly we define the notion of **submersion** and that of **local diffeomorphism**.

EXERCISE 3.7. Show that, in the previous definition, it is enough to check the required condition for (single!) one chart χ of M around p and one chart χ' of M' around $f(p)$.

Note that, since any open U inside a manifold M has a canonical induced smooth structure made up of the charts of M whose domain is inside U (simple exercise!), the notion of local diffeomorphism at p can also be rephrased without reference to charts as: there exists an open neighborhood U_p of p in M and $U'_{f(p)}$ of $f(p)$ in N such that $f|_{U_p} : U_p \rightarrow U'_{f(p)}$ is a diffeomorphism. The rephrasing of the immersion and submersion conditions independently of charts (therefore providing another proof of (actually insight into) the previous exercise) will be possible once we discuss that notion of tangent spaces.

From the submersion and immersion theorems on Euclidean spaces, i.e. Theorem 2.11 and Theorem 2.12 from the previous chapter, we immediately deduce:

THEOREM 3.11 (the submersion theorem). *If $f : M \rightarrow N$ is a smooth map between two manifolds which is a submersion at a point $p \in M$, then there exist a charts χ of M around p and χ' of N around $f(p)$ such that $f'_\chi = \chi' \circ f \circ \chi^{-1}$ is given by*

$$f'_\chi(x_1, \dots, x_n, x_{n+1}, \dots, x_m) = (x_1, \dots, x_n).$$

THEOREM 3.12 (the immersion theorem). *If $f : M \rightarrow N$ is a smooth map between two manifolds which is a submersion at a point $p \in M$, then there exist a charts χ of M around p and χ' of N around $f(p)$ such that $f'_\chi = \chi' \circ f \circ \chi^{-1}$ is given by*

$$f'_\chi(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0).$$

Moreover, the charts $\chi : U \rightarrow \Omega \subset \mathbb{R}^m$ and $\chi' : U' \rightarrow \Omega' \subset \mathbb{R}^n$ can be chosen so that

$$f(U) = \{q \in U' : \chi'_{m+1}(q) = \dots = \chi'_n(q) = 0\}.$$

Combining also with the inverse function theorem we deduce:

LEMMA 3.13. *Consider a smooth function $f : M \rightarrow N$, $p \in M$. Then the following are equivalent:*

- (1) f is a local diffeomorphism around p .
- (2) f is both a submersion as well as an immersion at p .
- (3) f is a submersion at p and $\dim(M) = \dim(N)$.
- (4) f is an immersion at p and $\dim(M) = \dim(N)$.

Moreover, f is a diffeomorphism if and only if it is a local diffeomorphism as well as a bijection.

PROOF. The equivalence between (2), (3) and (4) is immediate from linear algebra: for a linear map $A : V \rightarrow W$ between two finite dimensional vector spaces, looking at the conditions: A is injective, A is surjective, $\dim(V) = \dim(W)$, any two implies the third. Since these actually imply that A is an isomorphism, making use also of the inverse function theorem, we obtain the equivalence with (1) as well.

The last part should be clear (otherwise should be treated as an easy exercise!). \square

4. Variations

There are several rather obvious variations on the notion of smooth manifold. For instance, keeping in mind that

$$\text{smooth} = \text{of class } \mathcal{C}^\infty,$$

one can consider a \mathcal{C}^k -version of the previous definitions for any $1 \leq k \leq \infty$. E.g., instead of talking about smooth compatibility of two charts, one talks about \mathcal{C}^k -compatibility, which means that the change of coordinates is a \mathcal{C}^k -diffeomorphism. One arises at the notion of **manifold of class \mathcal{C}^k** , or **\mathcal{C}^k -manifold**. For $k = \infty$ we recover smooth manifolds, while for $k = 0$ we recover topological manifolds.

Yet another possibility is to require "more than smoothness"- e.g analyticity. That gives rise to the notion of **analytic manifold**. Looking at manifolds of dimension $m = 2n$, hence modelled by $\mathbb{R}^m = \mathbb{C}^n$, one can also restrict even further- to maps that are holomorphic. That gives rise to the notion of **complex manifold**. Etc.

Another possible variation is to change the "model space" \mathbb{R}^m . The simplest and most standard replacement is by the upper-half planes

$$\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geq 0\}.$$

This gives rise to the notion of **smooth manifold with boundary**: what changes in the previous definition is the fact that the charts are homeomorphisms into opens inside \mathbb{H}^m .

REMARK 3.14. *Another variation that one sometimes encounters in the literature, usually in order to avoid reference to Topology, is the definition of manifolds as a structure on a set M rather than a topological space M . One problem with such an approach is that its "directness" (avoiding topology) is only apparent, and the various topological notions that inevitably show up over and over again would look rather mysterious and even more complicated. Another problem with such an approach (even more important than the previous one) is the fact that the various manifolds that we encounter are, before anything, topological spaces in a very natural way. Not making any reference to such natural topologies would be artificial and less intuitive.*

However, here is an exercise that indicates how one could (but should not) proceed.

EXERCISE 3.8. Let M be a set and let \mathcal{A} be a collection of bijections

$$\chi : U \rightarrow \chi(U) \subset \mathbb{R}^m$$

between subsets $U \subset M$ and opens $\chi(U) \subset \mathbb{R}^m$. We look for topologies on M with the property that each $\chi \in \mathcal{A}$ becomes a homeomorphism. We call them topologies compatible with \mathcal{A} .

- (i) If the domains of all the $\chi \in \mathcal{A}$ cover M , show that M admits at most one topology compatible with \mathcal{A} .
- (ii) Assume that, furthermore, for any $\chi, \chi' \in \mathcal{A}$, $\chi' \circ \chi^{-1}$ (defined on $\chi(U \cap U')$, where U is the domain of χ and U' is of χ') is a homeomorphism between opens in \mathbb{R}^m . Show that M admits a topology compatible with \mathcal{A} .

(Hint: try to define a topology basis).

5. Example: the Euclidean spaces \mathbb{R}^m

The **standard smooth structure on the Euclidean space** \mathbb{R}^m is the one defined by the atlas consisting of the (single) chart $\text{Id} : \mathbb{R}^m \rightarrow \mathbb{R}^m$.

In general, \mathbb{R}^m has many smooth structures:

EXERCISE 3.9. For a homeomorphism $\chi : \mathbb{R}^m \rightarrow \mathbb{R}^m$, consider

$$\mathcal{A}_\chi = \{\chi\},$$

(the atlas on \mathbb{R}^m consisting of one single chart, namely χ itself).

Show that, for a general homeomorphism χ , \mathcal{A}_χ defines a smooth structure different than the standard one, but \mathbb{R}^m endowed with the resulting smooth structure is diffeomorphic to the standard one.

As the previous exercise shows, the interesting question for \mathbb{R}^m , and as a matter of fact for any topological space M , is:

how many non-diffeomorphic smooth structures does a topological space M admit?

Even for $M = \mathbb{R}^m$ this is a highly non-trivial question, despite the fact that the answer is deceptively simple: the spaces \mathbb{R}^m with $m \neq 4$ admit only one such smooth structure, while \mathbb{R}^4

admits an infinite number of them (called "exotic" smooth structures on \mathbb{R}^4)!

In the rest of the course, whenever making reference to an Euclidean space \mathbb{R}^m as a manifold, we always consider it endowed with the standard smooth structure. And similarly for opens $U \subset \mathbb{R}^m$ (using the atlas consisting of the identity map $\text{Id}_U : U \rightarrow U$). In particular, this is relevant when talking about smooth functions whose domain or codomain are (opens inside) Euclidean spaces. For instance, when looking at smooth maps

$$\gamma : I \rightarrow M$$

defined on an open interval $I \subset M$ we are talking about **(smooth) curves** in M . Here M is a manifold and the smoothness of γ is checked using charts (U, χ) for M and the resulting representation of γ in the chart χ :

$$\gamma^\chi := \chi \circ \gamma : I_\chi \subset \mathbb{R}^m$$

(with domain $I_\chi = \gamma^{-1}(U) \subset I$).

Similarly, given a map

$$f : M \rightarrow \mathbb{R}^n$$

defined on a manifold M (and $n \geq 1$ integer), we may talk about its smoothness. Using the minimal atlas of \mathbb{R}^n (and Exercise 3.6!) we see that the smoothness of f is checked by using charts (U, χ) for M and looking at the representation of f w.r.t. f

$$f_\chi = f \circ \chi^{-1} : \chi(U) \rightarrow \mathbb{R}^n$$

(without making reference to the chart that we use for \mathbb{R}^n which is, of course, $\text{Id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$). In particular, for $n = 1$ we can talk about real-valued functions of M , and this gives rise to **the algebra of smooth functions on M** :

$$\mathcal{C}^\infty(M) \subset \mathcal{C}(M).$$

EXERCISE 3.10. Show that, indeed, $\mathcal{C}^\infty(M)$ is a sub-algebra of $\mathcal{C}(M)$. Then show that $\mathcal{C}^\infty(M)$ is closed under locally finite sums.

For the purpose of obtaining **smooth partitions of unity**, i.e. partitions of unity that are smooth, one is left with showing that $\mathcal{C}^\infty(M)$ is normal- and this is where Theorem 1.16, from Chapter I, comes in handy. Actually, the proof of the existence of smooth partitions of unity on \mathbb{R}^m (Theorem 2.7, Chapter I) applies right away to any manifold M ; we obtain:

THEOREM 3.15. *On any manifold M and for any open cover \mathcal{U} , there exists a smooth partition of unity on M subordinated to \mathcal{U} .*

COROLLARY 3.16. *Any compact manifold M can be embedded in \mathbb{R}^n for a sufficiently large integer n , i.e. M is diffeomorphic to an embedded submanifold M' of \mathbb{R}^n .*

6. Example: The spheres S^m

The next example we would like to mention is that of the m -dimensional sphere, realized as the subspace of \mathbb{R}^{m+1} at distance 1 from the origin:

$$S^m = \{(x_0, \dots, x_m) \in \mathbb{R}^{m+1} : (x_0)^2 + (x_1)^2 + \dots + (x_m)^2 = 1\}.$$

As with any subspace of an Euclidean space, we endow it with the Euclidean topology. This example will also fit into the general discussion of embedded submanifolds of Euclidean spaces (see below) but it is instructive to consider it separately.

Intuitively it should be clear that, locally, S^m looks like (opens inside) \mathbb{R}^m - and that is something that we use everyday (we do live on some sort of sphere, remember?). For instance,

drawing small disks on the sphere and projecting them on planes through the origin, one can easily build charts. For instance,

$$U = \{(x_0, \dots, x_m) \in S^m : x_m > 0\}$$

(not even so small!) and the projection into the horizontal plane $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{m+1}$ gives rise to

$$\chi : U \rightarrow \mathbb{R}^m, \quad \chi(x_0, \dots, x_m) = (x_0, \dots, x_m).$$

If you follow your intuition to build such charts, than any charts that you obtain should be smoothly compatible- therefore you will obtain a natural smooth structure on S^m .

Here is a more precise, and more elegant way to do it: via the stereographic projections. This allows one to define the smooth structure made of (only!) two atlases. For that we use the stereographic projections w.r.t. the north and the south poles,

$$p_N = (0, \dots, 0, 1), \quad p_S = (0, \dots, 0, -1) \in S^n,$$

respectively. The one w.r.t. to p_N is the map

$$\chi_N : S^m \setminus \{p_N\} \rightarrow \mathbb{R}^n$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 1).

FIGURE 1.

Computing the intersections, we find the precise formula:

$$\chi_N : S^m \setminus \{p_N\} \rightarrow \mathbb{R}^n, \quad \chi_N(p_0, p_1, \dots, p_m) = \left(\frac{p_0}{1 - p_m}, \dots, \frac{p_{m-1}}{1 - p_m} \right)$$

(while for χ_S we find a similar formula, but with +s instead of the -s). Reversing the process (i.e. computing its inverse), we find

$$\chi_N^{-1} : \mathbb{R}^m \rightarrow S^m \setminus \{p_N\}, \quad \chi_N^{-1}(x_1, \dots, x_n) = \left(\frac{2x_1}{|x|^2 + 1}, \dots, \frac{2x_m}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \right)$$

and we deduce that χ_N is a homeomorphism. And similarly for χ_S . Computing the change of coordinates between the two charts we find

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\}, \quad \chi_S \circ \chi_N^{-1}(x) = \frac{x}{|x|^2}.$$

This is clearly smooth, composed with itself is the identity, therefore it is a diffeomorphism (... therefore also solving Exercise 2.19 from Chapter 1). We deduce that the two charts

$$(6.1) \quad (S^m \setminus \{p_N\}, \chi_N), \quad (S^m \setminus \{p_S\}, \chi_S)$$

define a smooth m -dimensional atlas on S^m . The resulting smooth structure is called **the standard smooth structure on S^m** .

EXERCISE 3.11. Show that, if we add to (6.1) the chart (U, χ) described above, we obtain the same smooth structure on S^m . Or, in other words, (U, χ) belongs to the maximal atlas induced by (6.1).

EXERCISE 3.12. Using your intuition, describe a smooth structure on the 2-torus. Then show that your smooth structure can be described by an atlas consisting of two only charts. (you should think on the picture. If you want to write down explicit formulas, you can use the following concrete realizations of the 2-torus:

$$(6.2) \quad T^2 = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\},$$

with $R > r > 0$).

As for the Euclidean spaces, it becomes very interesting (and exciting!) when we ask about the existence of other smooth structures on S^m (endowed with the Euclidian topology). Of course, for the reasons we explained, we only look at non-diffeomorphic smooth structures. Again, the answers are deceivingly simple (and the proofs higher non-trivial):

- except for the exceptional case $m = 4$ (see below), for $m \leq 6$ the standard smooth structure on S^m is the only one we can find (up to diffeomorphisms).
- S^7 admits precisely 28 (!?) non-diffeomorphic smooth structures.
- S^8 admits precisely 2.
- ...
- and S^{11} admits 992, while S^{12} only one!
- and S^{31} more than 16 million, while S^{61} only one (and actually, next to the case S^m with $m \leq 6$, S^{61} is the only odd-dimensional sphere that admits only one smooth structure).
- ...

Moreover, some of the exotic (i.e. non-diffeomorphic to the standard) spheres can be described rather simply. For instance, fixing $\epsilon > 0$ small enough,

$$W_k := \{(z_1, z_2, z_3, z_4, z_5) : z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2\} \subset \mathbb{C}^5 \cong \mathbb{R}^{10}$$

(one for each integer $k \geq 1$) are all homeomorphic to S^7 , they are all endowed with smooth structures induced from the standard (!) one on \mathbb{R}^{10} , but the first 28 of them are each two non-diffeomorphic (and after that they start repeating). In this way one obtains a rather explicit description of all the 28 smooth structures on S^7 .

While the number of smooth structures on S^m is well understood for $m \neq 4$, the case of S^4 remains a mystery: is there just one smooth structure? Is there a finite number of them (and how many?)? Or there is actually an infinite number of them? The smooth Poincare conjecture says that there is only one; however, nowadays it is believed that the conjecture is false.

7. The projective spaces \mathbb{P}^m

Probably the simplest example of a smooth manifold that does not sit *naturally* inside an Euclidean space (therefore for which the abstract notion of manifold is even more appropriate)

is the m -dimensional projective space \mathbb{P}^m . Recall that it consists of all lines through the origin in \mathbb{R}^{m+1} :

$$\mathbb{P}^m = \{l \subset \mathbb{R}^{m+1} : l - \text{one dimensional vector subspace}\}.$$

As you have seen in the course on Topology (but we recall below), it comes with a natural topology. This topology can be described in several ways. The bottom line is that each point $x \in \mathbb{R}^{m+1} \setminus \{0\}$ defines a line: the one through x and the origin:

$$l_x = \mathbb{R}x = \{\lambda x : \lambda \in \mathbb{R}\} \subset \mathbb{R}^{m+1};$$

and each line arises in this way. More formally, we have a natural surjective map

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \rightarrow \mathbb{P}^m, \quad x \mapsto l_x,$$

and then we can endow \mathbb{P}^m with the induced quotient topology: the smallest one which makes π into a continuous map. Explicitly, a subset U of \mathbb{P}^m is open if and only if $\pi^{-1}(U)$ is open in \mathbb{R}^{m+1} (think on a picture!).

Noticing that two different points x, y define the same line if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}^*$, we see that we are in the situation of group actions: the multiplicative group \mathbb{R}^* acts on $\mathbb{R}^{m+1} \setminus \{0\}$ and \mathbb{P}^m is the resulting quotient

$$\mathbb{P}^m = (\mathbb{R}^{m+1} \setminus \{0\}) / \mathbb{R}^*,$$

endowed with the quotient topology. And one can do a bit better by making use of the fact that each line intersects the unit sphere, i.e. can be written as l_x with $x \in S^m$. The restriction of π defines a surjection

$$\pi_0 : S^m \rightarrow \mathbb{P}^m$$

which is continuous (or, if we did not define the smooth structure on \mathbb{P}^m yet, can be used to define it again as a quotient topology). This "improvement" shows in particular that \mathbb{P}^m is compact (as the image of a compact by a continuous map).

Moreover, since $l_x = l_y$ with $x, y \in S^m$ happens only when $y = \pm x$, we see that we deal with an action of \mathbb{Z}_2 on S^m and

$$\mathbb{P}^m = S^m / \mathbb{Z}_2.$$

As a quotient of a Hausdorff space modulo an action of a finite group, \mathbb{P}^m is Hausdorff (however, that should be clear to you right away!).

From now on we will use the more standard notation

$$[x_0 : x_1 : \dots : x_m]$$

to represent that elements of \mathbb{P}^m (hence $[x]$ is the line through x). The notation $[\cdot]$ may remind you that we may think of a line as equivalence classes, while the use of $:$ in the notation should suggest "division". Keep in mind that

$$[x_0 : x_1 : \dots : x_m] = [\lambda \cdot x_0 : \lambda \cdot x_1 : \dots : \lambda \cdot x_m]$$

with $\lambda \in \mathbb{R}^*$, are the only type of identities that we have in \mathbb{P}^m .

The natural smooth structure is obtained by starting from a simple observation: the last equality allows us in principle to make the first coordinate equal to 1:

$$[x_0 : x_1 : \dots : x_m] = \left[1 : \frac{x_1}{x_0} : \dots : \frac{x_m}{x_0} \right],$$

i.e. to use just coordinates from \mathbb{R}^m ; with one little problem- when $x_0 = 0$ (this "little problem" is what forces us to use more than one chart). We arrive at a very natural chart for \mathbb{P}^m : with domain

$$U_0 = \{[x_0 : x_1 : \dots : x_m] \in \mathbb{P}^m : x_0 \neq 0\}$$

and defined as

$$\chi_0 : U_0 \rightarrow \mathbb{R}^m, \chi_0([x_0 : x_1 : \dots : x_m]) = \left(\frac{x_1}{x_0}, \dots, \frac{x_m}{x_0} \right).$$

And similarly when trying to make the other coordinates equal to 1:

$$\chi_i : U_i \rightarrow \mathbb{R}^m, \chi_i([x_0 : x_1 : \dots : x_m]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_m}{x_i} \right),$$

defined on U_i defined by $x_i \neq 0$.

DEFINITION 3.17. *Show that*

$$(U_0, \chi_0), (U_1, \chi_1), \dots, (U_m, \chi_m)$$

defines a smooth structure on \mathbb{P}^m .

Note that this is also a first example in which, at least intuitively, we need many more than just two charts to form an atlas. The question, valid for any manifold M , asking what is the minimal number of charts one needs to obtain an atlas of M , is a very interesting one. In general one can show that one can always find an atlas consisting of no more than $m + 1$ charts where m is the dimension of M (not very deep, but non-trivial either!). Therefore, denoting by $N_0(M)$ the minimal number of charts that we can find, one always has $N_0(M) \leq \dim(M) + 1$. The atlas described above confirms this inequality for \mathbb{P}^m . However, in concrete examples, we can always do better. E.g. we have seen that $N_0(S^m) = 2$ (well, why can't it be 1?). The same holds for all compact orientable surfaces. Thinking a bit (but not too long) about \mathbb{P}^m one may expect that $N_0(\mathbb{P}^m) = m + 1$; however, that is not the case. The precise computation was carried out by M. Hopkins, except for the cases $n = 31$ and $n = 47$. For those cases we know that $N_0(\mathbb{P}^{31})$ is either 3 or 4, while $N_0(\mathbb{P}^{46})$ is either 5 or 6. For all the other cases, writing $m = 2^k a - 1$ with a odd, one has

$$N_0(\mathbb{P}^m) = \begin{cases} \max\{2, a\} & \text{if } k \in \{1, 2, 3\} \\ \text{the least integer } \geq \frac{m+1}{2(k+1)} & \text{otherwise} \end{cases}$$

Returning to the basics, note that there is a complex analogue of \mathbb{P}^m ; for that reason \mathbb{P}^m is sometimes denoted $\mathbb{R}\mathbb{P}^m$ and called the real projective space. The m -dimensional complex projective space $\mathbb{C}\mathbb{P}^m$ is defined completely analogously but using complex lines $l_z \subset \mathbb{C}^{m+1}$, i.e. 1-dimensional complex subspaces of \mathbb{C}^{m+1} (and 1-dimensional is in the complex sense). Again, one can write

$$\mathbb{C}\mathbb{P}^m = \{[z_0 : z_1 : \dots : z_m] : (z_0, z_1, \dots, z_m) \in \mathbb{C}^m \setminus \{0\}\}$$

with the (only) equalities:

$$[z_0 : z_1 : \dots : z_m] = [\lambda \cdot z_0 : \lambda \cdot z_1 : \dots : \lambda \cdot z_m] \quad \text{for } \lambda \in \mathbb{C}^*.$$

Analogously to the real case,

$$\mathbb{C}\mathbb{P}^m = (\mathbb{C}^{m+1} \setminus \{0\}) / \mathbb{C}^*.$$

We can also intersect the (complex) lines with the sphere $S^{2m+1} \subset \mathbb{C}^m$ i.e., in formulas, realize each line as $[z]$ with $z \in S^{2m+1}$; two like this, $[z_1]$ and $[z_2]$, are equivalent if and only if $z_2 = \lambda \cdot z_1$, where this time $\lambda \in S^1$; i.e. the group \mathbb{Z}_2 from the real case is replaced by the group S^1 of complex numbers of norma 1 (endowed with the usual multiplication). We obtain

$$\mathbb{C}\mathbb{P}^m = S^{2m} / S^1.$$

As for the smooth structure one proceeds completely analogously, keeping in mind the identification $\mathbb{C}^m = \mathbb{R}^{2m}$: we get a (smooth) atlas made of $m + 1$ charts. Note that the change of coordinates is actually given by holomorphic maps- therefore $\mathbb{C}\mathbb{P}^m$ is also a complex manifold (see Section 4). Note that m is the complex dimension of $\mathbb{C}\mathbb{P}^m$; as a manifold, it is $2m$ -dimensional.

As a curiosity: for the minimal number of charts needed to cover $\mathbb{C}\mathbb{P}^m$, the answer is much simpler in the complex case:

$$N_0(\mathbb{C}\mathbb{P}^m) = \begin{cases} m+1 & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}.$$

8. Embedded submanifolds of \mathbb{R}^m

First of all, we would like to point out that all embedded submanifolds of \mathbb{R}^m in the sense of Analysis (as discussed in Section 5 of Chapter II) can naturally be turned into manifolds in the general sense of this chapter. We will change a bit the notation from Chapter II: while there we had d -dimensional submanifolds M or \mathbb{R}^m , we will now be talking about n -dimensional submanifolds N of \mathbb{R}^m (hence d and M there now become n and N , respectively).

THEOREM 3.18. *If $N \subset \mathbb{R}^m$ is an n -dimensional embedded submanifold of \mathbb{R}^m then the collection of n -dimensional charts of N (in the sense of (1) from Section 5, Chapter II) define an atlas, and therefore a smooth structure on N .*

The resulting smooth structure will be called the induced smooth structure on N . The proof of the theorem is immediate since the composition and inverses of diffeomorphisms (between subsets of Euclidean spaces) are automatically diffeomorphisms.

EXERCISE 3.13. Show that the induced smooth structure on $S^m \subset \mathbb{R}^{m+1}$ coincides with the one defined by the stereographic projections.

While in Chapter 2 (Section 5) we had four characterizations of submanifolds, labelled there as (1)-(4), it was the first one that made the previous theorem immediate. As we pointed out in Chapter 2, condition (4) was the one that contained most information (as it used both the parametrizations as well as the implicit equations); hence there is no wonder that (4) is usually the most useful one. Moreover, it will also be the one that will allow us to proceed more generally and talk about embedded submanifolds N of an arbitrary manifold M .

Let us now check that the general notion of smoothness from this chapter is compatible with the one from Chapter 2.

PROPOSITION 3.19. *Let $f : M \rightarrow N$ be a function between two embedded submanifolds*

$$M \subset \mathbb{R}^{\tilde{m}}, \quad N \subset \mathbb{R}^{\tilde{n}}.$$

Then f is smooth as a function between subsets of Euclidean spaces (Definition 2.13, Chapter 2) if and only if it is smooth as a map between manifolds (Definition 3.9).

PROOF. Let us call by "A-smoothness" the notion from Chapter 2 and smoothness the one from this chapter. It is clear that the two coincide when M and N are open in their ambient Euclidean spaces.

Assume first that f is A-smooth. We will consider charts $\tilde{\chi} : \tilde{U} \rightarrow \Omega$ of $\mathbb{R}^{\tilde{m}}$ adapted to M (i.e. satisfying condition (4) in the definition of smoothness from Chapter 2, and similarly $\tilde{\chi}' : \tilde{U}' \rightarrow \Omega'$ adapted to N ; they induce charts $\chi := \tilde{\chi}|_{U \cap M}$ for M , and similarly χ' for N . To prove that f is smooth around a point $p \in M$ it suffices to show that for any $\tilde{\chi}$ around p and $\tilde{\chi}'$ around $f(p)$, as above, $f_{\tilde{\chi}}^{\tilde{\chi}'}$ is smooth. Since f is A-smooth, we may assume that $f|_U = \tilde{f}|_U$ with $\tilde{f} : \tilde{U} \rightarrow \tilde{U}'$ is smooth; in turn, \tilde{f} induces $F := \tilde{f}_{\tilde{\chi}}^{\tilde{\chi}'} : \Omega \rightarrow \Omega'$, whose restriction to $\Omega \cap (\mathbb{R}^m \times \{0\}) \subset \mathbb{R}^{\tilde{m}}$ is precisely $f_{\chi}^{\chi'}$ (and takes values in $\Omega' \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^{\tilde{n}}$). Hence we are in the situation of having a smooth map $F : \Omega \rightarrow \Omega'$ between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, taking $\Omega_0 = \Omega \cap (\mathbb{R}^m \times \{0\})$ to $\Omega'_0 = \Omega' \cap (\mathbb{R}^n \times \{0\})$ and we want to show that $F|_{\Omega_0} : \Omega_0 \rightarrow \Omega'_0$ is smooth as a map between opens in \mathbb{R}^m and \mathbb{R}^n ; that should be clear.

Assume now that f is smooth. To check that it is A-smooth, we fix $p \in M$ and consider charts $\tilde{\chi}$ and $\tilde{\chi}'$ and proceed as above and with the same notation. This time but we do not have \tilde{f} , but having it is equivalent to having F extending $F_0 = f_{\tilde{\chi}'}^{\chi'}$. Then we are in a similar general situation as above, when we have a smooth map $F_0 : \Omega_0 \rightarrow \Omega'_0$ between opens in \mathbb{R}^m and \mathbb{R}^n and we want to extend it to a smooth map F between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, defined in a neighborhood of $\chi(p)$. Using the decomposition $\mathbb{R}^{\tilde{m}} = \mathbb{R}^m \times \mathbb{R}^{\tilde{m}-m}$ and similarly for $\mathbb{R}^{\tilde{n}}$, we just set $F(x, y) = (F_0(x), 0)$. \square

Note that in most text-books the notion of submersion and immersion is defined using tangent spaces; we will come back to this once tangent spaces will be discussed in full generality. Here we discuss the case of embedded submanifolds of Euclidean spaces, since their tangent spaces have already been discussed. Start with two embedded submanifolds

$$M \subset \mathbb{R}^{\tilde{m}}. \quad N \subset \mathbb{R}^{\tilde{n}}.$$

For $p \in M$ consider the tangent space $T_p M \subset \mathbb{R}^{\tilde{m}}$ as defined in Chapter 2 (Definition 2.21). For a smooth map $f : M \rightarrow N$ and $p \in M$ we consider the differential f at p (cf. e.g. the first part of Exercise 2.10):

$$(df)_p : T_p M \rightarrow T_{f(p)} N.$$

EXERCISE 3.14. For a smooth map $f : M \rightarrow N$ as above (between embedded submanifolds of Euclidean spaces) and $p \in M$,

- f is a submersion at p if and only if $(df)_p : T_p M \rightarrow T_{f(p)} N$ is surjective.
- f is an immersion at p if and only if $(df)_p : T_p M \rightarrow T_{f(p)} N$ is injective.

9. Embedded submanifolds and embeddings

As we have already mentioned, the characterization (4) of embedded submanifolds of Euclidean spaces (Section 5 of Chapter 2) allows us to proceed more generally and talk about embedded submanifolds N of an arbitrary manifold M . Indeed, condition (4) has an obvious generalization to this context:

DEFINITION 3.20. *Given an m -dimensional manifold M and a subset $N \subset M$, we say that N is an embedded n -dimensional submanifold of M if for any $p \in N$, there exists a chart*

$$\chi : U \rightarrow \Omega \subset \mathbb{R}^m$$

for M around p with the property that

$$\chi(U \cap N) = \Omega \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^m$$

or, more explicitly,

$$(9.1) \quad \chi(U \cap N) = \{p \in U : \chi_{n+1}(p) = \dots = \chi_m(p) = 0\}.$$

The charts (U, χ) of M satisfying this equality are called **charts of M adapted to N** .

Any chart χ as in the definition induces an open $\Omega_N \subset \mathbb{R}^n$ defined by

$$\chi(U \cap N) = \Omega_N \times \{0\},$$

and then a chart

$$\chi_N : U \cap N \rightarrow \Omega_N \subset \mathbb{R}^n$$

for N . With these we obtain an induced smooth structure on N .

EXERCISE 3.15. Check that, indeed, the collection of all charts of type χ_N obtained from charts χ of M that are adapted to N , define a smooth structure on N .

EXERCISE 3.16. Assume that $N \subset M$ is an embedded submanifold and let P be another manifold. Show that:

- (1) a function $f : P \rightarrow N$ is smooth if and only if it is smooth as an M -valued function.
- (2) if a function $f : N \rightarrow P$ admits a smooth extension to M , then f is smooth.
- (3) if N is closed in M , then the previous statement holds with "if and only if".

(Note: the last point is more difficult because the hint is: use partitions of unity for M (and don't forget that $M \setminus N$ is already open in M).

Here is an extremely very useful method for constructing embedded submanifolds (... whose usefulness is heard to overestimate). In principle, it should work in the explicit examples given by equations. The setting is as follows: we are looking at a smooth map

$$f : M \rightarrow N$$

and we are interested in its fiber above a point $q \in N$:

$$F = f^{-1}(q), \quad (\text{with } q \in N).$$

The condition that we need here is that q is a **regular value of f** in the sense that f is a submersion at all points $p \in f^{-1}(q)$.

THEOREM 3.21 (the regular value theorem). *If $q \in N$ is a regular value of a smooth map*

$$f : M \rightarrow N,$$

then the fiber above q , $f^{-1}(q)$, is an embedded submanifold of M of dimension

$$\dim(f^{-1}(q)) = \dim(M) - \dim(N).$$

PROOF. In principle, this is just another face of the submersion theorem. Let $d = m - n$, where m and n are the dimension of M and N , respectively. We check the submanifold condition around an arbitrary point $p \in F = f^{-1}(q)$. For that we apply the submersion theorem to f near p to find charts $\chi : U \rightarrow \Omega$ and $\chi' : U' \rightarrow \Omega'$ of M around p and of N around $f(p)$, respectively, such that

$$f_{\chi'}^{\chi}(x) = (x_1, \dots, x_n).$$

After changing χ and χ' by a translation, we may assume that $\chi(p) = 0$ and $\chi'(q) = 0$. We claim that, up to a reindexing of the coordinates, χ is a chart of M adapted to F . Indeed, we have

$$\chi(U \cap F) = \{\chi(p') : p' \in U, f(p') = q\} = \{x \in U : f_{\chi'}^{\chi}(x) = 0\},$$

or, using the form of $f_{\chi'}^{\chi}$,

$$\chi(U \cap F) = \{x = (x_1, \dots, x_m) \in U : x_1 = \dots = x_n = 0\},$$

proving that F is an $m - n$ -dimensional embedded submanifold of M . □

EXERCISE 3.17. Show that

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$$

is a 2-dimensional embedded submanifold of \mathbb{R}^3 .

The notion of embedded submanifold allows us to introduce the smooth version of the notion of topological embedding. Recall that a map $f : N \rightarrow M$ between two topological spaces is a topological embedding if it is injective and, as a map from N to $f(N)$ (where the second space is now endowed with the topology induced from M), is a homeomorphism. The difference between the topological and the smooth case is that, while subspaces of a topological space inherit a natural induced topology, for smooth structures we need to restrict to embedded submanifolds.

DEFINITION 3.22. A smooth map $f : N \rightarrow M$ between two manifolds N and M is called a **smooth embedding** if:

- (1) $f(N)$ is an embedded submanifold of M .
- (2) as a map from N to $f(N)$, f is a diffeomorphism.

Here is an useful criterion for smooth embeddings.

THEOREM 3.23. A map $f : N \rightarrow M$ between two manifolds is a smooth embedding if and only if it is both a topological embedding as well as an immersion.

PROOF. The direct implication should be clear, so we concentrate on the converse. We first show that $f(N)$ is an embedded submanifold. We check the required condition at an arbitrary point $q = f(p) \in M$, with $p \in N$. For that use the immersion theorem (Theorem 3.12) around p ; we consider the resulting charts $\chi : U \rightarrow \Omega \subset \mathbb{R}^n$ for N around p and $\chi' : U' \rightarrow \Omega' \subset \mathbb{R}^m$ for M around q ; in particular,

$$f(U) = \{q \in U' : \chi'_{m+1}(q) = \dots = \chi'_n(q) = 0\}.$$

Since f is a topological embedding, $f(U)$ will be open in $f(N)$, i.e. of type

$$f(U) = f(U) \cap W$$

for some open neighborhood W of $q \in M$. It should be clear now that $U'' := U' \cap W$ and $\chi'' := \chi'|_{U''}$ defines a chart for M adapted to N .

We still have to show that, as a map $f : N \rightarrow f(N)$, f is a diffeomorphism (where $f(N)$ is with the smooth structure induced from M). It should be clear that this map continues to be an immersion. Since N and $f(N)$ have the same dimension (an injective maps between vectors spaces of the same dimension are automatically bijective), it follows from Lemma 3.13 that $f : N \rightarrow f(N)$ is a diffeomorphism. \square

And here is a very useful consequence:

COROLLARY 3.24. Let

$$f : N \rightarrow M$$

be a smooth map between two manifolds N and M , with the domain N being compact. Then f is an embedding if and only if it is an injective submersion.

PROOF. Use one of the main properties of compact spaces: injective maps from compacts to Hausdorff spaces are automatically topological embeddings! \square

In particular, using the same arguments as in the topological case (see Theorem 1.12 from Chapter 1) we find:

THEOREM 3.25. Any (smooth) compact manifold can be smoothly embedded into some Euclidean space.

PROOF. We use the same map as in the proof of Theorem 1.12 from Chapter 1, just that we now make use of smooth partitions of unity (which exist by Theorem 3.15). It suffices to show that the resulting topological embedding

$$i = (\eta_1, \dots, \eta_k, \tilde{\chi}_1, \dots, \tilde{\chi}_k) : M \rightarrow \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

is also an immersion. We check this at an arbitrary point $p \in M$. Since $\sum_i \eta_i = 1$, we find an i such that $\eta_i(p) \neq 0$. We may assume that $i = 1$. In particular, p must be in U_1 , and then we can choose the chart χ_1 around p and look at the representation i_{χ_1} with respect to this chart; we will check that it is an immersion at $x := \chi_1(p)$. Hence assume that the differential of i_{χ_1} at

x kills some vector $v \in \mathbb{R}^d$ (and we want to prove that $v = 0$). Then the differential of all the components of i_{χ_1} (taken at x) must kill v . But looking at those components, we remark that:

- the first \mathbb{R} -component is $\eta := \eta_1 \circ \chi_1^{-1} : U_1 \rightarrow \mathbb{R}$.
- the first \mathbb{R}^d component is the linear map $f : U_1 \rightarrow \mathbb{R}^d, f(u) = \eta(u) \cdot u$.

Hence we must have in particular:

$$(d\eta)_x(v) = 0, \quad (df)_x(v) = 0.$$

From the formula of f we see that $(df)_x(v) = (d\eta)_x(v) \cdot x + \eta(x) \cdot v$; hence, using that previous equations we find that $\eta(x) \cdot v = 0$. But $\eta(x) = \eta_1(p)$ was assumed to be non-zero, hence $v = 0$ as desired. \square

10. General (immersed) submanifolds

In Topology the leading principle for talking about "a subspace A of a space X " was to make sure that the inclusion $i : A \rightarrow X$ was continuous, and we chose "the best possible topology" on A doing that- and that gave rise to the induced (subspace) topology on any subset $A \subset X$. Looking for the analogous notion of "subspace" in the smooth context (i.e. a notion of "submanifold"), the first remark is that "inclusions" in the smooth context are expected to be immersions. Furthermore, starting with a subset N of a manifold M , there are several possible ways to proceed:

- (1) try to make the set N into a manifold (in particular into a topological space) such that the inclusion $i : N \hookrightarrow M$ is smooth (immersion).
- (2) in the previous point choose "the best possible" manifold structure on N .
- (3) try to make the space N , endowed with the topology induced from M , into a manifold such that the inclusion $i : N \hookrightarrow M$ is smooth (immersion)?

The previous section took care of the last possibility- which gave rise to the notion of embedded submanifold. The first possibility (i.e.e item (1) above) gives rise to the notion of immersed submanifold.

DEFINITION 3.26. *Given a manifold M , and **immersed submanifold** of M is a subset $N \subset M$ together with a structure of smooth manifold on N , such that the inclusion $i : N \rightarrow M$ is an immersion.*

We emphasize: an immersed submanifold is not just the subset $N \subset M$, but also the auxiliary data of a smooth structure on N (and, as the next exercise shows, given N there may be several different smooth structures on N that make it into an immersed submanifold). Moreover, the required smooth structure on N induces, in particular, a topology on N ; but we insist: this topology does not have to coincide with the one induced from M !

EXERCISE 3.18. Consider the figure eight in the plane \mathbb{R}^2 . Show that it is not an embedded submanifold of \mathbb{R}^2 , but it has at least two different smooth structures that make it into an immersed submanifold of \mathbb{R}^2 .

Immersed sub manifolds may look a bit strange/pathological, but they do arise naturally and one does have to deal with them. In general, any injective immersion

$$f : N \rightarrow M$$

gives rise to such an immersed submanifold: $f(N)$ together with the smooth structure obtained by transporting the smooth structure from N via the bijection $f : N \rightarrow f(N)$ (i.e. the charts of $f(N)$ are those of type $\chi \circ f^{-1}$ with χ a chart of N). And often immersed sub manifolds do arise naturally in this way. For instance, the two immersed submanifold structure on the figure

eight arise from two different injective immersions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ (with the same image: the figure eight), as indicated in the picture.

If N (... together with a smooth structure on it) is an immersed submanifold of M , the immersion theorem tells us that there exist charts (U, χ) of M around arbitrary points $p \in N$ such that $U_N := N \cap \{q \in U : \chi_{n+1}(q) = \dots = \chi_m(q) = 0\}$ is open in N and $\chi_N = \chi|_{U_N}$ is a chart for N . However, when saying that " U_N is open in N ", we are making use of the topology on N induced by the smooth structure we considered on N , and not with respect to the topology induced from M (look again at the previous exercise!). This is the difference with the notion of "embedded submanifold".

PROPOSITION 3.27. *If N is an embedded submanifold of the manifold M , then the set N admits precisely one smooth structure such that the inclusion into N becomes an immersion.*

Thinking about item (2) above, a rather natural class of subsets N of the manifold M to consider is obtained by requiring that the set N admits a unique smooth structure that makes it into an immersed submanifold of M . Such N s will be called **immersed submanifolds with unique smooth structure**. This is precisely the class of immersed submanifolds which are encoded just in the subset $N \subset M$ and no extra-data. By the previous proposition, embedded submanifolds have this property; however, the converse is not true- for instance the dutch figure eight is not embedded, but does have the uniqueness property (exercise).

However, it turns out that there is a smaller and more interesting class of immersed submanifolds that have the uniqueness property (and still includes the embedded submanifolds, as well as most of the other interesting examples).

DEFINITION 3.28. *An immersed submanifold N of a manifold M is called **an initial submanifold** if the following condition holds: for any*

- manifold P ,
- map $f : P \rightarrow N$,

one has that f is smooth if and only if it is smooth as a map into M .

PROPOSITION 3.29. *For a subset N of a manifold M one has the following:*

embedded submanifold \implies initial submanifold \implies submanifold with unique smooth structure.

PROOF. Consider the inclusion $i : N \hookrightarrow M$. For the first implication the main point is to show that if N is embedded and $f : P \rightarrow N$ has the property that $i \circ f : P \rightarrow M$ is smooth, then f is smooth. To check that f is smooth, we use arbitrary charts χ of P and charts χ' of M adapted to N . Note that

$$(i \circ f)\chi^{\chi'} = f_{\chi'}^{\chi_N} : \Omega \rightarrow \mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \subset \mathbb{R}^m,$$

for some open $\Omega \subset \mathbb{R}^p$. Hence it suffices to remark that if a function $g : \Omega \rightarrow \mathbb{R}^n$ is smooth as a map with values in the larger space \mathbb{R}^m , then it is smooth.

For the second part assume that N is initial. In particular, it comes with a smooth structure defined by some maximal atlas \mathcal{A} , making the inclusion into M an immersion. To prove the uniqueness property, we assume that we have a second smooth structure with the same property, with associated maximal atlas denoted \mathcal{A}' . Applying the initial condition to $P = (N, \mathcal{A}')$ with f being the inclusion into M , we find that the identity map $\text{id} : (N, \mathcal{A}') \rightarrow (N, \mathcal{A})$ is smooth. Applying the definition of smoothness, we see that any chart in \mathcal{A}' must be compatible with any chart in \mathcal{A} . Therefore $\mathcal{A}' = \mathcal{A}$. \square

Using similar arguments, you should try the following:

EXERCISE 3.19. Let N be a subset of the manifold M . Then:

- (i) For any topology on N , the resulting space N can be given at most one smooth structure that makes it into an immersed submanifold of M .
- (ii) If we endow N with the quotient topology, then the space N can be given a smooth structure that makes it into an immersed submanifold of M if and only if N is an embedded submanifold of M ; moreover, in this case the smooth structure on N is unique.

In general, for any two manifolds M and N of dimensions m and n , their product has a canonical structure of $(m+n)$ -dimensional manifold: if $\chi : U \rightarrow \Omega \subset \mathbb{R}^m$ is a smooth chart of M and $\chi' : U' \rightarrow \Omega' \subset \mathbb{R}^n$ one of N , then

$$\chi \times \chi' : U \times U' \rightarrow \Omega \times \Omega' \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}, \quad (p, q) \mapsto (\chi(p), \chi'(q))$$

will define a smooth chart of $M \times N$ (we do not ask you to prove anything here).

Take now $M = N = S^1$ the unit circle with the standard smooth structure, and we consider the resulting manifold $S^1 \times S^1$. While $S^1 \subset \mathbb{R}^2$, $S^1 \times S^1$ sits naturally inside $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. We now want to embed it in \mathbb{R}^3 , i.e. find a submanifold T of \mathbb{R}^3 that is diffeomorphic to $S^1 \times S^1$.

EXERCISE 3.20. We define $T \subset \mathbb{R}^3$ as the image of the map:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad f(\alpha, \beta) = ((2 + \cos \alpha) \cos \beta, (2 + \cos \alpha) \sin \beta, \sin \alpha).$$

Do the following:

- (1) write T as $g^{-1}(0)$ (the zero-set of g) for some smooth map $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ which is a submersion at each point in T . This ensures that T is indeed a submanifold of \mathbb{R}^3 .
- (2) show that f is an immersion.
- (3) using f find a homeomorphism $f_0 : S^1 \times S^1 \rightarrow T$ and prove that it is an immersion.
- (4) then show that $f_0 : S^1 \times S^1 \rightarrow T$ is a diffeomorphism.
- (5) if you haven't done that already: draw a picture of T inside \mathbb{R}^3 .

(Hint: try to avoid ugly computations and use what we have already discussed in the class).

11. More examples: classical groups and ... Lie groups

One very interesting (and special) class of manifolds are the so-called Lie groups: they are both groups, as well as manifolds, and the two structures are compatible (see below for the precise definition). The classical examples are groups of matrices (endowed with the usual product of matrices). They sit inside the space of all $n \times n$ matrices (for some n natural number)- which is itself and Euclidean space (just that the variables are arranged in a table rather than in a row):

$$\mathcal{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

for matrices with real entries, and

$$\mathcal{M}_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$$

for matrices with complex entries. Since we are interested in groups w.r.t. multiplication of matrices (and in a group one can talk about inverses), we have to look inside the so called **general linear group**:

$$GL_n(\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) : \det(A) \neq 0\},$$

and similarly $GL_n(\mathbb{C})$. Since the determinant is a polynomial function, it is also continuous, hence the groups GL_n are opens inside the Euclidean spaces \mathcal{M}_n - therefore they are manifolds in a canonical way (and the usual entries of matrices serves as global coordinates). Inside these there are several interesting groups such as:

- **the orthogonal group:**

$$O(n) = \{A \in GL_n(\mathbb{R}) : A \cdot A^T = I\}.$$

- **the special orthogonal group:**

$$SO(n) = \{A \in O(n) : \det(A) = 1\}.$$

- **the special linear group:**

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}.$$

- **the unitary group:**

$$U(n) = \{A \in GL_n(\mathbb{C}) : A \cdot A^* = I\}.$$

- **the special unitary group:**

$$SU(n) = \{A \in GL_n(\mathbb{C}) : A \cdot A^* = I, \det(A) = 1\}.$$

- **the symplectic group:**

$$Sp_n(\mathbb{R}) := \{A \in GL_{2n}(\mathbb{R}) : A^T J A = J\},$$

where

$$(11.1) \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in GL_{2n}(\mathbb{R}).$$

Recall that A^T denotes the transpose of A , A^* the conjugate transpose.

We would like to point out that they carry a natural smooth structure (making them into Lie groups); actually, they are all embedded submanifolds of the corresponding (Euclidean) spaces of matrices. Since they are all given by (algebraic) equations, there is a natural to proceed: use the regular value theorem (Theorem 3.21). As an illustration, let us look at the case of $O(n)$.

EXAMPLE 3.30 (the details in the case of $O(n)$). The equation defining $O(n)$ suggest that we should be using the function

$$f : GL_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R}), \quad f(A) = A \cdot A^T.$$

However, it is important to note that the values of this function takes values in a smaller Euclidean space. Indeed, since in general, $(A \cdot B)^T = B^T \cdot A^T$, the matrices of type $A \cdot A^T$ are always symmetric. Therefore, denoting by \mathcal{S}_n the space of symmetric $n \times n$ matrices (again an Euclidean space, of dimension $\frac{n(n+1)}{2}$), we will deal with f as a map

$$f : GL_n(\mathbb{R}) \rightarrow \mathcal{S}_n.$$

(If we didn't remark that f was taking values in \mathcal{S}_n , we would not have been able to apply the regular value theorem; however, the problem that we would have encountered would clearly indicate that we have to return and make use of \mathcal{S}_n from the beginning; so, after all, there is no mystery here).

Now, f is a map from an open in an Euclidean space (and we could even use f defined on the entire $\mathcal{M}_n(\mathbb{R})$), with values in another Euclidean space; and it is clearly smooth since it is given by polynomial expressions. So, at each point $A \in GL_n(\mathbb{R})$, the differential of f at A ,

$$(df)_A : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{S}_n$$

can be computed by the usual formula:

$$(df)_A(X) = \left. \frac{d}{dt} \right|_{t=0} f(A+tX) = \left. \frac{d}{dt} \right|_{t=0} (A \cdot A^T + t(A \cdot X^T + A^T \cdot X) + t^2 X \cdot X^T) = A \cdot X^T + X \cdot A^T.$$

Since $O(n) = f^{-1}(\{I\})$, we have to show that $(df)_A$ is surjective for each $A \in O(n)$. I.e., for $Y \in \mathcal{S}_n$, show that the equation $A \cdot X^T + X \cdot A^T = Y$ has a solution $X \in \mathcal{M}_n(\mathbb{R})$. Well, it does, namely: $X = \frac{1}{2}YA$ (how did we find it?). Therefore $O(n)$ is a smooth sub manifold of GL_n of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Note that it also follows that the tangent space of $O(n)$ at the identity,

$$T_I O(n) \subset T_I GL_n(\mathbb{R}) = \mathcal{M}_n(\mathbb{R}),$$

coincides with the kernel of $(df)_I$, i.e. it is the space of antisymmetric matrices, usually denoted:

$$o(n) := \{X \in \mathcal{M}_n(\mathbb{R}) : X + X^T = 0\}.$$

The tangent space of such groups, taken at the identity matrix, play a very special role and are known under the name of *Lie algebra of the group* (well, the name also refers to some extra-structure they inherit- but we will come back to that later, after we discuss tangent vectors and vector fields). For the other groups one proceeds similarly and we find that they are all embedded submanifolds of GL_n s, with:

- $SO(n)$: of dimension $\frac{n(n-1)}{2}$, with

$$T_I SO(n) = o(n) = \{X \in \mathcal{M}_n(\mathbb{R}) : X + X^T = 0\}.$$

- $SL_n(\mathbb{R})$: of dimension $n^2 - 1$, with

$$T_I SL_n(\mathbb{R}) = sl_n(\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) : Tr(X) = 0\}.$$

- $U(n)$: of dimension n^2 , with

$$T_I U(n) = u(n) = \{X \in \mathcal{M}_n(\mathbb{C}) : X + X^* = 0\}.$$

- $SU(n)$: of dimension $n^2 - 1$, with

$$T_I SU(n) = su(n) = \{X \in \mathcal{M}_n(\mathbb{C}) : X + X^* = 0, Tr(X) = 0\}.$$

EXERCISE 3.21. Do the same for the other groups in the list. At least for one more.

EXERCISE 3.22. The aim of this exercise is to show how $GL_n(\mathbb{C})$ sits inside $GL_{2n}(\mathbb{R})$, and similarly for the other complex groups. Well, there is an obvious map

$$j : GL_n(\mathbb{C}) \rightarrow GL_{2n}(\mathbb{R}), \quad A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Show that:

- (1) j is an embedding, with image $\{X \in GL_{2n}(\mathbb{R}) : J \cdot X = X \cdot J\}$, where J is the matrix (11.1).
- (2) j takes $U(n)$ into $O(n)$.
- (3) actually j takes $U(n)$ into $SO(n)$.

(Hints:

- for (2): first check that $j(Z^*) = j(Z)^T$ for all $Z \in GL_n(\mathbb{C})$.

- for (3): First prove that $\det(j(Z)) = |\det(Z)|^2$ for all $Z \in GL_n(\mathbb{C})$. For this: show that there exists a matrix $X \in GL_n(\mathbb{R})$ such that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1} \\ -I & X \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Instead of handling each of the groups separately, there is a much stronger result that ensures that such groups are smooth manifolds:

THEOREM 3.31. *Any closed subgroup of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ is automatically an embedded submanifold.*

Of course, this theorem is very useful and one should at least be aware of it. The proof is not tremendously difficult, but it would require one entire lecture- and that we cannot afford. But, in principle, the idea is simple: starting with $G \subset GL_n(\mathbb{R})$ closed subgroups, one looks at $\mathfrak{g} := T_I G \subset \mathcal{M}_n$, one shows that it is a vector subspace, and one uses the exponential of matrices ($\exp : \mathcal{M}_n \rightarrow GL_n$) restricted to \mathfrak{g} to produce a chart for G around $\exp(0) = I$ (the identity matrix); for charts around other points, one uses the group structure to translate the chart around I to a chart around any point in G .

Finally, since we have mentioned "Lie groups", here is the precise definition:

DEFINITION 3.32. *A Lie group G is a group which is also a manifold, such that the two structures are compatible in the sense that the multiplication and the inversion operations,*

$$m : G \times G \rightarrow G, m(g, h) = gh, \quad \iota : G \rightarrow G, \iota(g) = g^{-1},$$

are smooth.

An isomorphism between two Lie groups G and H is any group homomorphism $f : G \rightarrow H$ which is a diffeomorphism.

Of course, all the groups that we mentioned are Lie groups (just think e.g. that the multiplication is given by a polynomial formula!); and, the conclusion of the last theorem is that all closed subgroups are actually Lie groups.

EXAMPLE 3.33. The unit circle S^1 , identified with the space of complex numbers of norm one,

$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

is a Lie group with respect to the usual multiplication of complex numbers. Actually, we see that

$$S^1 = SU(1).$$

Also, looking at $SO(2)$, we see that it consists of matrices of type

$$\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$

and we obtain an obvious diffeomorphism (an isomorphism of Lie groups) $SO(2) \cong S^1$.

EXAMPLE 3.34. Similarly, the 3-sphere S^3 can be made into a (non-commutative, this time) Lie group. For that we replace \mathbb{C} by the space of quaternions:

$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$

where we recall that the product in \mathbb{H} is uniquely determined by the fact that it is \mathbb{R} -bilinear and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Recall also that for

$$u = x + iy + jz + kt \in \mathbb{H}$$

one defines

$$u^* = x - iy - jz - kt \in \mathbb{H}, \quad |u| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2} \in \mathbb{R}.$$

Then, the basic property $|u \cdot v| = |u| \cdot |v|$ still holds and we see that, identifying S^3 with the space of quaternionic numbers of norm 1, S^3 becomes a Lie group.

EXERCISE 3.23. Show that

$$F : S^3 \rightarrow SU(2), \quad F(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where we interpret S^3 as $\{(\alpha, \beta) \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\}$, is an isomorphism of Lie groups.

By analogy with the previous example, one may expect that S^3 is isomorphic also to $SO(3)$ (at least the dimensions match!). However, the relation between these two is more subtle:

EXERCISE 3.24. Recall that we view S^3 inside \mathbb{H} . We also identify \mathbb{R}^3 with the space of pure quaternions

$$\mathbb{R}^3 \xrightarrow{\sim} \{v \in \mathbb{H} : v + v^* = 0\}, \quad (a, b, c) \mapsto ai + bj + ck.$$

For each $u \in S^3$, show that

$$A_u(v) := u^*vu$$

defines a linear map $A_u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which, as a matrix, gives an element

$$A_u \in SO(3).$$

Then show that the resulting map

$$\phi : S^3 \rightarrow SO(3), \quad u \mapsto A_u$$

is smooth, is a group homomorphism, is a surjective local diffeomorphism, but each fiber has two elements (it is a 2-1 cover).

Then deduce that $SO(3)$ is diffeomorphic to the real projective space \mathbb{P}^3 .

REMARK 3.35. One may wonder: which spheres can be made into Lie groups? Well, it turns out that S^0 , S^1 and S^3 are the only ones!

On the other hand, it is interesting to understand what happens with the arguments we used for S^1 and S^3 (and which were very similar to each other) in higher dimensions. The main point there was the multiplication on \mathbb{C} and \mathbb{H} and the presence of a norm such that

$$(11.2) \quad |x \cdot y| = |x| \cdot |y|$$

(so that, for two elements in the sphere, i.e. of norm one, their product is again in the sphere). So, to handle S^n similarly, we would need a "normed division algebra" structure on \mathbb{R}^{n+1} , by which we mean a multiplication "." on \mathbb{R}^{n+1} that is bilinear and a norm satisfying the previous condition. Again, it is only on \mathbb{R} , \mathbb{R}^2 and \mathbb{R}^4 that such a multiplication exists. If we do not insist on the associativity of the multiplication, there is one more possibility: \mathbb{R}^8 (the so called octonions). But nothing else! And this was known since the 19th century! But why did people care about such operations in the 19th century? Well ... it was number theory and the question of which numbers (integers) can be written as a sum of two, three, etc squares. For sum of two squares the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2 + y^2)(a^2 + b^2) = (xa - yb)^2 + (xb + ya)^2.$$

Or, in terms of the complex numbers $z_1 = x + iy$, $z_2 = a + ib$, the norm equation (11.2). The search for similar "magic formulas" for sum of three squares never worked, but it did for four:

$$(x^2 + y^2 + z^2 + t^2)(a^2 + b^2 + c^2 + d^2) =$$

$$(xa + yb + zc + td)^2 + (xb - ya - zd + tc)^2 + \\ +(xc + yd - za - tb)^2 + (xd - yc + zb - ta)^2.$$

This is governed by the quaternions and its norm equation (11.2).