CHAPTER 5

Differential forms

1. Differential forms of degree one (1-forms)

For a (real) vector space V, its dual V^* is the vector space consisting of all linear maps $\xi : V \to \mathbb{R}$. While the elements of V are vectors of V, the elements of V^* are thought of as "covectors of V". And vectors and covectors pair via the evaluation map

$$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}, \quad (\xi, v) \mapsto \langle \xi, v \rangle := \xi(v).$$

Moreover, the equations

$$\langle e^i, e_j \rangle = \delta^i_j$$

show how to produce a basis $e^1, \ldots, e^m$ of V^* from a basis $e_1, \ldots, e_m$ of M (and the other way around).

Differential 1-forms on a manifold M arise similarly, in duality with vector fields. First of all, for each $p \in M$, we define **the cotangent space** of M at p as the dual of the tangent space

$$T^*_p M := (T_p M)^*.$$

The elements of $T^*_p M$ are called **cotangent vectors at** p.

EXAMPLE 5.1. For any smooth function $f : M \to \mathbb{R}$ its differential at any point $p \in M$,

$$(df)_p : T_p M \to \mathbb{R}$$

can now be interpreted as a cotangent vector at p,

$$(df)_p \in T^*_p M.$$

Of course, for this to make sense we only need f to be defined on an open containing p.

While a chart χ around p gives rise to a basis of $T_p M$ (for p in the domain of χ), it will also give rise to a basis of $T^*_p M$: just apply the previous example to the components $\chi_i : U \to \mathbb{R}$ of χ to obtain cotangent vectors at p

$$(d\chi_1)_p, \ldots, (d\chi_m)_p.$$

To see that this is a basis one can just remark that this is dual to the similar basis of $T_p M$; more precisely, it follows right away from the definitions of all the term involved that

$$\langle (d\chi_i)_p, \left(\frac{d}{d\chi_j} \right)_p \rangle = \delta^i_j$$

for all i and j.

Similar to set-theoretical vector fields on M we can now talk about set-theoretical cotangent vector fields on M, by which we mean maps

$$\omega : M \ni p \mapsto \omega_p \in T^*_p M.$$

To make sense of their smoothness one proceeds like for vector fields: with respect to any chart $\chi : U \to \Omega \subset \mathbb{R}^m$, we can write

$$\omega_p = \sum_i \omega^i_\chi(\chi(p))(d\chi_i)_p,$$

with $\omega^i_\chi : \Omega \to \mathbb{R}$ called the coefficients of ω with respect to the chart χ.

DEFINITION 5.2. *A **differential 1-form** on M is any map*

$$\omega : M \ni p \mapsto \omega_p \in T^*_p M$$

whose coefficients w.r.t. any chart are smooth. We denote by $\Omega^1(M)$ the vector space of all such 1-forms (with the vector space structure defined by $(\omega + \omega')_p = \omega_p + \omega'_p$, etc).

EXAMPLE 5.3 (differentials of functions). Continuing the previous example, we see that for any function $f \in \mathcal{C}^\infty(M)$ its differential now becomes a 1-form

$$df \in \Omega^1(M)$$

(why is it smooth?). In this way, the differentiation of $\mathbb{R}$-valued functions becomes a linear operator

$$d : \mathcal{C}^\infty(M) \to \Omega^1(M).$$

EXAMPLE 5.4 (expressions of type $\sum f_i dg_i$). As for vector fields, 1-forms ω can be multiplied by smooth functions f to produce a new 1-form $f \cdot \omega$ (whose value at $p \in M$ is $f(p) \cdot \omega_p$). More abstractly, one has an operation

$$\mathcal{C}^\infty(M) \times \Omega^1(M) \to \Omega^1(M)$$

which makes the space of 1-forms into a $C^\infty(M)$-module.

Combining with differentials of functions, we see see that expressions of type

$$\sum_{i=1}^k f_i \cdot dg_i$$

now make sense as 1-forms. Actually, any 1-form is locally representable by such an expression (of course, in a non-unique way). Looking in a chart, this becomes a property of $\mathbb{R}^m$; and it should be clear that any 1-form ω on $\mathbb{R}^m$ can be uniquely written as

$$\omega = \sum f_i \cdot dx_i \tag{1.1}$$

with $f_i \in \mathcal{C}^\infty(\mathbb{R}^m)$ (where x_i are the coordinate functions $\in \mathcal{C}^\infty(\mathbb{R}^m)$),

EXERCISE 5.1. Show that:

1. For any function $f \in \mathcal{C}^\infty(\mathbb{R}^m)$, one has
$$df = \sum_i \frac{\partial f}{\partial x_i} \cdot dx_i.$$
2. Show that the 1-form on $\mathbb{R}^2$ given by
$$\omega = ydx - xdy$$
is not exact, i.e. cannot be written as df for some smooth function f.
3. Show that, for a 1-form $\omega \in \Omega^1(M)$ on a manifold M to be exact, i.e. to be of type df for some smooth function f, a necessary (but, as we shall see, not sufficient) condition is that
$$\omega([X,Y]) = L_X(\omega(Y)) - L_Y(\omega(X)) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

4. How is item 3. related to item 2?

EXAMPLE 5.5 (restrictions to submanifolds). There is a "restriction operation" of 1-forms $\omega \in \Omega^1(M)$ to arbitrary submanifolds $N \subset M$. Indeed, since any tangent vector $X_p \in T_pN$ can be viewed also as a tangent vector to M, one can just evaluate ω_p on any $X_p \in T_pN$. The resulting 1-form on N is called the restriction of ω to N and is denoted $\omega|_N$. Hence

$$(\omega|_N)_p (X_p) := \omega_p(X_p)$$

defines the restriction operation

$$\text{restr} : \Omega^1(M) \to \Omega^1(N), \quad \omega \mapsto \omega|_N.$$

This operation is particularly useful when N is an embedded submanifold of an Euclidean space $\mathbb{R}^k$: one can produce 1-forms on N by starting with 1-forms on $\mathbb{R}^k$ (necessarily of type (1.1)); and, even more, any 1-form on N arises in this way (but this last statement is not completely obvious). In other words, one can represent 1-form on N by 1-forms on $\mathbb{R}^k$. E.g., for $N = S^2$, we can now talk about

$$(x^2 + y) \cdot dx + dy + xyz \cdot dz \text{ on } S^2,$$

by which we mean the restriction of $(x^2 + y) \cdot dx + dy + xyz \cdot dz$ from $\mathbb{R}^3$ to $S^2 \subset \mathbb{R}^3$.

However, one should be aware that different forms on $\mathbb{R}^k$ may give the same restriction to N. E.g., for $S^2 \subset \mathbb{R}^3$, there are expressions that look like being non-zero (and they are non-zero as 1-forms on $\mathbb{R}^3$) but which represent the zero form on S^2. For instance,

$$x \cdot dx + y \cdot dy + z \cdot dz \text{ on } S^2$$

is zero. Indeed, viewing it as a 1-form on S^2, it means we evaluate it only on vectors tangent to S^2; and, for $p = (x, y, z) \in S^2$ and an arbitrary tangent vector

$$V_p = a \left(\frac{\partial}{\partial x}\right)_p + b \left(\frac{\partial}{\partial y}\right)_p + c \left(\frac{\partial}{\partial z}\right)_p \in T_pS^2$$

one has

$$(x \cdot dx + y \cdot dy + z \cdot dz)(V_p) = ax + by + cz = 0$$

since V was tangent to S^2. In other words,

$$(x \cdot dx + y \cdot dy + z \cdot dz)|_{S^2} = 0.$$

EXERCISE 5.2. Show that the 1-form on S^1 given by

$$\omega = (ydx - xdy)\,|_{S^1}$$

is not exact, i.e. cannot be written as df with $f \in \mathcal{C}^\infty(S^1)$.

EXERCISE 5.3. Show that for any $f \in \mathcal{C}^\infty(M)$ and any chart (U, χ) one has, on U,

$$df = \sum_i \frac{\partial f}{\partial \chi_i} \cdot \chi_i.$$

Deduce that for any two coordinate charts (U, χ) and (U', χ') one has, on $U \cap U'$,

$$d\chi_i' = \sum_j \frac{\partial \chi_i'}{\partial \chi_j} \cdot d\chi_j. \tag{1.2}$$

Then, using the duality between $\frac{\partial}{\partial \chi_i}$ and $d\chi_i$, show again that, over $U \cap U'$, one has:

$$\frac{\partial}{\partial \chi_i} = \sum \frac{\partial \chi_j'}{\partial \chi_i} \cdot \frac{\partial}{\partial \chi_j'}.$$

And here is one very nice feature of 1-forms: unlike vector fields, 1-forms can be pulled-back along *any* smooth map. Actually, an instance of this operation is the restriction to submanifolds that we have just described- which corresponds to pull-backs via the inclusion map.

To explain this, we start with a smooth map $F : M \to N$ and $\omega \in \Omega^1(N)$. Then at each point $p \in M$ we can apply the differential $(dF)_p : T_pM \to T_{F(p)}N$ followed by $\omega_{F(p)} : T_{F(p)}N \to \mathbb{R}$ to obtain a covector on M at p:

$$T_pM \ni X_p \mapsto \omega_{F(p)}\left((dF)_p(X_p)\right) \in \mathbb{R}$$

(compare with vector fields!). The resulting 1-form on M is denoted by

$$F^*(\omega) \in \Omega^1(M)$$

and is called **the pull-back of ω by F**.

EXERCISE 5.4. Show that, indeed, $F^*(\omega)$ is smooth. Also check that, when $M \subset N$ and F is the inclusion, then $F^*(\omega) = \omega|_M$.

Therefore, any smooth map $F : M \to N$ gives rise to an operation

$$F^* : \Omega^1(N) \to \Omega^1(M).$$

EXERCISE 5.5. Show that for any $F : M \to N$ smooth, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{C}^\infty(N) & \xrightarrow{d} & \Omega^1(N) \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ \mathcal{C}^\infty(M) & \xrightarrow{d} & \Omega^1(M) \end{array}.$$

While covectors of a vector space V take vectors to real numbers, 1-forms take vector fields to smooth functions: given ω as above and $X \in \mathfrak{X}(M)$, evaluating ω_p on X_p for each $p \in M$ we obtain a smooth function

$$\omega(X) \in \mathcal{C}^\infty(M);$$

(why smooth?). When we vary X it is clear that the resulting map

$$\mathfrak{X}(M) \to \mathcal{C}^\infty(M), \quad X \to \omega(X)$$

is $C^\infty(M)$-linear, i.e. it is linear and

$$\omega(f \cdot X) = f \cdot \omega(X) \quad \text{for all } f \in \mathcal{C}^\infty(M), X \in \mathfrak{X}(M).$$

We denote by

$$\mathrm{Hom}_{\mathcal{C}^\infty}(\mathfrak{X}(M), \mathcal{C}^\infty(M))$$

the space of all $\mathcal{C}^\infty(M)$-linear maps from $\mathfrak{X}(M)$ to $\mathcal{C}^\infty(M)$.

THEOREM 5.6. *Interpreting any $\omega \in \Omega^1(M)$ as a map $\mathfrak{X}(M) \to \mathcal{C}^\infty(M)$, we obtain an isomorphism*

$$\Omega^1(M) \cong Hom_{\mathcal{C}^\infty}(\mathfrak{X}(M), \mathcal{C}^\infty(M)).$$

PROOF. Note first that, by the same arguments as in Proposition 4.18 from Chapter 4, for a set-theoretical 1-form ω its smoothness is equivalent to the fact that $\omega(X)$ is a smooth function on M for all $X \in \mathfrak{X}(M)$.

Another ingredient that we need in the proof is the fact that, for each $p \in M$, the evaluation of vector fields at p,

$$\mathrm{ev}_p : \mathfrak{X}(M) \to T_pM, \quad X \mapsto X_p,$$

is surjective. To show this, we start with an arbitrary tangent vector $v \in M$ (p is fixed now). Using a coordinate chart around p, we find a vector field Y defined on an open neighborhood U of p such that $Y_p = v$. As in the proof of Lemma 4.12 from Chapter 4, choose a smooth function η on M, supported inside U and with $\eta(p) = 1$. Then $X := \eta \cdot Y$ is well-defined and smooth on M and $X_p = \eta(p) \cdot v = v$.

We now move to the actual proof. For the injectivity of the map we have to show that, if $\omega_p(X_p) = 0$ for all $X \in \mathfrak{X}(M)$ and all $p \in M$, then $\omega_p = 0$ for all $p \in M$. But this clearly follows from the surjectivity of ev_p.

We now prove the surjectivity of the map in the statement. Let $A : \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ be a $\mathcal{C}^\infty(M)$-linear map. We fix $p \in M$ arbitrary and we define

$$\omega_p : T_pM \to \mathbb{R}, \quad \omega_p(v) := A(Y)(p)$$

where A is any vector field with $Y_p = v$. If we check that this definition does not depend on the choice of Y then we are done: linearity of each ω_p follows right away, while the resulting set-theoretical 1-form is also smooth since, by construction, $\omega(X) = A$.

To see that $A(Y)(p)$ does not depend on the choice of Y, assume that Y' is another vector field with $Y'_p = v$. Then $X := Y - Y'$ vanishes at p and we have to show that $A(X)(p) = 0$. Using the $\mathcal{C}^\infty(M)$-linearity of A, this will follow from the following:

LEMMA 5.7. *If a vector field $X \in \mathfrak{X}(M)$ vanishes at a point $p \in M$, then we can write*

$$X = \sum_{j=0}^{k} f_j \cdot X^j$$

with $k \geq 1$ integer, $X_j \in \mathfrak{X}(M)$ and f_j smooth functions that vanish at p.

PROOF. We first look in a coordinate chart (U, χ) around p which sends p to $\chi(p) = 0 \in \mathbb{R}^m$. Then writing

$$X|_U = \sum_{i=1}^{m} F_i \cdot \frac{\partial}{\partial \chi_i},$$

the coefficients F_i are smooth functions on U that vanish at p. By the the proof of Theorem 4.15 from Chapter 4 applied to $F_i \circ \chi^{-1}$, we see that each F_i can be written as a finite sum $\sum_j \chi_j \cdot g_i^j$ with $g_i^j : U \to \mathbb{R}$ smooth. From this we immediately deduce that we can find some smooth functions $f_1, \ldots, f_k$ on U and vector fields $Y_1, \ldots, Y_k$ (on U) such that

$$X|_U = \sum_{j=1}^{k} f_j \cdot Y^j.$$

To go from U to the entire M, the main idea is to write $X = \eta \cdot X + (1 - \eta) \cdot X$, for some carefully chosen smooth function η.

Note that, after eventually making U smaller, we may assume that $f_i = \tilde{f}_i|_U$ for some smooth function $\tilde{f}_i \in \mathcal{C}^\infty(M)$; to see this, just use the second dotted item in the proof of Lemma 4.13 from Chapter 4. Furthermore, as in that proof, we choose a smooth function $\eta \in C^\infty(M)$ which is supported inside U and is 1 in some neighborhood V of p. Since η is supported inside U, each $\eta|_U \cdot Y^j \in \mathfrak{X}(U)$ can be promoted to a vector field $X^j \in \mathfrak{X}(M)$ by declaring it to be zero outside

U. With the inspiration from the "main idea" mentioned above, we now remark that

$$X = (1-\eta)\cdot X + \sum_{j=1}^{k} \tilde{f}_j \cdot \tilde{X}^j$$

on the entire M. Indeed, at $p \in U$ the right hand side is

$$(1-\eta(p))\cdot X_p + \eta(p)\cdot \sum_{j=1}^{k} f_j(p)\cdot X_p^j = (1-\eta(p))\cdot X_p + \eta(p)\cdot X_p = X_p,$$

while at p outside U we obtain $(1-0)\cdot X_p + \sum_j \tilde{f}_j(p)\cdot 0 = X_p$.

Therefore the desired decomposition is achieved if we take $f_0 = 1-\eta$ and $X^0 = X$. □

□

And, still as in the case of tangent vectors, one can form the cotangent bundle

$$T^*M := \sqcup_{p\in M} T_p^*M = \{(p,\omega_p) : p \in M, \omega_p \in T_p^*M\}$$

and put a smooth structure on T^*M so that the smoothness of a set-theoretical 1-form ω is equivalent to the smoothness of ω as a map from M to T^*M.

Exercise 5.6. Fill in the details, i.e. state and prove a version of Proposition 4.21 but for T^*M instead of TM.

2. Differential forms: the linear algebra behind them

The passage from 1-forms to arbitrary k-forms is basically a passage from linear maps $T_pM \to \mathbb{R}$ to k-multilinear skew-symmetric maps. Here we discuss the linear algebra involved.

Definition 5.8. *Given a vector space V and $k \in \mathbb{N}$ a natural number, a k-covector on V is any map*

$$\eta : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$$

that is multiplinear (i.e. linear in each argument). We say that ω if a linear k-form on V if

$$\omega(v_\sigma(1), \ldots, v_{\sigma(k)}) = sign(\sigma)\omega(v_1, \ldots, v_k)$$

for any $v_1, \ldots, v_k \in V$ and any permutation $\sigma \in S_k$. We denote by $\Lambda^k V^*$ the (vector) space of all such linear k-forms on V.

Exercise 5.7. Show that ω is skew-symmetric if and only if the expression $\omega(v_1, \ldots, v_k)$ changes sign whenever two entries v_i and v_j are interchanged.

Although our interest is $\Lambda^k V^*$, it will be useful to consider more general k-multiplinear maps (without being skew-symmetric); we denote the space of such by $T^k V^*$. Note that, while

$$\Lambda^k V^* \subset T^k V^*,$$

there is also a skew-symmetrization map going the other way,

$$\text{Alt} : T^k V^* \to \Lambda^k V^*$$

where for $\omega \in T^k V^*$, $\text{Alt}(\omega)$ is given by

$$\text{Alt}(\omega)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma)\omega(v_\sigma(1), \ldots, v_{\sigma(k)}).$$

EXERCISE 5.8. Show that, for $\omega \in T^k V^*$, one has:

$$\omega \in \Lambda^k V^* \Longleftrightarrow \mathrm{Alt}(\omega) = \omega.$$

The main operation on forms is that of wedge-product. Before giving the precise definition/formula, note first that there a pretty obvious operation involving the spaces $T^k V^*$, namely:

$$T^k V^* \times T^l V^* \to T^{k+l} V^*, \quad (\omega, \eta) \mapsto \omega \bullet \eta,$$

where $\omega \bullet \eta$ is given by

$$(\omega \bullet \eta)(v_1, \ldots, v_{k+l}) := \omega(v_1, \ldots, v_k) \cdot \eta(v_{k+1}, \ldots, v_{k+l}).$$

EXERCISE 5.9. Do the following:

(1) If $\omega \in T^2 V^*$ is symmetric and $\eta \in T^l V^*$ is arbitrary, show that $\mathrm{Alt}(\omega \bullet \eta) = 0$.
(2) Prove the same but now assuming that $\omega \in T^k V^*$ satisfies $\mathrm{Alt}(\omega) = 0$.
(3) Deduce that for any $\omega \in T^k V^*$ and $\eta \in T^l V^*$ one has $\mathrm{Alt}(\omega \bullet \eta) = \mathrm{Alt}(\mathrm{Alt}(\omega) \bullet \eta)$.

However, if ω and η are skew-symmetric, $\omega \cdot \eta$ may fail to be skew-symmetric. Therefore, to obtain a skew-symmetric element, we will apply the skew-symmetrization map Alt. To avoid un-necessary coefficients in the final formula we will do a small rescaling (see also below):

DEFINITION 5.9. *Given $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$, define the element*

$$\omega \wedge \eta \in \Lambda^{k+l} V^*,$$

called the ***wedge product of*** ω ***and*** η*, by*

$$\omega \wedge \eta = \frac{(k+l)!}{k! \cdot l!} Alt(\omega \bullet \eta).$$

REMARK 5.10. When computing $\mathrm{Alt}(\omega \cdot \eta)$, the fact that ω and η are skew-symmetric will be reflected in the fact that many of the terms will repeat. When we avoid repetitions, we get sums not over all permutations $\sigma \in S_{k+l}$ but only over those with the property that

$$\sigma(1) < \ldots < \sigma(k), \quad \sigma(k+1) < \ldots < \sigma(k+l).$$

Such permutations will be called (k,l)**-shuffles** and we denote by $S_{k,l} \subset S_{k+l}$ the resulting set. All together, we find the following explicit formula for the wedge product:

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+l}) := \sum_{\sigma \in S_{k,l}} \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

For instance, for $\omega, \eta \in V^*$ one obtains $\omega \wedge \eta \in \Lambda^2 V^*$ given by

$$(\omega \wedge \eta)(v, w) = \omega(v)\eta(w) - \omega(w)\eta(v).$$

Or, for $\omega, \in \Lambda^2 V^*, \eta \in V^*$,

$$(\omega \wedge \eta)(u, v, w) = \omega(u, v)\eta(w) + \omega(v, w)\eta(u) - \omega(u, w)\eta(v).$$

The reason for the coefficients $k!$ used in the definition of the wedge was precisely obtain such formulas without coefficients (incidentally, let us point out that the resulting formula now makes send over any field, not necessarily of characteristic zero). Actually, replacing $k!$ by any coefficients c_k would still produce (modified) wedge-products with similar properties (see the next proposition) but, in some (precise) sense, they are all "isomorphic". However, simple choices of c_ks, such as $c_k = 1$, would produce uglier final formulas.

In this way we obtain operations

$$\cdot \wedge \cdot : \Lambda^k V^* \times \Lambda^l V^* \to \Lambda^{k+l} V^*.$$

Here are the main properties of these operations:

PROPOSITION 5.11. *The wedge operation is bilinear, and:*

(1) *graded-commutativity: for any $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$,*

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

(2) *associativity: for any linear forms ω, η and ζ one has*

$$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta).$$

PROOF. For graded-commutativity, we look at $\eta \wedge \omega(v_1, \ldots, v_{k+l})$. The sum defining it has expressions of type

$$\text{sign}(\sigma)\eta(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \cdot \omega(v_{\sigma(l+1)}, \ldots, v_{\sigma(l+k)})$$

where σ is an (l,k)-shuffle. Now, any such σ defines another permutation $\hat{\sigma}$ by

$$\hat{\sigma}(1) = \sigma(l+1), \ldots, \hat{\sigma}(k) = \sigma(l+k), \quad \hat{\sigma}(k+1) = \sigma(1), \ldots, \hat{\sigma}(k+l) = \sigma(l)$$

which is a (k,l)-shuffle. A moment of reflection shows that

$$\text{sign}(\hat{\sigma}) = (-1)^{kl}\text{sign}(\sigma).$$

All these clearly imply the graded-commutativity.

For associativity, up to a constant that is the same on both sides, we have to show that

$$\text{Alt}(\text{Alt}\,(\omega \bullet \theta) \bullet \zeta) = \text{Alt}(\omega \bullet \text{Alt}\,(\theta \bullet \zeta))\,.$$

Using the last part of Exercise 5.9 to rewrite the left hand side, and the similar argument for the right hand side, what we have to prove becomes:

$$\text{Alt}((\omega \bullet \theta) \bullet \zeta) = \text{Alt}(\omega \bullet (\theta \bullet \zeta))\,.$$

This follows right away since the operation $\bullet$ is clearly associative. □

EXAMPLE 5.12. For any linear 1-forms $\xi_1, \ldots, \xi_k \in V^*$, one obtains a k-form

$$\xi_1 \wedge \ldots \wedge \xi_k \in \Lambda^k V^*.$$

As we shall soon see, any linear k-form can be written as a sum of expressions of this type. Even more, this operation allows us to produce an explicit basis for the vector space $\Lambda^k V^*$ by starting with a basis

$$e_1, \ldots, e_m \in V$$

of V. First of all, we consider the dual basis

$$e^1, \ldots, e^m \in V^*.$$

Then we start taking wedges of such elements; to obtain k-forms, we need to use k such (co)vectors. Given the graded commutativity of the wedge-operations, we can always re-order the indices in an increasing fashion. Therefore, we will consider sets of increasing indices

$$I = (i_1, \ldots, i_k), \quad \text{with1} \leq i_1 < \ldots < i_k \leq m;$$

the set of such Is will be denoted by $\text{Ord}_k(m)$. And, for $I \in \text{Ord}_k(m)$, consider

$$e^I := e^{i_1} \wedge \ldots \wedge e^{i_k} \in \Lambda^k V^*.$$

LEMMA 5.13. *The vectors e^I, with $I \in Ord_k(m)$, form a basis of $\Lambda^k V^*$. In particular, for $k \leq m$ (where m is the dimension of V) one has*

$$dim(\Lambda^k V^*) = Card\,(Ord_k(m)) = \frac{m!}{k!(m-k)!},$$

while for $k > m$ one has $\Lambda^k V^* = 0$.

PROOF. Note that, for any I as above and any other $J = (j_1, \ldots, j_k) \in \mathrm{ord}_k(M)$ one has

$$e^I(e_{j_1}, \ldots, e_{j_k}) = \delta^I_J$$

(i.e. 1 when $I =$ and zero otherwise). From this we deduce right away that the e^Is are linearly independent: if $\sum_I \lambda_I e^I = 0$ with $\lambda_I \in \mathbb{R}$, evaluating on $(e_{j_1}, \ldots, e_{j_k})$ we find that $\lambda_J = 0$ for all J.

To see that our family spans the entire $\Lambda^k V^*$, it suffices to show that, for any $\omega \in \Lambda^k V^*$),

$$\omega = \sum_{I \in \mathrm{Ord}_k(m)} \omega(e_{i_1}, \ldots, e_{i_k}) \cdot e^I.$$

To check this equality, we first note that the difference of the two sides, call it $\eta \in \Lambda^k V^*$, is zero when evaluated on all elements of type $(e_{i_1}, \ldots, e_{i_k})$. But this is easily seen to imply that $\eta = 0$: for an arbitrary expression $\eta(v_1, \ldots, v_k)$, decomposing each v_i with respect to the basis $\{e_1, \ldots, e_m\}$ and using the multilinearity of η and then the antisymmetry, we obtain a sum of expressions of type $\eta(e_{i_1}, \ldots, e_{i_k})$. □

3. Differential forms

We now pass to differential forms on an arbitrary m-dimensional manifold M. As for 1-forms, one first talks about **set-theoretical k-forms**, by which we mean maps

$$\omega : M \ni p \mapsto \omega_p \in \Lambda^k T^*_p M.$$

And, still as for 1-forms, one makes sense of their smoothness using charts (U, χ): any such chart gives rise to the basis

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$

of T_pM, the dual basis

$$(d\chi_1)_p, \ldots, (d\chi_m)_p$$

of T^*_pM and then, using Lemma 5.13, to the basis of $\Lambda^k T^*_p M$ given by

$$(d\chi)^I_p = (d\chi_{i_1})_p \wedge \ldots \wedge (d\chi_{i_k})_p$$

with $I \in \mathrm{Ord}_k(1, \ldots, m)$. Therefore, we can write our ω as

$$\omega_p = \sum_{I \in \mathrm{Ord}_k(m)} \omega^I_\chi(\chi(p))(d\chi)^I_p,$$

where each coefficient ω^I_χ is a function on $\Omega = \chi(U) \subset \mathbb{R}^m$. We say that ω is **smooth** if all its coefficients, with respect to any chart, are smooth functions.

A differential k-form on M, or a differential form of degree k, is any set-theoretical k-form that is smooth; we denote by

$$\Omega^k(M)$$

the space of all such k-forms.

It should be clear now that $\Omega^k(M)$ is a vector space, and it comes with a structure of $\mathcal{C}^\infty(M)$-module,

$$\mathcal{C}^\infty(M) \times \Omega^k(M) \to \Omega^k(M), \quad (f, \omega) \mapsto f \cdot \omega.$$

While, point-wise (at some $p \in M$), ω takes k tangent vectors (at p) to produce numbers, globally (when varying p), it takes k vector fields and produces a function. More explicitly, for $X^1, \ldots, X^k \in \mathfrak{X}(M)$, one obrains the function

$$\omega(X^1, \ldots, X^k) : M \to \mathbb{R}, \quad p \mapsto \omega_p(X^1_p, \ldots, X^k_p).$$

EXERCISE 5.10. Check that the smoothness of ω is equivalent to the fact that $\omega(X^1, \dots, X^k)$ is smooth for any $X^1, \dots, X^k \in \mathfrak{X}(M)$.

PROPOSITION 5.14. *The previous interpretation of k-forms gives a 1-1 correspondence between k-forms on M and multilinear maps*

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-times} \to \mathcal{C}^\infty(M)$$

with the following two properties:

- *ω is skew-symmetric.*
- *ω is $\mathcal{C}^\infty(M)$-linear in each argument X_i:*

$$\omega(X_1, \dots, X_{i_1}, X_i + X_i^{'}, X_{i+1}, \dots, X_k) = \omega(X_1, \dots, X_{i_1}, X_i, X_{i+1}, \dots, X_k) + \\ + \omega(X_1, \dots, X_{i_1}, X_i^{'}, X_{i+1}, \dots, X_k),$$

$$\omega(X_1, \dots, X_{i_1}, f \cdot X_i, X_{i+1}, \dots, X_k) = f \cdot \omega(X_1, \dots, X_k).$$

PROOF. For the moment, let denoted by $\hat{\omega}$ the version of ω that acts on vector fields. It is clear that it is skew-symmetric and $\mathcal{C}^\infty(M)$-multilinear. For the converse, let

$$A : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-\text{times}} \to \mathcal{C}^\infty(M)$$

a map with the same properties and we want to show that $A = \hat{\omega}$ for some differential k-form ω. As in the previous sections, for $p \in M$ we define ω_p on tangent vectors $v_p^1, \dots, v_p^k \in T_pM$ by:

$$\omega_p(v_p^1, \dots, v_p^k) := A(X^1, \dots, X^k)(p),$$

where $X^i \in \mathfrak{X}(M)$ are chosen so that, at p, they give back the vectors v_p^i. The main issue is, as for the similar discussion for 1-forms, to see that this definition does not depend on the choice of the X^is. But that follows exactly as for 1-forms (or one could even use that result inductively). The rest should be clear. □

EXAMPLE 5.15. In Exercise 5.1 we have seen that, for a 1-form $\omega \in \Omega^1(M)$ to be exact one would need in particular that the expression

$$-\omega([X, Y]) + L_X(\omega(Y)) - L_Y(\omega(X))$$

to be zero for all $X, Y \in \mathfrak{X}(M)$. Looking at the expression above one can check right away that it is skew-symmetric and $\mathcal{C}^\infty(M)$-bilinear in X and Y. Therefore it defines a 2-form on M- that is usually denoted by

$$d\omega \in \Omega^2(M);$$

as we have seen, it arises as the obstruction to ω being exact.

Note that, in this way, we extended the differential of functions to 1-forms:

$$\mathcal{C}^\infty(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M).$$

This, and its further continuation will be discussed in more detail in the next section ("De Rham differential").

Next, the wedge-product discussed in the previous section, applied at each point $p \in M$, gives rise to similar operations

$$\cdot \wedge \cdot : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M), \quad (\omega, \eta) \mapsto \omega \wedge \eta,$$

i.e. defined by

$$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p.$$

Or, using the interpretation of k-forms as in the previous proposition,

$$(\omega \wedge \eta)(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in S_{k,l}} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \eta(V_{\sigma(k+1)}, \ldots, V_{\sigma(k+l)}).$$

Of course, one still has:

PROPOSITION 5.16. *The wedge operation is bilinear, and:*

(1) *graded-commutativity: for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,*

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

(2) *associativity: for any differential forms ω, η and ζ on M one has*

$$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta).$$

PROOF. Just apply point-wise the similar linear algebra result. □

And, still as for 1-forms, forms of arbitrary degree can be pulled-back via any smooth map $F : M \to N$, giving rise to operations:

$$F^* : \Omega^k(N) \to \Omega^k(M).$$

Explicitly, for $\omega \in \Omega^k(N)$, **the pull-back of ω by F**, $F^*\omega \in \Omega^k(M)$, is given by

$$F^*(\omega)_p(X_p^1, \ldots, X_p^k) := \omega_{F(p)} \left((dF)_p(X_p^1), \ldots, (dF)_p(X_p^k) \right).$$

Again, when M is a submanifold of N and F is the inclusion map, the pull-back of ω via the inclusion will be denoted by

$$\omega|_M \in \Omega^k(M)$$

and will be called **the restriction of ω to M**. And of particular interest is the case when M is a submanifold of some Euclidean space $\mathbb{R}^d$; then:

- k-forms on $\mathbb{R}^d$ are always of type

$$\sum_{1 \leq i_1 < \ldots < i_k \leq d} f_{i_1, \ldots, i_k} \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

  with $f_{i_1, \ldots, i_k}$ smooth functions on $\mathbb{R}^d$.
- one obtains k-forms on M by restricting such k-forms on $\mathbb{R}^d$ to M.

EXERCISE 5.11. Show that the pull-back operation preserves the wedge-products: if $F : M \to N$ is smooth then, for any ω and η differential forms on N, one has

$$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta).$$

REMARK 5.17 (The graded world). It is customary to formally put together all the spaces $\Omega^k(M)$ and consider

$$\Omega^\bullet(M) := \bigoplus_k \Omega^k(M).$$

The notation "$\bullet$" is to indicate that we deal with a graded object; its elements are (finite) sums of homogeneous ones (the ones that live in one single degree). With this, the wedge product becomes now (after extending bi-linearly) a product

$$\cdot \wedge \cdot : \Omega^\bullet(M) \times \Omega^\bullet(M) \to \Omega^\bullet(M)$$

making $\Omega(M)$ into a (graded) algebra. Note the similarity with the algebra of polynomials in several variables, where each polynomial is a sum of homogeneous ones. And, as there, on sums of homogeneous elements, the wedge product is:

$$(\omega_0+\omega_1+\ldots)\wedge(\eta_0+\eta_1+\ldots) = (\omega_0\wedge\eta_0)+(\omega_0\wedge\eta_1+\omega_1\wedge\eta_0)+(\omega_0\wedge\eta_2+\omega_1\wedge\eta_1+\omega_2\wedge\eta_0)+\ldots$$

And here is one very useful slogan/principle when working with graded objects:

The sign rule: *looking at an expression involving elements that have a degree, when interchanging two such elements, say of degrees k and l, the expression changes sign by $(-1)^{kl}$.*

E.g., graded commutativity becomes precisely the one from Proposition 5.16.

4. The exterior (DeRham) differential

In Example 5.3 we mentioned the differential of functions (0-forms) as 1-forms, while in Example 5.15 we indicated a similar operation on from 1-forms to 2-forms. We now extend those to arbitrary degrees, i.e. a sequence of operators d:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \ldots$$

or, equivalently, an operator on the full algebra of differential forms (cf. Remark 5.17),

$$d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$$

which rises the degrees by 1 (as indicated by the notation $\bullet + 1$"). From the graded point of view (Remark 5.17) one should think of the symbol d as having degree 1 (and this will dictate some signs later on).

There are quite a few different ways to proceed to introduce d. But, before anything, let us point out the properties that we may want to be satisfied by d; we will see that some of them already force what d should be, or at least its uniqueness. Already at the start, we have to decide whether we want to discuss d (properties, definitions, existence, etc) on a fixed manifold M, or for all manifolds at once.

Of course, first of all we want:

DeRham-0: *d is $\mathbb{R}$-linear and, on 0-forms (functions), it is the usual differential of functions.*

Next, by thinking about the behaviour of d on the product of functions, we would like a similar Leibniz identity w.r.t. the wedge product $\omega\wedge\eta$:

DeRham-1: *d satisfies the (graded) Leibniz identity:*

$$d(\omega\wedge\eta) = d(\omega)\wedge\eta + (-1)^k\omega\wedge d(\eta).$$

for all $\omega\in\Omega^k(M)$, $\eta\in\Omega^l(M)$.

For an explanation of the sign, please see the sign rule from Remark 5.17 and remember that the symbol d is assigned degree 1.

One way to come across d is by applying inductively the reasoning described in Example 5.15: if d was defined on forms up to degree $k-1$, and we wonder when a k-form $\omega\in\Omega^k(M)$ is exact (i.e. of the type $d\eta$ for some $\eta\in\Omega^{k-1}(M)$), looking for "the natural conditions" that ω must satisfy, one discovers an expression $d\omega$ whose non-vanishing is an obstruction to ω being exact; this will be d on the k-forms. Of course, underlying the entire idea is one very simple property that d will have:

DeRham-2: $d \circ d = 0$.

EXAMPLE 5.18. On $M = \mathbb{R}^m$, if we assume these properties for d, we already know how to compute d on arbitrary forms. E.g., for

$$\omega = (x^2 + yz)dx \wedge dy,$$

applying the Leibinz rule we find

$$d\omega = d(x^2 + yz) \wedge dx \wedge dy + (x^2 + yz)d(dx \wedge dz).$$

For the term $d(x^2 + yz)$ we get, using "DeRham-0", $2xdx + ydz + zdy$. The term $d(dx \wedge dz)$ is, by Leibniz and $d^2 = 0$, zero. We obtain

$$d\omega = 2xdx \wedge dx \wedge dy + ydz \wedge dx \wedge dy + zdy \wedge dx \wedge dy.$$

Remembering that $dx \wedge dx = 0$ (by graded commutativity of the wedge), and similarly $zdy \wedge dx \wedge dy = 0$, we are left with

$$d\omega = ydz \wedge dx \wedge dy,$$

or, to keep the natural order of the variables,

$$d\omega = ydx \wedge dy \wedge dz.$$

Actually, a similar reasoning forces the definition of d on all k-forms on $\mathbb{R}^m$: any k-form is a sum of forms of type

$$f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

and, by the same arguments as above,

$$d(f dx_{i_1} \wedge \wedge dx_{i_k}) = \sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \wedge dx_{i_k}.$$

The fact that the d defined in this way does indeed satisfy (DeRham-1) and (DeRham-2) is now a simple check forms of this type.

Of course, exactly the same applies to any open subset $U \subset \mathbb{R}^m$ or to any domain U of a coordinate chart (U, χ) in any manifold M (we would use then the 1-forms $d\chi_i$ instead of dx_i); given the uniqueness of d satisfying the axioms, it follows that d doesn't even depend on the actual χ that we use to write down the formulas.

COROLLARY 5.19. *If M is a manifold, then for any $U \subset M$ that is the domain of a coordinate chart, there is and is unique an operator*

$$d_U : \Omega^\bullet(U) \to \Omega^{\bullet+1}(U)$$

satisfying (DeRham-0), (DeRham-1) and (DeRham-2). It is given by

$$d_U(f d\chi_{i_1} \wedge \wedge d\chi_{i_k}) = \sum_i \frac{\partial f}{\partial \chi_i} d\chi_i \wedge d\chi_{i_1} \wedge \wedge d\chi_{i_k}$$

where $\chi : U \to \Omega \subset \mathbb{R}^m$ is any coordinate chart defined on U (but independent of the actual χ).

Therefore, an obvious way to proceed to define d on a general M is to "proceed locally". That would mean that we would be looking for operators d that are local in the following sense:

DeRham-3: *Locality: if $U \subset M$ is open and $\omega \in \Omega^\bullet(M)$ is supported inside U, then so is $d\omega$.*

As for functions, the support of ω is the closure of the set of points $p \in M$ at which $\omega_p \neq 0$.

EXERCISE 5.12. Show that a linear operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is local if and only if for any open $U \subset M$ there is a linear operator $d|_U : \Omega^\bullet(U) \to \Omega^{\bullet+1}(U)$ with the property that

$$(d\,\omega)|_U = d|_U(\omega|_U) \quad \text{for all } \omega \in \Omega^\bullet(U).$$

$$\begin{array}{ccc} \Omega^\bullet(M) & \xrightarrow{d} & \Omega^{\bullet+1}(M) \\ {\scriptstyle \text{restr}}\downarrow & & \downarrow{\scriptstyle \text{restr}} \\ \Omega^\bullet(U) & \xrightarrow{d|_U} & \Omega^{\bullet+1}(U) \end{array}.$$

Then show that, given two local operators d and d', $d = d'$ if and only if M can be covered by opens U on which $d|_U = d^{'}|_U$.

COROLLARY 5.20. *On any manifold M there exists and is unique an operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ which is local and so that for any domain U of a coordinate chart, $d|_U$ coincides with d_U from the previous corollary. Moreover, d will automatically satisfy (DeRham-0), (DeRham-1) and (DeRham-2).*

PROOF. The definition is forced: for $p \in M$, choosing a coordinate chart (U, χ) around p we must have

$$(d\omega)_p = (d_U\omega|_U)_p.$$

Given the uniqueness property of the d_Us it follows that the right hand side does not depend on the coordinate chart that we use, hence it can be used to define $d\omega$ unambiguously. Since the conditions (DeRham-1) and (DeRham-2) are local and they hold for the d_Us, it follows that they hold for d as well. □

It is interesting to note that locality can be obtained from the rest of the axioms:

LEMMA 5.21. *On a manifold M, if $D : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ satisfies the condition that*

$$D(f \cdot \omega) = fD(\omega) + df \wedge \omega \quad \textit{for all } f \in \mathcal{C}^\infty(M), \omega \in \Omega^\bullet(M)$$

(in particular if D satisfies (DeRham-0) and (DeRham-1)) then D is local.

PROOF. We fix ω, we denote by A its support and we claim that the support of $d\omega$ is inside A. Note that A is the largest closed subset of M outside which ω vanishes. Therefore suffices to show that $d(\omega)$ vanishes outside A. Let $p \in M \setminus A$; since A is closed, $M \setminus A$ is open, hence we find a smooth function $\eta \in \mathcal{C}^\infty(M)$ that is zero on A and is 1 in a neighborhood of p. But then $\omega = (1 - \eta) \cdot \omega$ and, using the condition from the statement we obtain

$$d\omega = (1 - \eta)d\omega - d\eta \wedge d\omega.$$

Evaluating at the point p we obtain that $d\omega$ vanishes at p. □

In particular,

COROLLARY 5.22. *On any manifold M there exists and is unique an operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ which satisfies (DeRham-0) , (DeRham-1) and (DeRham-2).*

PROOF. One just needs to remark that, if d satisfies (DeRham-1) and (DeRham-2) then so do all its restrictions $d|_U$; for domains U of coordinate charts, due to the uniqueness of d_U, we obtain that $d|_U = d_U$. □

DEFINITION 5.23. *The resulting operator $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ is called the **DeRham operator on** M (or the exterior differential).*

While so far we concentrated on building $d = d_M$ on a given manifold M, one can proceed slightly differently and handle all the manifolds at once; then the natural axiom to impose is the behaviour of d_M when we change the manifold, i.e. its compatibility with pull-backs. In low degree this was seen in Exercise 5.5. It means that, for any smooth map $F : M \to N$ one has a commutative diagram

$$\begin{array}{ccc} \Omega^\bullet(N) & \xrightarrow{d_N} & \Omega^{\bullet+1}(N) \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ \Omega^\bullet(M) & \xrightarrow{d_M} & \Omega^{\bullet+1}(M) \end{array},$$

i.e. $F^* \circ d_N = d_M \circ F^*$. Note that this condition forces automatically that each d is local. Therefore:

COROLLARY 5.24. *If we are looking for operators $d_M : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$, one for each manifold M, such that they are compatible with pull-backs, we have uniqueness and existence under any of the following requirements:*

1. *on $\mathbb{R}^m$, $d_{\mathbb{R}^m}$ is the operator previously discussed.*
2. *all d_M satisfy (DeRham-0) and (DeRham-1).*

Of course, for any manifold M, d_M will be precisely the DeRham operator.

And here is another way to introduce the DeRham operator: just right away, by explicit formulas. One way to find those formulas would be by returning to the previous idea of proceeding like in Example 5.15 inductively, by looking for the natural conditions for a form to be exact.

PROPOSITION 5.25. *One has the following explicit formula for the DeRham operator on any manifold M: for $\omega \in \Omega^k(M)$, $d\omega \in \Omega^{k+1}(M)$ is given by*

$$\begin{aligned} d\omega(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i L_{X_i}(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_k)) \end{aligned} \tag{4.1}$$

where the elements under the hat are deleted.

PROOF. Note that the previous formula is skew-symmetric and $\mathcal{C}^\infty(M)$-multilinear (check that!). Hence it defines an operator d_M for each M. Looking back at the previous corollary, we see that all that is used is the compatibility with pull-backs by diffeomorphisms and by inclusions of opens inside manifolds. That is clearly true for the d_Ms. Hence we are left with checking that, for $\mathbb{R}^m$, this formula gives back the standard DeRham operator. Moreover, a simple check shows that $d(f \cdot \omega) = f \cdot d\omega + df \wedge \omega$ for any function f, hence it suffices to restrict our attention to forms on $\mathbb{R}^m$ of type $\omega = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ and we are left with showing that

$$d_{\mathbb{R}^m}(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = 0$$

(where recall that $d_{\mathbb{R}^m}$ now denotes the operator from the statement). This can be done in several ways:

- direct computation (straightforward but not so nice).
- check it first for $k = 1$ (obvious) and then check the Leibniz identity on expressions of form $\omega \wedge \eta$ where ω is a 1-form (and this computation is not that ugly and can easily be carried out on a general manifold M).

□

EXERCISE 5.13. Check by direct computation that, defining d by the formula (4.1), then it satisfies the Leibniz identity on expressions of form $\omega \wedge \eta$ where ω is a 1-form.

5. Cartan's magic formula $di_X + i_X d = \mathcal{L}_X$

We first discuss the Lie derivative of forms $\omega \in \Omega^\bullet(M)$ along vector fields $V \in \mathfrak{X}(M)$, giving rise to operations

$$\mathcal{L}_V : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

These Lie derivatives fit in the general philosophy explained and illustrated in Section 7 of the previous chapter:

DEFINITION 5.26. *For $\omega \in \Omega^k(M)$ and $V \in \mathfrak{X}(M)$ define **the Lie derivative of ω along V** as*

$$\mathcal{L}_V(\omega) := \left.\frac{d}{dt}\right|_{t=0} (\phi_V^t)^*\omega \in \Omega^k(M).$$

As before, when V is not complete, one has to look at each $p \in M$ and not that the resulting formula

$$\mathcal{L}_V(\omega)(X_p^1, \ldots, X_p^k) = \left.\frac{d}{dt}\right|_{t=0} \omega_{\phi_V^t(p)} \left((d\phi_V^t)_p(X_p^1), \ldots, (d\phi_V^t)_p(X_p^k) \right) \tag{5.1}$$

makes sense independent of whether V is complete or not.

Not also that, from the graded point of view, the operation $\mathcal{L}_V$ has degree 0 (because it does not change the degree of the forms).

PROPOSITION 5.27. *For any vector field $V \in \mathfrak{X}(M)$:*

1. *$\mathcal{L}_V : \Omega^\bullet \to \Omega^\bullet(M)$ is a degree zero derivation, i.e. it is linear and satisfies*
$$\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)$$
for any two differential forms ω and η.
2. *$\mathcal{L}_V$ commutes with d: $\mathcal{L}_V \circ d = d \circ \mathcal{L}_V$.*
3. *$\mathcal{L}_V$ is the only degree zero derivation on $\Omega^\bullet(M)$ which commutes with d and which, on functions, it is the usual derivative L_V.*

Moreover, it can be described by the explicit formula:

$$\mathcal{L}_V(\omega)(X_1, \ldots, X_k) = L_V\left(\omega(X_1, \ldots, X_k)\right) - \sum_{i=1}^k \omega(X_1, \ldots, X_{i-1}, [V, X_i], X_{i+1}, \ldots, X_k).$$

PROOF. The derivation identity follows from the standard one and the fact that $(\phi_V^t)^*$ is compatible with the wedge operation:

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_V^t)^*(\omega \wedge \eta) = \left.\frac{d}{dt}\right|_{t=0} \left((\phi_V^t)^*\omega \wedge (\phi_V^t)^*\eta \right) = \left(\left.\frac{d}{dt}\right|_{t=0} (\phi_V^t)^*\omega \right) \wedge \eta + \omega \wedge \left(\left.\frac{d}{dt}\right|_{t=0} (\phi_V^t)^*\eta \right).$$

That $\mathcal{L}_V$ commutes with d follows from the similar property of the pull-backs $(\phi - V^t)^*$.

For the uniqueness in (3), assume that D is another operators with the same properties. It follows that $\mathcal{L}_V(df) = dL_V(f)$, and we obtain that $D(\omega) = \mathcal{L}_V(\omega)$ on differential forms of type

$$\omega = \sum_i f_i \wedge dg_i^1 \wedge \ldots \wedge dg_i^k.$$

It would suffices to show that any k-form can be written in this way. This is true and it would follow for instance if we proved that M can be embedded in some Euclidean space (why would that be enough?). To avoid using such a result, one can first restrict to manifolds on which it is certainly true: domain of coordinate charts. In more detail: due to the Leibniz identity it

follows (as in the discussions about d) that D is local, we can talk about $D|_U$ and, on domains of coordinate charts, $D|_U$ still has the same properties as D. Hence $D|_U = \mathcal{L}_V|_U$, and then $D = \mathcal{L}_V$.

For the final formula, we start from the explicit formula (5.1) and we use that $\frac{d}{dt}|_{t=0}(\phi_V^t)^*(X) = [V, X]$ (Proposition 4.46 in the previous chapter) or, equivalently,

$$\left.\frac{d}{dt}\right|_{t=0} (d\phi_V^{-t})_{\phi_V^t(p)}(X_{\phi_V^t(p)}) = [V, X]_p$$

for all X. This allows us to write

$$(d\phi_V^{-t})_{\phi_V^t(p)}(X_{\phi_V^t(p)}) = X_p + t[V, X]_p + o(t^2)$$

for t near 0, and then

$$(d\phi_V^t)_p(X_p) = X_{\phi_V^t(p)} + t[V, X]_{\phi_V^t(p)} + o(t^2).$$

Inserting this in (5.1) and using also Proposition 4.45 from the previous chapter (applied to V and $f = \omega(X_1, \ldots, X_k)$ we find the formula from the statement. □

Another interesting operators induced by a vector field $V \in \mathfrak{X}(M)$, but simpler this time, is the interior product:

DEFINITION 5.28. *For $\omega \in \Omega^k(M)$ and $V \in \mathfrak{X}(M)$ define **the interior product** ω **by** V, denoted $i_V(\omega)$ as the differential form of degree $k-1$ given by*

$$i_V(\omega)(X_1, \ldots, X_{k-1}) = \omega(V, X_1, \ldots, X_{k_1}).$$

This defines an operator

$$i_V : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M);$$

since it lowers the degree by 1, we say it is of degree -1.

PROPOSITION 5.29. *For any vector field $V \in \mathfrak{X}(M)$,i_V is a degree -1 derivation, i.e. it is linear and satisfies*

$$i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)$$

for any two differential forms $\omega \in \Omega^k(M)$ and η. Actually, it is the only degree -1 derivation on $\Omega^\bullet(M)$ which, on 1-forms, is the evaluation on V.

PROOF. The derivation formula is a completely straightforward computation. For the uniqueness, one can just imitate the arguments used for the uniqueness of $\mathcal{L}_V$, just that it is a bit simpler now (being degree -1, it kills of the function, and the condition from the statement prescribes i_V on 1-forms hence, by the Leibniz identity, on all the sums of wedges of 1-forms. One can either show that any differential form can be written as such a sum, or just provide a local argument completely similar to that for $\mathcal{L}_V$. □

Next: the Cartan magic formula, which says that $\mathcal{L}_V$ is the graded commutator of d and i_V; given that d has degree 1 and i_V has degree -1, that means:

THEOREM 5.30. *For any vector field $V \in \mathfrak{X}(M)$ one has*

$$\mathcal{L}_V = d \circ i_V + i_V \circ d.$$

PROOF. A simple computation using the Laibinz identities for d and i_V (with special care at the signs involved) shows that $d \circ i_V + i_V \circ d$ is a derivation of degree zero. Since $d \circ d = 0$, we see that it also commutes with d. Moreover, on functions f it gives $i_V d(f) = L_V(f)$. Hence, by the uniqueness from Proposition 5.27, it must coincide with $\mathcal{L}_V$. □

EXERCISE 5.14. Assume that we have a right action of the circle S^1 on a manifold M, i.e. a smooth map

$$M \times S^1 \to M, \quad (\lambda, p) \mapsto \lambda \cdot p$$

such that $p \cdot 1 = p$ and $(p \cdot \lambda) \cdot \lambda' = p \cdot (\lambda\lambda')$ for all $\lambda, \lambda' \in S^1$, $p \in M$. Define the following vector field on M:

$$V_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{2\pi\, it}.$$

Assume now that $\omega \in \Omega^1(M)$ is a 1-form that is invariant in the sense that

$$R_\lambda^*(\omega) = \omega \quad \text{for all } \lambda \in S^1,$$

where $R_\lambda : M \to M$ is the right multiplication by λ. Show that:

(1) Also $d\omega$ is S^1-invariant.
(2) If $i_V(\omega) = 1$ then $i_V(d\omega) = 0$.

6. Integration on manifolds

In this section we explain how to extend standard integration from $\mathbb{R}^m$ to more general manifolds. By standard integration we mean the integral

$$\int : \mathcal{C}_c^\infty(\mathbb{R}^m) \to \mathbb{R} \quad f \mapsto \int f = \int_{\mathbb{R}^m} f(x)dx = \int_{\mathbb{R}^m} f(x_1, \ldots, x_m)dx_1 \ldots dx_m$$

defined (only) on the space of compactly supported smooth functions. For $U \subset \mathbb{R}^m$ open and $f \in \mathcal{C}_c^\infty(\Omega)$, we will use the notation

$$\int_\Omega f = \int_\Omega f(x)dx$$

for the integral $\int \tilde{f}$ where $\tilde{f}$ is f extended by zero outside Ω.

Actually, what we need to know about the standard integration is that:

- it is linear in f.
- change of variables formula: if $c : \Omega \to \Omega'$ is a diffeomorphism between two opens in $\mathbb{R}^m$ then, for any $f \in \mathcal{C}_c^\infty(\Omega')$,

$$\int_{\Omega'} f(x')dx' = \int_\Omega |\mathrm{Jac}_x(c)| \cdot f(c(x))dx$$

or, more compactly,

$$\int_{\Omega'} f = \int_\Omega |\mathrm{Jac}(c)| \cdot f \circ c.$$

Recall that the Jacobian of c is the function

$$\mathrm{Jac}(c) : \Omega \to \mathbb{R}, \quad x \mapsto \mathrm{Jac}_x(c) := \det\left(\frac{\partial c_i}{\partial x_j}(x)\right).$$

Note that its presence in the change of variable formula shows that the integral of a function f is not independent of the coordinates. For that reason, the objects that can (naturally) be integrated on manifolds will not be functions (we will be able to integrate functions only when a certain extra-choice will be made). Instead, one has to look for objects which locally (in coordinate charts) can be represented by smooth functions, and so that when we change the coordinates the representing functions change by a Jacobian factor. This brings us to differential forms of top degree m (also making more sense of the symbol of the expression "$f(x_1, \ldots, x_m)dx_1 \ldots dx_m$" in the integration).

To give the details, we fix an m-dimensional manifold M and we look at differential forms of top degree

$$\omega \in \Omega^m(M).$$

Note that, for any coordinate chart $\chi : U \to \Omega \subset \mathbb{R}^m$, one can write

$$\omega|_U = f_\chi^\omega \circ \chi \cdot d\chi_1 \wedge \ldots \wedge d\chi_m$$

for a unique function $f_\chi^\omega \in \mathcal{C}^\infty(\Omega')$; this will be called **the representing function of ω w.r.t. the chart** (U, χ). How does the representing function change when we change the chart? Well, if $\chi' : U' \to \Omega'$ is another chart, denoting by $c = \chi' \circ \chi^{-1} : \Omega \to \Omega'$ the change of coordinates, we know that

$$d\chi_i^{'} = \sum_j \frac{\partial \chi_i^{'}}{\partial \chi_j} \cdot d\chi_j = \sum_j \frac{\partial c_i}{\partial x_j} \circ \chi \cdot d\chi_j.$$

From this, and the graded commutativity of the wedge-product, we deduce that

$$d\chi_1^{'} \wedge \ldots \wedge d\chi_m^{'} = \det\left(\frac{\partial \chi_i^{'}}{\partial \chi_j}\right) \cdot d\chi_1 \wedge \ldots \wedge d\chi_m,$$

where the coefficient is

$$\det\left(\frac{\partial \chi_i^{'}}{\partial \chi_j}\right) = \det\left(\frac{\partial c_i}{\partial x_j} \circ \chi\right) = \mathrm{Jac}(c \circ \chi).$$

We deduce right away that

$$f_\chi = \mathrm{Jac}(c) \cdot f_{\chi'} \circ c.$$

This is "almost" the expression in the change of variables formula- only the absolute value is missing. For that reason we stop for a bit and look into this issue.

6.1. Digression: oriented manifolds. We start with the linear discussion. For a finite dimensional vector space V, we denote by $\mathrm{Fr}(V)$ the set of all frames (i.e. ordered bases) of V. For two frames $u, u' \in \mathrm{Fr}(V)$ we denote by $[u : u']$ the matrix of coordinate changes from u to u' (an $m \times m$ invertible matrix with entires real numbers, where $m = \dim(V)$) and by $\mathrm{sign}(u : u')$ the sign of its determinant:

$$\mathrm{sign}(u : u') := \mathrm{sign}\left(\det([u : u'])\right) \in \{-1, +1\}.$$

An orientation on V is basically a partition of the set of frames into two: positively and negatively oriented frames, so that two frames u and u' are of the same type if and only if $\mathrm{sign}(u : u') > 0$.

One way to formalize this is as follows: call an **orientation** on V any map

$$\mathcal{O} : \mathrm{Fr}(V) \to \{-1, +1\}$$

with the property that

$$\mathcal{O}(u') = \mathrm{sign}(u : u')\mathcal{O}(u) \cdot \quad \text{for all } u, u' \in \mathrm{Fr}(V).$$

We also say that $(V, \mathcal{O})$ is an oriented vector space, and then

$$\mathrm{Fr}(V) = \mathrm{Fr}_+(V, \mathcal{O}) \cup \mathrm{Fr}_-(V, \mathcal{O}), \quad \text{where } \mathrm{Fr}_\pm(V, \mathcal{O}) = \{u \in \mathrm{Fr}(V) : \mathcal{O}(u) = \pm 1\}$$

(positively and negatively oriented frames of $(V, \mathcal{O})$).

Note that any $u \in \mathrm{Fr}(V)$ gives rise to a unique orientation $\mathcal{O}_u$ by the condition that u is positively oriented for $\mathcal{O}_u$; explicitly,

$$\mathcal{O}_u : \mathrm{Fr}(V) \to \{-1, +1\}, \quad u' \mapsto \mathrm{sign}(u : u').$$

And, moreover, for $u_0, u_1 \in \mathrm{Fr}(V)$,

$$\mathcal{O}_{u_0} = \mathcal{O}_{u_1} \iff \mathrm{sign}(u_0 : u_1) > 0.$$

Hence an orientation on V can also be seen as an equivalence class of frames with respect to the equivalence relation based on the condition $\text{sign}(u_0 : u_1) > 0$.

Moving to a manifold M, we look at families

$$\mathcal{O} = \{\mathcal{O}_p : p \in M\} \quad \text{with } \mathcal{O}_p - \text{orientation on } T_pM. \tag{6.1}$$

We say that $\mathcal{O}$ is **smooth** if for any $p_0 \in M$ there is a chart (U, χ) around p_0 such that, for any $p \in U$,

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$

is a (positively) oriented frame of $(T_pM, \mathcal{O}_p)$. Such a chart is said to be **adapted to** $\mathcal{O}$.

Definition 5.31. *An **orientation on the manifold** M is any family $\mathcal{O}$ of orientations $\mathcal{O}_p$ on the tangent spaces T_pM which is smooth in the sense described above.*

*We also say that $(M, \mathcal{O})$ is an oriented manifold, and the charts adapted to $\mathcal{O}$ will be called **positively oriented charts** of $(M, \mathcal{O})$.*

Of course, on an oriented manifold, we will only use positively oriented charts. From the previous discussion it should be clear now that

$$\text{Jac}(\chi' \circ \chi^{-1}) > 0 \quad \text{for all } \chi, \chi' - \text{positively oriented.}$$

Exercise 5.15. Let M be a manifold.

(1) for an atlas $\mathcal{A}$ on M show that the existence of an orientation $\mathcal{O}_\mathcal{A}$ on M with the property that each $\chi \in \mathcal{A}$ is adapted to $\mathcal{O}_\mathcal{A}$ is equivalent to

$$\text{Jac}(\chi' \circ \chi^{-1}) > 0 \quad \text{for all } \chi, \chi' \in \mathcal{A}. \tag{6.2}$$

(2) show that two atlases $\mathcal{A}$ and $\mathcal{A}'$ with this property induce the same orientation iff

$$\text{Jac}(\chi' \circ \chi^{-1}) > 0 \quad \text{for all } \chi \in \mathcal{A}, \chi' \in \mathcal{A}'. \tag{6.3}$$

In other words, to chose an orientation on M is the same thing as choosing an atlas $\mathcal{A}$ satisfying (6.2), and two such atlases $\mathcal{A}$ and $\mathcal{A}'$ induce the same orientation on M if an only if (6.3) holds.

Exercise 5.16. If M is connected and orientable (i.e. it admits an orientation), show that it admits exactly two orientations.

End of the digression on oriented manifolds

Returning to our discussion on integration, it should be clear now that the choice of an orientation on M is precisely what we need to solve our small problem with the absolute value of the Jacobians; here is the final outcome:

Theorem 5.32. *On any oriented manifold $(M, \mathcal{O})$, there is a well-defined map*

$$\int_M : \Omega^m_c(M) \to \mathbb{R}$$

which is uniquely determined by the following two properties:

(1) *$\int_M$ is linear.*
(2) *for any positively oriented chart $\chi : U \to \Omega$, if $\omega \in \Omega^m_c(M)$ is supported in U then*

$$\int_M \omega = \int_\Omega f^\omega_\chi(x)dx \quad (\textit{standard integration over } \Omega \subset \mathbb{R}^m), \tag{6.4}$$

where f^ω_χ is the representing function of ω w.r.t. the chart (U, χ).

PROOF. We divide the proof into three steps:

Step 1: integrating over domains of charts: It should now be clear from the previous discussions that for any $U \subset M$ which is inside the domain of some chart of M one obtains an integration map

$$\int_U : \Omega^m_c(U) \to \mathbb{R}$$

(defined on the space of m-forms whose support is compact and included in U); it is defined simply by the right hand side of (6.4), but the point is that it does not depend on the choice of a positively oriented chart χ defined on U.

Note also that for any open $V \subset U$, one has a commutative diagram

$$\begin{array}{ccc} \Omega^m_c(V) & \hookrightarrow & \Omega^m_c(U) \\ & {\scriptstyle \int_V} \searrow \quad \swarrow {\scriptstyle \int_U} & \\ & \mathbb{R} & \end{array} \quad \text{, i.e.} \quad \int_V \omega = \int_U \omega \quad \text{for all } \omega \in \Omega^m_c(V).$$

This implies that we can safely write $\int_M \omega$ for any $\omega \in \Omega^m_c(M)$ which is supported inside the domain of a chart: it actually does not depend on which domain we use.

Step 2: from domains of charts to the entire M: Now we claim that for any $\omega \in \Omega^m_c(M)$, can be written as a finite sum

$$\omega = \sum_{i=1}^k \omega^i$$

where each $\omega_i \in \Omega^m_c(M)$ is supported in the domain of a chart (and k is some integer). Once we prove this, we will define

$$\int_M \omega := \sum_i \int_M \omega_i. \tag{6.5}$$

To prove the claim one uses a partition of unity argument: if K is the support of ω, since K is compact it can be covered by a finite number of opens $U_1, \dots, U_k$, each of them a domain of a coordinate chart, and each one of them relatively compact (in the sense that they have compact closures) has compact closure. Then $\{M \setminus K, U_1, \dots, U_k\}$ is an open cover of M and we choose a partition of unity $\eta_0, \eta_1, \dots, \eta_k$ with η_0 supported inside $M \setminus K$ and η_i supported inside U_i. Note that, since U_i is relatively compact, each η_i is compactly supported. Also, since η_0 is supported in $M \setminus K$, one has $\eta_0 \cdot \omega = 0$. Therefore, multiplying the identity $1 = \sum_{i=0}^k \eta_i$ by ω we obtain

$$\omega = \sum_{i=1}^k \omega_i \quad \text{where } \omega_i = \eta_i \cdot \omega \in \Omega^m_c(M).$$

Step 3: $\int_M \omega$ does not depend on the decomposition $\omega = \sum_i \omega_i$: Taking now (6.5) as the definition of $\int_M$, we have to check that the outcome does not depend on the way we write ω as a sum of forms supported inside domains of charts.

To prove this, assume that we have two such writings

$$\omega = \sum_{i=1}^k \omega_i = \sum_{j=1}^{k'} \omega'_j$$

Let K be a compact that contains the supports of all ω_is and ω_js. As before, we can find a finite family $\{\eta_\alpha\}_{\alpha \in I}$ of compactly supported smooth functions, each supported inside a chart domain V_α, such that $\sum_\alpha \eta_\alpha = 1$. Then $\omega_i = \sum_\alpha \eta_\alpha \cdot \omega_i$. Note also that we already know that $\int_M$

behaves linearly when looking at forms that are supported inside a coordinate chat. Applying this for the coordinate charts U_i, and then V_α, we find that $\sum_{i=1}^k \int_M \omega_i$ equals to

$$\sum_{i=1}^{k}\sum_{\alpha}\int_M \eta_\alpha\cdot\omega_i=\sum_{\alpha}\sum_{i=1}^{k}\int_M \eta_\alpha\cdot\omega_i=\sum_{\alpha}\int_M \eta_\alpha\cdot\omega.$$

Of course, the same applies to $\sum_{j=1}^{k'}\omega_j'$, therefore proving the claim.

Finally, note also that $\int_M$ defined in this way is obviously linear. □

THEOREM 5.33 (Stokes- version without bounday). *If M is an m-dimensional oriented manifold then, for any $\eta\in\Omega_c^{m-1}(M)$ one has*

$$\int_M d\eta=0.$$

In particular, on a compact oriented manifold, $\int_M\omega=0$ whenever ω is exact.

PROOF. Writing (as above) η as a sum of forms supported in domains of coordinate charts, we may assume that $M=\mathbb{R}^m$. Then η is of type

$$\eta=\sum_i f_i\cdot dx_1\wedge\ldots dx_{i-1}\wedge dx_{i+1}\wedge\ldots\wedge dx_m, \tag{6.6}$$

so that we can write

$$d\eta=\left(\sum_i(-1)^{i-1}\frac{\partial f_i}{\partial x_i}\right)dx_1\wedge\ldots\wedge dx_m.$$

Hence it suffices to show that, for any $f\in\mathcal{C}^\infty(\mathbb{R}^m)$ compactly supported and any i one has

$$\int_{\mathbb{R}^m}\frac{\partial f}{\partial x_i}dx_1\wedge\ldots\wedge dx_m=0.$$

Choosing an interval $[a,b]$ such that the support of f is strictly inside $[a,b]^m$, everything follows from

$$\int_a^b\frac{\partial f_i}{\partial x_i}dx_i=f(b)-f(a)=0. \tag{6.7}$$

□

EXERCISE 5.17. Do again Exercise 5.2.

6.2. Volume forms; integration of functions. We now return to manifolds M that are compact without boundary. We have seen that an orientation $\mathcal{O}$ on M allows us to talk about the integral of m-forms (m being the dimension of M),

$$\int_M:\Omega^m(M)\to\mathbb{R}.$$

Therefore, an integration of functions could be defined as soon as we fix a top-form

$$\mu\in\Omega^m(M),$$

by considering the integral of $f\cdot\mu$:

$$\mathcal{C}^\infty(M)\to\mathbb{R},\quad f\mapsto\int_M f\cdot\mu.$$

In some sense, this is precisely how the standard integration on $\mathbb{R}^m$ (of functions!) is obtained: it is associated to the choice of the the top-form

$$\mu=dx_1\wedge\ldots\wedge dx_m$$

(note the compatibility with the standard notation $\int_{R^m} f(x)dx_1 \dots dx_m$ for the integration of functions!).

Something similar can be done on any manifold, once a "volume form" is fixed.

DEFINITION 5.34. *A volume form on an m-dimensional manifold M is any top form*

$$\mu \in \Omega^m(M)$$

with the property that $\mu_p \neq 0$ for all $p \in M$.

EXAMPLE 5.35. Of course, on $\mathbb{R}^{m+1}$,

$$\mu_{\mathbb{R}^{m+1}} := dx_0 \wedge dx_1 \wedge \ldots \wedge dx_m$$

is a volume form. Taking the interior product with the radial vector field $R = \sum_i x_i \cdot \frac{\partial}{\partial x_i}$ we obtain an m-form

$$i_R(\mu_{\mathbb{R}^{m+1}}) = \sum_{i=0}^{m} (-1)^i x_i \cdot dx_0 \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m \in \Omega^m(\mathbb{R}^m).$$

Its restriction to the sphere $S^m \subset \mathbb{R}^{m+1}$,

$$\mu_{S^m} := \left(\sum_{i=0}^{m} (-1)^i x_i \cdot dx_0 \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m \right) \Bigg|_{S^m} \in \Omega^m(S^m)$$

is a volume form on S^m.

EXERCISE 5.18. Show that, indeed, μ_{S^m} is a volume form on S^m.

The relationship with orientations is:

LEMMA 5.36. *A manifold M is orientable if and only if it admits a volume form $\mu \in \Omega^m(M)$. Moreover, any such volume form μ induces an orientation $\mathcal{O}_\mu$ on M, uniquely determined by the condition that, at any $p \in M$, a frame*

$$v_p^1, \ldots, v_p^m$$

of T_pM is positively oriented if and only if

$$\mu_p(v_p^1, \ldots, v_p^m) > 0.$$

Furthermore, using the integration induced by this orientation, the volume of (M, μ),

$$Vol(M, \mu) := \int_M \mu,$$

is strictly positive.

Therefore, once one fixes a volume form μ, one has an induced orientation $\mathcal{O}_\mu$, this orientation gives rise to an integration $\int_M : \Omega^m(M) \to \mathbb{R}$ and, using again μ, one obtains an integration of functions

$$f \mapsto \int_M f \cdot \mu.$$

And the integration of the constant function 1 is $\mathrm{Vol}(M, \mu)$.

PROOF. The orientation $\mathcal{O}_\mu$ can be defined, at each $p \in M$, as the map

$$\mathcal{O}_{\mu,p} : \mathrm{Fr}(T_pM) \to \{-1, +1\},$$

$$\mathcal{O}_{\mu,p}(v_p^1, \ldots, v_p^m) := \mathrm{sign}(\mu(v_p^1, \ldots, v_p^m)).$$

The fact that $\mu_p \neq 0$ ensures that the expression under the sign is non-zero.

The main thing still to prove is the converse: if M is orientable, then it admits a volume form. While volume forms clearly exist locally, the question is how to "glue them" to global volume forms. In principle, we would consider a family $\{\omega_i\}$ of volume forms on domains U_i of coordinate charts covering M indexed by a set I, a partition of unity $\{\eta_i : i \in I\}$ subordinated to the cover and set

$$\omega := \sum_i \eta_i \cdot \omega_i \in \Omega^m(M).$$

For this to be a volume form we would need to check that, for $p \in M$, ω_p is non-zero. Denoting by $J \subset I$ the set of indices j with the property that $p \in \text{supp}(\eta_j)$ (a finite number of them!),

$$c_j = \eta_j(p) \in \mathbb{R}, \quad V = T_pM, \quad \mu_j = \omega_{i,p} \in \Lambda^m V^*,$$

we end with a finite sum

$$\sum_{j\in J} c_j \cdot \mu_j \in \Lambda^m V^*, \quad \text{with} \quad \sum_j c_j = 1, \quad \mu_j \neq 0$$

(an affine combination of non-zero elements in the top exterior power). The problem is that this sum may be zero- and it is here that we should the use the orientation $\mathcal{O}$ of M. Indeed, if we knew that V comes with an orientation and each μ_j was not only non-zero, but positively oriented, then the above sum is clearly still positively oriented, hence non-zero. Hence, all we have to do in the previous argument is to start right from the beginning with a family $\{\omega_i\}$ of volume forms on domains U_i which are positively oriented there with respect to the original orientation (i.e, at each $p \in U_i$, $\omega_{i,p}$ is so w.r.t. the orientation $\mathcal{O}_p$). □

6.3. Stokes on manifolds with boundary. The Stokes formula discussed above, when applied to functions on $M = \mathbb{R}$, or on intervals $(a, b) \subset \mathbb{R}$, boils down to the fact that if $f : \mathbb{R} \to \mathbb{R}$ is supported inside the interval (a, b) then

$$\int_{\mathbb{R}} \frac{\partial f}{\partial x}(x)dx = \int_a^b \frac{\partial f}{\partial x}(x)dx = f(b) - f(a) = 0.$$

However, such formulas are especially useful when f is supported inside (or just defined on) a closed interval $[a, b]$, without assuming that $f(a) = f(b) = 0$. This shows that a Stokes theorem would be (even) more interesting when formulated on manifolds with boundary.

The notion of manifold with boundary was briefly mentioned in Section 3 of Chapter 3: the main point was to allow charts that compare opens in M with opens inside the half space

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}.$$

Since any open inside $\mathbb{R}^m$ is diffeomorphic to an open inside

$$\text{Int}(\mathbb{H}^m) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m > 00\},$$

that means that we are allowing more local models. Of course, the new addition is due to the presence of the boundary

$$\partial(\mathbb{H}^m) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 0\}.$$

Definition 5.37. *Given a manifold with boundary, a point $p \in M$ is called a boundary point if there exists a chart $\chi : U \to \Omega \subset \mathbb{H}^m$ around p such that $\chi(p) \in \partial(\mathbb{H}^m)$.*

*The **bounday of** M, denoted ∂M, is the set of boundary points.*

One can check that, in the previous definition, the condition $\chi(p) \in \partial(\mathbb{H}^m)$ does not depend on the choice of χ. Also, it follows that

$$\chi|_{\partial(U)} : \partial(U) \to \partial(\Omega) = \Omega \cap \partial(\mathbb{H}^m)$$

(where the last space is open in $\partial(\mathbb{H}^m) \cong \mathbb{R}^{m-1}$) can be used as a chart for ∂M. These make ∂M into an $(m-1)$-dimensional manifold (fill in the details!).

The definitions we gave in the previous lectures for:

- smooth maps,
- tangent vectors and tangent spaces,
- differential forms and Deham differential,
- orientations

apply word by word to all manifolds with boundary. Care would be required when formulating the regular value theorem or when representing tangent vectors as speeds of curves (e.g., for points $p \in \partial M$, one has to allow also curves $\gamma : [0, \epsilon) \to M$). Note that, at points $p \in \partial M$, the tangent space $T_p\partial M$ (a tangent space of a manifold without boundary!) sits inside T_pM:

$$T_p\partial M \subset T_pM.$$

Even more, among the tangent vectors that are not tangent to the boundary, there is a special class- the ones pointing outwards.

DEFINITION 5.38. *For $p \in \partial M$, a tangent vector $v \in T_pM$ is said to be an outward tangent vector if $v \notin T_p\partial M$ and if there exists a curve $\gamma : (-\epsilon, 0] \to M$ with $\gamma(0) = p$ and such that $v = \frac{d\gamma}{dt}(0)$.*

In terms of coordinate charts, these are the tangent vectors $v \in T_pM$ with the property that, for any coordinate chart $\chi : U \to \Omega \subset \mathbb{H}^m$ around p, the corresponding vector

$$v^\chi \in \mathbb{R}^m$$

has strictly negative last coordinate (think on a picture!).

The notion of outward pointing vector is important for our discussion because it allows us to obtain an orientation of ∂M out of an orientation of M.

DEFINITION 5.39. *Given an oriented manifold with boundary $(M, \mathcal{O})$, define* ***the induced orientation*** *$\partial\mathcal{O}$ on the boundary as follows: for $p \in \partial M$, declare a frame $(v_1, \ldots, v_{m-1})$ of $T_p\partial M$ to be positively oriented if, for a (or, equivalently, for any) outwards pointing tangent vector $v \in T_pM$, $(v, v_1, \ldots, v_{m-1})$ is a positively oriented frame of $(T_pM, \mathcal{O}_p)$.*

With this, whenever we have an oriented manifold M with boundary, its boundary ∂M will be automatically considered to be oriented with the induced orientation. Therefore, the integrations $\int_M$ and $\int_{\partial M}$ are unambiguously defined.

THEOREM 5.40 (Stokes on manifolds with boundary). *For any m-dimensional oriented compact manifold with boundary M one has*

$$\int_M d\eta = \int_{\partial M} \eta|_{\partial M} \quad \text{for all } \eta \in \Omega^{m-1}(M).$$

PROOF. We will prove the previous formula without assuming that M is compact, but under the condition that η is compactly supported. Covering the support of η by a finite number of coordinate charts $\chi_i : U_i \xrightarrow{\sim} \Omega_i$ where each Ω_i is either $\mathbb{R}^m$ or $\mathbb{H}^m$, and decomposing η as a sum of forms supported inside U_i, we may assume that $M = \mathbb{R}^m$ or $M = \mathbb{H}^m$. The first situation was already treated, so let us assume that $M = \mathbb{H}^m$. As in the case without boundary, we write η in the fom (6.6) and see that the terms corresponding to $i \neq m$ do not contribute to $\int d\eta$, and neither to $\eta|_{\mathbb{H}^m}$. Hence we may assume that

$$\eta = f \cdot dx_1 \wedge \ldots \wedge dx_{m-1},$$

so that we can write

$$\int_{\mathbb{H}^m} d\eta = (-1)^{m-1} \int_{\mathbb{H}^m} \frac{\partial f}{\partial x_m}(x_1, \dots, x_m) dx_1 \dots dx_m.$$

Writing

$$\int_{\mathbb{H}^m} \frac{\partial f}{\partial x_m}(x_1, \dots, x_m) dx_1 \dots dx_m = \int_{\mathbb{R}^{m-1}} \left(\int_0^\infty \frac{\partial f}{\partial x_m}(x_1, \dots, x_m) dx_m \right) dx_1 \dots dx_{m-1}$$

we find that

$$\int_{\mathbb{H}^m} d\eta = (-1)^m \int_{\mathbb{R}^{m-1}} f(x_1, \dots x_{m-1}, 0) dx_1 \dots dx_{m-1}.$$

Note that, in the last equality, $\int_{R^{m-1}}$ is the standard integration- i.e. it uses the standard orientation on $\mathbb{R}^{m-1}$, for which the canonical basis $\{e_1, \dots, e_{m-1}\}$ is positively oriented. However, the orientation induced by the one on $\mathbb{H}^m$ is defined by the condition that $\{-e_m, e_1, \dots, e_{m-1}\}$ is positively oriented; hence the induced orientation differs from the canonical one by a factor of $(-1)^m$; hence

$$\int_{\mathbb{H}^m} d\eta = \int_{\partial \mathbb{H}^m} f(x_1, \dots x_{m-1}, 0) dx_1 \dots dx_{m-1} = \int_{\partial \mathbb{H}^m} \eta|_{\partial \mathbb{H}^m}.$$

□

7. De Rham cohomology

We now return to the DeRham differential and its property that $d \circ d = 0$. The principle underlying "Homological algebra" tells us that, whenever we have a sequence of operators d satisfying $d \circ d = 0$, the first thing we do is to look at the resulting cohomology.

DEFINITION 5.41. *A cochain complex of vector spaces is a sequence*

$$\xrightarrow{d} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \dots$$

consisting of vector spaces and linear maps between them satisfying $d \circ d = 0$. We say, in short, that $(C^\bullet, d)$ ***is a complex****.*

Given such a complex, an element $\omega \in \mathcal{C}^k$ is called

- ***closed*** *if $d\omega = 0$; denote by Z^k the set of such.*
- ***exact*** *if $\omega = d\eta$ for some $\eta \in C^{k-1}$; denote by B^k the resulting set.*

While the condition $d \circ d = 0$ means simply that $B^k \subset Z^k$, we define the k-th cohomology space of $(C^\bullet, d)$ as the resulting quotient vector space

$$H^k(C^\bullet, d) := Z^k / B^k.$$

Of course, the central example for us is provided by differential forms and DeRham differential; the resulting complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

(with 0 in negative degrees) is called the **DeRham complex of the manifold** M.

DEFINITION 5.42. *The* ***DeRham cohomology spaces of*** *M, denoted $H^k(M)$, are the cohomology spaces of the DeRham complex $(\Omega^\bullet(M), d)$. For a closed form $\omega \in \Omega^k(M)$ we will denote by*

$$[\omega] \in H^k(M)$$

the induced cohomology class.

When all the cohomology spaces are finite dimensional (e.g. when M is compact-see below) , define ***the Euler characteristic*** *of M as*

$$\chi(M) = \sum_i (-1)^i dim(H^i(M)).$$

EXERCISE 5.19. Show that for any connected compact manifold M, one has $H^k(M) = 0$ if $k > \dim(M)$ and $H^0(M) \cong \mathbb{R}$ if M is connected. What if M is not connected?

EXERCISE 5.20. Show that the wedge product of forms induces a well-defined operation

$$H^k(M) \times H^l(M) \to H^{k+l}(M), \quad ([\omega], [\eta]) \mapsto [\omega] \cdot [\eta] := [\omega \wedge \eta].$$

Deduce that

$$H^\bullet(M) := \bigoplus_k H^k(M)$$

has the structure of a ring (actually an algebra).

Next, the fact that differential forms can be pulled-back allows us to compare the cohomology of different manifolds. Underlying this discussion is the notion of chain map: given two complexes $(\mathcal{C}^\bullet, d_\mathcal{C})$ and $(\mathcal{D}^\bullet, d_\mathcal{D})$, a **chain map** between them is a sequence of linear maps $f : \mathcal{C}^k \to \mathcal{D}^k$ which commute with the operators d:

$$f \circ d_\mathcal{C} = d_\mathcal{D} \circ f.$$

In this situation it is an easy (but must do!) exercise to see that f induces maps

$$[f] : H^k(\mathcal{C}, d_\mathcal{C}) \to H^k(\mathcal{D}, d_\mathcal{D})$$

by sending the class of an element ω into the class of $f(\omega)$.

Again, the basic example is the pull-back map

$$F^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

induced by a smooth map $F : M \to N$ between two manifolds. The maps induced in cohomology is denoted by the same letter:

$$F^* : H^\bullet(N) \to H^\bullet(M).$$

EXERCISE 5.21. Chek that $(G \circ F)^* = F^* \circ G^*$ in cohomology; deduce that if $F : M \to N$ is a diffeomorphism, than the map induced by F in cohomology is an isomorphism of vector spaces.

EXERCISE 5.22. With respect to the ring/algebra structure on cohomology from Exercise 5.20, show that the map $F^* : H^\bullet(N) \to H^\bullet(M)$ is a ring/algebra homomorphism.

So far we have discussed the cohomology groups H^k as "functors" that allow us to pass from the world of manifolds (with smooth maps between them) to that of vector spaces (with linear maps between them) (or, with the previous exercises in mind, to the category of algebras with algebra homomorphisms between them). We now look at some of the more specific properties, properties that are extremely useful for explicit computations.

7.1. Poincare lemma. In short, the Poincare lemma says that any closed form on $\mathbb{R}^m$ of strictly positive degree must be exact. In other words, it provides the first complete computation of cohomology:

THEOREM 5.43 (Poincare lemma). *One has*

$$H^k(\mathbb{R}^m) = \begin{cases} 0 & \text{if } k \neq 0 \\ \mathbb{R} & \text{if } k = 0 \end{cases}.$$

Moreover, we will see there is even more to learn from its proof.

PROOF. We proceed by induction on m. For $m = 0$ this is clearly true; we assume it to hold for m and prove it for $m + 1$. Write

$$\mathbb{R}^{m+1} = \mathbb{R} \times \mathbb{R}^m$$

with coordinates denoted by $(t, x_1, \ldots, x_m)$. We will treat t as a special coordinate that we try to eliminate: we will show that any closed k-form on $\mathbb{R}^{m+1}$ is an exact one plus one that depends only on $(x_1, \ldots, x_m)$ (so that we can apply the induction hypothesis).

An arbitrary k-form on $\mathbb{R}^{m+1}$ can be decomposed as

$$\omega = \alpha_t + dt \wedge \beta_t \tag{7.1}$$

where

$$\mathbb{R} \ni t \mapsto \alpha_t \in \Omega^k(\mathbb{R}^m), \quad \mathbb{R} \ni t \mapsto \beta_t \in \Omega^{k-1}(\mathbb{R}^m)$$

are smooth families of forms on $\mathbb{R}^m$ and they are interpreted as forms on $\mathbb{R}^m \times \mathbb{R}$ without dt-terms. To such families one can apply the DeRham differential in two ways:

- apply the DeRham differential in $\mathbb{R}^m$ to each α_t. In this case we use the notation $d\alpha_t$- again a smooth family of forms, and again interpreted as a form on $\mathbb{R}^{m+1}$ without dt-component.
- interpret from the beginning α_t as a form on $\mathbb{R}^{m+1}$ and apply the DeRham differential for $\mathbb{R}^{m+1}$.

To distinguish the two, we will denote by d_{tot} the DeRham differential for $\mathbb{R}^{m+1}$. The relationship between the two is simply:

$$d_{\text{tot}}(\alpha_t) = d(\alpha_t) + dt \wedge \dot{\alpha}_t, \tag{7.2}$$

where $\dot{\alpha}_t$ is the derivative with respect to t (of $t \mapsto \alpha_t$). We now start with a $\omega \in \Omega^k(\mathbb{R}^{m+1})$ closed; writing it as (7.1), we find (using also that $dt \wedge dt = 0$)

$$0 = d_{\text{tot}}(\omega) = d(\alpha_t) + dt \wedge (\dot{\alpha}_t - d\beta_t)) .$$

So, first of all,

$$d\alpha_t = 0 \quad \text{for all } t.$$

We will be using this only for $t = 0$ (i.e. α_0 will be the closed form on $\mathbb{R}^m$ we are looking for). Secondly, we obtain

$$\dot{\alpha}_t = d\beta_t \tag{7.3}$$

for all t and, therefore, we can write

$$\alpha_t = \alpha_0 + \int_0^t d\beta_t. \tag{7.4}$$

Denoting $\gamma_t = \int_0^t \beta_t$ (again a family of forms on $\mathbb{R}^m$, interpreted as a form on $\mathbb{R}^{m+1}$ with no dt-term), the general formula (7.2) gives us

$$d_{\text{tot}}(\gamma_t) = \int_0^t d\beta_t + dt \wedge \beta_t.$$

Plugging the resulting formula for $dt \wedge \beta_t$, as well as the formula (7.4) for α_t, in (7.1), we find:

$$\omega = \alpha_0 + d_{\text{tot}}(\gamma_t).$$

And this is precisely what we wanted to achieve (just to make sure: α_0 is a closed form on $\mathbb{R}^m$ hence, by induction hypothesis, it is $d\eta$ for some $\eta \in \Omega^k(\mathbb{R}^m)$; of course, since η has no dependence on t, one has $d\eta = d_{\text{tot}}\eta$, hence $\omega = d_{\text{tot}}(\eta + \gamma_t)$ will be exact.) □

REMARK 5.44. The previous proof is written in such a way that it is not so important that we work with $\mathbb{R}^m$: instead, we can replace it by an arbitrary manifold M and $\mathbb{R}^{m+1}$ by $M \times \mathbb{R}$; hence the argument above shows that if a manifold M has no cohomology in degree k, then the same is true for $M \times \mathbb{R}$; and with a bit more effort one can prove that, for any M, $M \times \mathbb{R}$ and M have isomorphic cohomologies (see below). However, let us state here what we get for general M without any effort (and which is at the heart of the homotopy invariance discussed below).

COROLLARY 5.45. *[of the proof] For any manifold M, the maps*

$$i_0^*, i_1^* : H^\bullet(M \times \mathbb{R}) \to H^\bullet(M)$$

induced by the "inclusions" $i_0, i_1 : M \to M \times \mathbb{R}$, $i_\epsilon(p) = (p, \epsilon)$ ($\epsilon \in \{0, 1\}$), coincide.

PROOF. Decomposing a k-form on $M \times \mathbb{R}$ as before, i.e. as in (7.1), then $i_\epsilon^* \omega = \alpha_\epsilon$. Hence the statement is that, if ω is closed, then $\alpha_0 = \alpha_1$ is an exact form on M; but that is clear from (7.4) since we get

$$\alpha_1 - \alpha_0 = \int_0^1 d\beta_t = d\left(\int_0^1 \beta_t\right).$$

□

7.2. Homotopy invariance. We now push the previous discussion to a more conceptual level. While the heard work has already been done with the proof of the Poincare lemma, the conceptual gain is substantial: it says that $H^\bullet(M)$ depends only on the "homotopy type" of M (which is much weaker than the diffeomorphism type).

To explain this, we first recall the notion of homotopy and homotopy equivalence. These notions are usually introduced in the general context of topological spaces; of ocurse, here we restrict to the smooth context. Moreover, while for topological spaces (and also intuitively) it is natural to use the interval

$$I = [0, 1]$$

(e.g. to parametrize deformations), in the smooth context that would force us right away to allow for manifolds with boundary. Allowing manifold with boundary is not a real problem, but not absolutely necessary in order to achieve the main result; more precisely, we can just use $\mathbb{R}$ instead of $[0, 1]$.

Therefore, in what follows, one may assume that I is either $[0, 1]$ (if you are ok with manifolds with boundary), or $I = \mathbb{R}$ (if you want to stick to manifolds without boundary but being willing to a accept a less natural framework).

DEFINITION 5.46. *Two smooth maps $F_0, F_1 : M \to N$ between two smooth manifolds M and N are said to be **homotopic** if there exists a smooth map*

$$\tilde{F} : M \times I \to N$$

such that $F_0(\cdot) = \tilde{F}(\cdot, 0)$, $F_1(\cdot) = \tilde{F}(\cdot, 1)$.

*We say that a smooth map $F : M \to N$ is a **homotopy equivalence** if there exists a smooth map $G : N \to M$ with the property that $G \circ F$ is homotopic to Id_M and $F \circ G$ is homotopic to Id_N. When such a map exists, we say that M and N are **homotopy equivalent**.*

While Topology studies topological spaces and continuous maps between them (the "category" of topological spaces) and Differential Geometry studies manifolds and smooth maps between them, Homotopy Theory studies topological spaces and homotopy classes of maps between them. "Isomorphisms" are: homeomorphisms in Topology, diffeomorphisms in Differential Geometry and ... homotopy equivalences in Homotopy Theory. Therefore, the next result is called "homotopy invariance":

THEOREM 5.47 (homotopy invariance). *For any two manifolds M and N:*

(1) *If $F_0, F_1 : M \to N$ are homotopic, then they induce the same maps in cohomology.*
(2) *If $F : M \to N$ is a homotopy equivalence, then the maps induced in cohomology, $F^* : H^\bullet(N) \to H^\bullet(M)$, are isomorphisms.*

PROOF. The first part follows right away from Corollary 5.45: if $\tilde{F} : M \times I \to N$ relates F_0 and F_1 as in the definition, we can write $F_0 = \tilde{F} \circ i_0$ and $F_1 = \tilde{F} \circ i_1$, where i_ϵ are the inclusions from the corollary. Therefore, in cohomology,

$$F_0^* = i_0^* \circ \tilde{F}^* = i_1^* \circ \tilde{F}^* = F_1^*.$$

And the second part follows easily from the first part: if $G : N \to M$ is as in the definition of homotopy equivalence we get

$$F^* \circ G^* = (G \circ F)^* = \mathrm{Id}$$

and similarly for $G^* \circ F^*$, hence F^* is an isomorohism. □

EXAMPLE 5.48. $M = \mathbb{R}^m$ and $N = \{0\}$ are homotopic equivalent. In one direction, there is only one map $F : \mathbb{R}^m \to \{0\}$; in the other direction, we use the map G that maps N to the origin. The only thing we have to show is that

$$f : \mathbb{R}^m \to \mathbb{R}^m, \quad f(x) = 0,$$

is homotopic to the identity. For that take $\tilde{f}(x,t) = t \cdot x$.

Therefore, the previous theorem will imply the Poincare lemma.

EXERCISE 5.23. By a similar argument show that, for any manifold M, the projection $\mathrm{pr} : M \times I \to M$ induces isomorphisms in cohomology

$$\mathrm{pr}^* : H^\bullet(M) \xrightarrow{\sim} H^\bullet(M \times I).$$

EXAMPLE 5.49. Similarly, one gets

$$H^\bullet(\mathbb{R}^m \setminus \{0\}) \cong H^\bullet(S^{m-1})$$

(and the last group will be computed in the next subsection). In other words, we claim that the two spaces involved are homotopy equivalent. To see this, we consider the inclusion

$$i : S^{m-1} \hookrightarrow \mathbb{R}^m \setminus \{0\}$$

and, in the other direction, the rescaling

$$r : \mathbb{R}^m \setminus \{0\} \to S^{m-1}, \quad r(x) = \frac{1}{||x||} \cdot x.$$

While $r \circ i$ is the identity map of S^{m-1}, we claim that $i \circ r$ is homotopic to the identity of $\mathbb{R}^m \setminus \{0\}$. Indeed,

$$F : \mathbb{R}^m \setminus \{0\} \times I \to \mathbb{R}^m \setminus \{0\}, \quad F(x,t) = t \cdot x + (1-t) \cdot i(r(x))$$

defined the desired homotopy.

7.3. Stokes theorem revisited. Stokes' theorem tells us that integration of top-forms interacts nicely with cohomology.

COROLLARY 5.50 (Stokes' theorem reformulated). *For a compact oriented manifold M, the integration $\int_M : \Omega^m(M) \to \mathbb{R}$ descends to a linear map*

$$\int_M : H^m(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega.$$

Using orientations induced by a volume form μ (Lemma 5.36), the resulting integration gives $\int_M \mu > 0$ (why?), therefore:

COROLLARY 5.51. *On a compact manifold, any volume form $\mu \in \Omega^m(M)$ defines a non-trivial cohomology class*

$$[\mu] \neq 0 \in H^m(M).$$

EXAMPLE 5.52. In particular, we see that the m-th cohomology group of the sphere S^m is non-zero, with with the volume form μ_{S^m} from Example 5.35 inducing a non-zero cohomology class.

We no show that this fact alone has some very interesting applications.

COROLLARY 5.53 (Brower's fixed point theorem). *Any smooth map from the closed unit disk to itself, $f : D^m \to D^m$, has at least one fixed point.*

PROOF. Assume there is a map f with no fixed point. Then for each $x \in D^m$, the line through x and $f(x)$ intersects the boundary sphere S^{m-1} in two points. Let $r(x)$ the point with the property that x belongs to the segment $[f(x), r(x)]$. Then we obtain a smooth map

$$r : D^m \to S^{m-1} \quad \text{with } r|_{S^{m-1}} = \mathrm{Id}$$

(why is the map r smooth)? Denoting by $i : S^{m-1} \hookrightarrow D^m$ the inclusion, we have $r \circ i = \mathrm{Id}$ hence, if the degree $m-1$ cohomology, the composition

$$H^{m-1}(S^{m-1}) \xrightarrow{r^*} H^m(D^m) \xrightarrow{i^*} H^{m-1}(S^{m-1})$$

is the identity map. But, since $H^{m-1}(D^m) = 0$ (why?), that would imply that $H^{m-1}(S^{m-1}) = 0$. So no such f exists. □

COROLLARY 5.54 (Nowhere vanishing vector fields on spheres). *The sphere S^m admits a non-where vanishing vector field if and only if m is odd.*

PROOF. Assume that such a vector field exists V exists. Hence

$$V : S^m \to \mathbb{R}^m, \quad V(p) \perp p \text{ for all } p \in S^m.$$

By rescaling, we may assume $||V(p)|| = 1$ at all p. But then

$$H(p,t) = \cos(\pi\, t)\, p + \sin(\pi\, t)\, V(p) \in S^m \quad \text{for all } p \in S^m, t \in I$$

and we obtain a homotopy

$$H : S^m \times I \to S^m$$

between Id and $\tau := -\mathrm{Id}$. Hence, by homotopy invariance, $-\tau$ will induce the identity map in the m-cohomology group of S^m; in particular,

$$[\tau^*(\mu_{S^m})] = [\mu_{S^m}].$$

But, given the explicit formula of μ_{S^m}, it is clear that

$$\tau^*(\mu_{S^m}) = (-1)^{m+1}\mu_{S^m}$$

already as forms (hence also at the level of cohomology classes). Since the cohomology class of μ_{S^m} is non-zero, we deduce that $1 = (-1)^{m+1}$, hence m must be odd. Of course, when m is odd, one can just build a simple nowhere vaninshing vector fields on S^m- e.g.

$$V := \left(x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0}\right) + \left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right) + \dots$$

□

REMARK 5.55 (for the interested students). For any oriented compact manifold M, using the product operation on cohomology induced by the wedge-product of forms (cf. e.g. Exercise 5.20) and the integration, we obtain a map

$$\beta : H^k(M) \times H^{m-k}(M) \to \mathbb{R}, \quad \beta([\omega],[\eta]) = \int_M \omega \wedge \beta$$

(for each k, where $m = \dim(M)$). This is know under the name of **the Poincare pairing**. The Poincare duality theorem says that this map is non-degenerate or, equivalently, that the induced map

$$H^k(M) \to H^{m-k}(M)^*, \quad u \mapsto \beta(u,\cdot)$$

is an isomorphism. Using the fact that the cohomology spaces of compact manifolds are finite dimensional (see below), this will imply that the groups $H^k(M)$ and $H^{m-k}(M)$ have the same dimension, for all k. In particular, we deduce that $\chi(M) = 0$ for all odd dimensional compact oriented manifolds M.

While the non-vanishing of $\int_M$ was enough to get such results, let us now describe the full information that it provides on the top cohomology spaces.

THEOREM 5.56. *For any connected compact oriented manifold M,*

$$\int_M : H^m(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

is an isomorphism.

PROOF. We have to show that a top-form $\omega \in \Omega^m(M)$ is zero in cohomology (exact) if and only if $\int_M \omega = 0$. Actually a version of this is true on any oriented M (compact or not), but for compactly supported form $\omega \in \Omega^m_c(M)$: $\omega = d\eta$ for some compactly supported η if and only if $\int_M \omega = 0$. We first prove that this is true for Euclidean spaces. On $\mathbb{R}$ the statement is immediate. In general, we proceed by induction on m; we assume it to hold for m and we prove it for $m+1$. Of course, this is similar to the proof of the Poincare lemma (Theorem 5.43), and we use the same notations as there. Hence let $\omega \in \Omega^{m+1}_c(\mathbb{R}^{m+1})$ whose integral is zero and write it in the form (7.1). Note that, from (7.3) and the fact that ω has compact support (hence $\alpha_t = 0$ for t very large or very small), it follows that

$$\int_{\mathbb{R}} d\beta_s = 0.$$

Hence the form $\int_{\mathbb{R}} \beta_s \in \Omega^m(\mathbb{R}^m)$ is closed, and clearly with compact supports. Moreover, its integral over $\mathbb{R}^m$ equals $\int_{\mathbb{R}^{m+1}} \omega = 0$. By the induction hypothesis, we can write it as

$$\int_{\mathbb{R}} \beta_s \, ds = d\gamma \quad \text{for some } \gamma \in \Omega^{m-1}_c(\mathbb{R}^{m-1}).$$

Consider now a function $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(s)\, ds = 1$ and consider $\psi(t) = \int_{-\infty}^t \phi(s)\, ds$. And define the following m-form on $\mathbb{R}^{m+1}$:

$$\eta := \int_{-\infty}^t \beta_s \, ds - \psi(t) \cdot \int_{-\infty}^{\infty} \beta_s \, ds - \phi \cdot dt \wedge \gamma.$$

By the choice of ϕ and the fact that γ has compact support in $\mathbb{R}^m$, η is compactly supported. We now compute its differential; using $\psi'(t) = \phi(t)$ and $\int_{\mathbb{R}} d\beta_s = 0$ (see also 7.2), we find

$$d_{\text{tot}}\eta = \int_{-\infty}^t d\beta_s \, ds + dt \wedge \left(\beta_t - \phi(t) \int_{-\infty}^{\infty} \beta_s ds + \phi(t) d\gamma \right).$$

From the choice of γ, the last two terms that are wedged with dt cancel. And, using (7.3) again, $\int_{-\infty}^{t} d\beta_s\, ds = \alpha_t$. Hence we see that $d_{\text{tot}}\eta = \alpha_t + dt \wedge \beta_t = \omega$.

We now return to the original proof. Hence M is compact and oriented, $\int_M \omega = 0$ and we want to prove that ω is exact. The strategy is to write ω as a sum of forms, each supported in a coordinate chart $U_i \cong \mathbb{R}^m$, and each with zero integral; by the previous result for $\mathbb{R}^m$, we will be done. To implement this idea, we first cover M with a finite number of domains of coordinate charts $U_i \cong \mathbb{R}^m$, $i \in \{0, 1, \ldots, k\}$. After eventually adding some more opens, we may assume that

$$U_0 \cap U_1 \neq \emptyset, \quad U_1 \cap U_2 \neq \emptyset, \quad \ldots U_{k-1} \cap U_k \neq \emptyset.$$

(this is an exercise; hint: for U and V two consecutive domains, if they do not intersect, choose a path that starts in U and ends in V, then cover the path carefully so that each two consecutive charts intersect).

Choose now a partition of unity $\{\phi_0, \phi_1, \ldots, \phi_k\}$ subordinated to this cover and let $c_i = \int_M \phi_i \cdot \omega$. Choose also forms $\alpha^i \in \Omega^m(M)$, supported inside $U_i \cap U_{i+1}$, with $\int_M \alpha_i = 1$ and consider the following forms

$$\begin{aligned}
\omega_0^{'} &= \phi_0 \cdot \omega - c_0\, \alpha_0 \\
\omega_1^{'} &= \phi_1 \cdot \omega - (c_0 + c_1)\, \alpha_1 + c_0\, \alpha_0 \\
\omega_2^{'} &= \phi_2 \cdot \omega - (c_0 + c_1 + c_2)\, \alpha_2 + (c_0 + c_1)\, \alpha_1, \quad \text{etc}
\end{aligned}$$

Note that, since $\sum_i c_i = \int_M \omega = 0$, and $\sum \phi_i = 1$, the sum of these forms is precisely ω. Moreover, by construction, each $\omega_i^{'}$ is supported inside U_i. Hence this provides the desired decomposition of ω. □

7.4. Mayer-Vietoris. We now discuss another useful tool for computing cohomology- the Mayer-Vietoris sequence. Roughly speaking, it provides a way to compute that cohomology of a manifold M by breaking M into pieces. It is formulated in terms of long exact sequences in the following sense.

Definition 5.57. *A sequence of vector spaces with linear maps between them,*

$$\ldots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \ldots$$

is said to be ***exact at*** *B if $Ker(g) = Im(g)$. We say that it is an* ***exact sequence*** *is it is exact at all entries.*

Example 5.58. A sequence of vector spaces of type

$$A \xrightarrow{f} B \longrightarrow 0$$

is exact if and only if f is surjective, while a sequence of type

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact if and only of f is injective.

Exercise 5.24. Exact sequences of type

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

are called **short exact sequences**. Show that, for such a short exact sequence, B is finite dimensional if and only if A and C are. Moreover, in this case prove that

$$\dim(B) = \dim(A) + \dim(C).$$

EXERCISE 5.25. Show that a sequence of vector spaces with linear maps between them,

$$\xrightarrow{d} C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \xrightarrow{d} \dots,$$

is exact if and only if it is a cochain complex with vanishing cohomology.

Moreover, when this happens, if the vector spaces C^i are finite dimensional and only a finite number of them are non-zero, show that

$$\sum_i (-1)^i \dim(C^i) = 0.$$

The Mayer-Vietoris theorem (and its application to the computation of cohomology of spheres) is probably the best illustration of the usefulness of long exact sequences.

To state the main result, assume we have a manifold M written as

$$M = U \cup V, \quad \text{with } U, V \subset M \text{ opens.}$$

There are some natural maps relating the cohomologies of M, U, V and $U \cap V$; they are built from the inclusions

$$i_U : U \to M, \quad i_V : U \to M,$$
$$j_U : U \cap V \to U, \quad j_U : U \cap V \to V.$$

Using the maps induced in cohomology, we consider

$$H^\bullet(M) \xrightarrow{(i_U^*, i_V^*)} H^\bullet(U) \oplus H^\bullet(V) \xrightarrow{j_U^* - j_V^*} H^\bullet(U \cap V).$$

Note that the composition of these two maps is zero.

THEOREM 5.59. *There exist linear maps*

$$\delta : H^\bullet(U \cap V) \to H^{\bullet+1}(M)$$

such that the following sequence is exact

$$\dots \xrightarrow{\delta} H^k(M) \xrightarrow{(i_U^*, i_V^*)} H^k(U) \oplus H^\bullet(V) \xrightarrow{j_U^* - j_V^*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \longrightarrow \dots.$$

PROOF. We first claim that, for each k, the sequence

$$0 \to \Omega^k(M) \xrightarrow{(i_U^*, i_V^*)} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_U^* - j_V^*} \Omega^k(U \cap V) \to 0 \tag{7.5}$$

is exact. Exactness at $\Omega^k(M)$ means the injectivity of the map (i_U^*, i_V^*), i.e. that, for $\omega \in \Omega^k(M)$, if $\omega|_U = 0$ and $\omega|_V = 0$ then $\omega = 0$. And this is clear.

Next, the exactness in the middle. Since the composition of the two maps is clearly zero, the image of the first is inside the kernel of the second. To prove the reverse inclusion, let

$$(\omega_U, \omega_V) \in \mathrm{Ker}(j_U^* - j_V^*).$$

That means that

$$\omega_U|_{U \cap V} = \omega_V|_{U \cap V}.$$

We have to show that there exists $\omega \in \Omega^k(M)$ whose restrictions to U and V is ω_U and ω_V, respectively. But that should be clear (the desired condition tells us how to define ω).

We are left with showing the exactness at $\Omega^k(U \cap V)$, i.e. the surjectivity of the map $j_U^* - j_V^*$. So, let $\omega \in \Omega^k(U \cap V)$ and we are looking for $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ so that

$$\omega = \omega_U|_{U \cap V} - \omega_V|_{U \cap V}.$$

For that choose partition of unity $\{\eta_U, \eta_V\}$ subordinated to $\{U, V\}$ and set $\omega_U = \eta_U \cdot \omega$, $\omega_V = -\eta_V \cdot \omega$.

Therefore (7.5) is exact. We now claim that the rest is a consequence of a general property of short exact sequence of complexes:

LEMMA 5.60. *Consider a short exact sequence of chain complexes*

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0, \tag{7.6}$$

i.e. $(A^\bullet, d_A)$, $(B^\bullet, d_B)$ *and* $(C^\bullet, d_C)$ *are chain complexes,* f *and* g *are chain maps, and the previous sequence is exact in each degree. Then there exist linear maps*

$$\delta : H^\bullet(C, d_C) \to H^{\bullet+1}(A, d_A)$$

such that the following sequence is exact

$$\ldots \xrightarrow{\delta} H^k(A, d_A) \xrightarrow{[f]} H^k(B, d_B) \xrightarrow{[g]} H^k(C, d_C) \xrightarrow{\delta} H^{k+1}(A, d_A) \longrightarrow \ldots.$$

PROOF. Let us describe the map δ on an arbitrary element $[c] \in H^k(C, d_C)$. Hence $c \in C^k$,

$$d_C(c) = 0.$$

The exactness of (7.6) at C ensures that g is surjective, hence we can write

$$c = g(b), \quad \text{with } b \in B^k.$$

Now, since $d_C(c) = 0$ and g is a chain map, we obtain $g(d_B(b)) = 0$, i.e. $d_B(b) \in \mathrm{Ker}(g)$. Using again the exactness of (7.6), this time at B (and in degree $k+1$) we can write

$$d_B(b) = f(a), \quad \text{with } a \in A^{k+1}.$$

Applying d_B we also find $f(d_A(a)) = 0$ and, since f is injective (the exactness of (7.6) at A), one has $d_A(a) = 0$. Hence a defines an element in the cohomology of A and we set:

$$\delta([c]) := [a].$$

We have to check that $[a]$ not depend on the choices involved: if $[c] = [c']$ and we also write

$$c' = g(b'), \quad d_B(b') = f(a'),$$

then a and a' represent the same cohomology class. To see this, since $[c] = [c']$, we can write $c' = c + d_C(z)$ with $z \in C^{k-1}$. Using again the surjectivity of g, write $z = g(y)$ for some $y \in B^{k-1}$. Then $c' = c + g(d_B(y))$ and, using the choice of b and b' we obtain

$$b' - b = d_B(y) \in \mathrm{Ker}(g)$$

hence we can write

$$b' = b + d_B(y) + f(x), \quad \text{with } x \in A^k.$$

Then $d_B(b') = d_B(b) + f(d_A(x))$ hence, using the way we chose a and a', $f(a') = f(a) + f(d_A(x))$. Hence, by the injectivity of f again, $a' = a + d_A(x)$, therefore $[a'] = [a]$ in cohomology.

With all the maps defined now, one still has to prove the exactness of the sequence. This is not so difficult and it is an extremely useful exercise (a really "must do" one!). Therefore, we check here the exactness at one spot only. Say at $H^k(C, d_C)$:

$$\mathrm{Im}[g] = \mathrm{Ker}(\delta).$$

(but we insist: it may be more instructive, and even easier, to do it yourself than reading the argument below!).

For the direct inclusion, we start with $[c] \in H^k(C, d_C)$ with the property that $c = g(b_0)$ for some $b_0 \in B^k$ closed. Recall that $\delta([c]) = [a]$ where a is the result of the choices of b with $c = g(b)$ and of a with $d_B(b) = f(a)$. But we can just choose $b = b_0$ and then, since $d_B(b_0) = 0$, one can choose $a = 0$. Hence $\delta([c]) = 0$. For the reverse inclusion, let $[c] \in C^k(C, d_C)$ such that $\delta([c]) = 0$. That means that we can write

$$c = g(b), \quad d_B(b) = f(a), \quad a = d_A(a_0)$$

for some $b \in B^k$, $a \in A^{k+1}$, $a_0 \in A^k$. The last two equations give $d_B(b) = fd_A(a_0) = d_B f(a_0)$ hence, denoting $b' = b - f(a_0)$, we have

$$b = b' + f(a_0) \quad \text{withd}_{\text{B}}(\text{b}') = 0/$$

Applying g and using that $g \circ f = 0$ and $c = g(b)$, we find

$$c = g(b') \quad \text{with } d_B(b') = 0.$$

Hence $[b'] \in H^k(B, d_B)$ is defined and is sent by $[g]$ to $[c]$. □

□

EXAMPLE 5.61. Let us compute the cohomology of the spheres S^m. We claim that

$$H^k(S^m) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = m \\ 0 & \text{otherwise} \end{cases}.$$

The strategy is to decompose

$$S^m = U \cup V$$

where U and V are basically the half hemispheres, extended so that they become open. E.g. they could be taken the entire domains of the stereographic projections with respect to the north and south pole, or just

$$U = \{(x_0, \dots, x_m) \in S^m : x_m > -\epsilon\}, \quad V = \{(x_0, \dots, x_m) \in S^m : x_m < -\epsilon\},$$

with $\epsilon \in (0, 1)$ any number. Note that both U and V are diffeomorphic to a disk (hence also to a Euclidean space), hence

$$H^k(U) \cong H^k(V) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

On the other hand, $U \cap V$ is homotopy equivalent to S^{m-1}- more precisely, the inclusion

$$i : S^{m-1} \hookrightarrow U \cap V$$

is a homotopy equivalence (prove this, if necessary with some inspiration from Example 5.49). Hence

$$H^k(U \cap V) \cong H^k(S^{m-1}).$$

Let us look at the case $m = 1$ first. This is slightly different than the rest because, in this case, $S^{m-1} = S^0 = \{-1, 1\}$ has two connected components, hence $H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$ (and zero in other degrees), hence the first part of the Mayer-Vietoris sequence,

$$0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(S^1) \to 0$$

will become (isomorphic to)

$$0 \to \mathbb{R} \xrightarrow{i} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j} \mathbb{R}^2 \to H^1(S^1) \to 0$$

Independent of what the maps i and j are, this can happen if and only of $H^1(S^1) \cong \mathbb{R}$ (exercise!). While $H^k(S^1) = 0$ for $k > 1$ (why?), we are done with the case $m = 1$. For $m \geq 2$, the Mayer-Vietoris will become

$$\xrightarrow{\delta} H^1(S^m) \longrightarrow 0 \longrightarrow H^1(S^{m-1}) \xrightarrow{\delta} H^2(S^m) \longrightarrow 0 \longrightarrow \dots$$

$$\dots$$

$$\dots \xrightarrow{\delta} H^{k-1}(S^m) \to 0 \longrightarrow H^{k-1}(S^{m-1}) \xrightarrow{\delta} H^k(S^m) \longrightarrow 0 \longrightarrow \dots$$

from which we deduce that

$$H^k(S^m) \cong H^{k-1}(S^{m-1}) \quad \forall k \geq 2, m \geq 1.$$

Hence

$$H^m(S^m) \cong H^{m-1}(S^{m-1}) \cong \ldots \cong H^1(S^1) \cong \mathbb{R}$$

and, similarly, $H^k(S^m) = 0$ for $k < m$, while for $k > 0$ we already know that the cohomology vanishes.

And here is another application of the Mayer-Vietoris sequence: the finite dimensionality of the cohomology groups of any manifold. However, for the proof we will need one extra-ingredient that will not be proven here:

PROPOSITION 5.62. *Any compact manifold admits a* ***finite good cover****, i.e. a finite open cover $\mathcal{U}$ with the property that, for each opens $U_1, \ldots, U_k \in \mathcal{U}$, the intersection $U_1 \cap \ldots \cap U_k$ is either empty or diffeomorphic to $\mathbb{R}^m$.*

Appealing to your intuition for imagining what a triangulation of a manifold is, a triangulation (which always exists) gives rise to such a finite good cover (one just has to "fatten" each simplex a bit). The standard way to prove the proposition above is by using a "Riemannian metric" on the manifold and taking geodesically convex neighborhoods of points. What we are able to prove here is:

COROLLARY 5.63. *Any manifold M which admits a finite good cover (in particular, any compact manifold) has finite dimensional cohomology group).*

This corollary allows us to define **the Euler characteristic** of any compact manifold M:

$$\chi(M) := \sum_i (-1)^i \dim \left(H^i(M)\right).$$

PROOF. We prove by induction on k the following statement: if M has a good cover consisting of k opens, then its cohomology is finite dimensional. The situation is clear when $k = 1$, since then $M \cong \mathbb{R}^m$. Assume that the statement is true for k and we prove it for $k + 1$. Hence assume that M admits a good cover $\{U_1, \ldots, U_{k+1})$. Consider

$$U = U_1 \cup \ldots \cup U_k, \quad V = U_{k+1}.$$

Note that the induction hypothesis can be applied to U, V and $U \cap V$ (for the last one note that $\{U_1 \cap V, \ldots, U_k \cap V\}$ is a good cover); hence their cohomologies are finite dimensional. Given the Mayer-Vietoris, it suffices to show that, if one has an exact sequence

$$A \xrightarrow{f} B \xrightarrow{\delta} C \xrightarrow{g} D \xrightarrow{h} E$$

where A, B, D and E are finite dimensional, then so is C. To see this, note that g takes values in

$$D' = \mathrm{Ker}(h)$$

δ descends to the quotient

$$B' := B/\mathrm{Im}(f),$$

and the exactness of the sequence gives a short exact sequence

$$0 \to B' \xrightarrow{\delta} C \xrightarrow{g} D' \to 0,$$

where B' and D' are finite dimensional (as a subspaces of the finite dimensional B, and a quotient of the finite dimensional D). Form this is follows easily that C is finite dimensional, of dimension equal to the sums of the dimensions of B' and D'. □

EXERCISE 5.26. Using the last part of Exercise 5.25, shows that the Euler characteristic $\chi(M)$ of a compact manifold M can be computed with the help of a finite good cover $\mathcal{U} = \{U_i : i \in I\}$ as follows:

$$\chi(M) = \sum_k (-1)^k n_k(\mathcal{U})$$

where, for each k, $n_k(\mathcal{U})$ is the cardinality of the set

$$\{J \subset I : |J| = k,\ \bigcap_{j \in J} U_j \neq \emptyset\}.$$

Then use this to compute the Euler characteristic of S^2.