HOMEWORK 3 (FOR OCTOBER 6, 2021)

Similar to what was discussed in the class in the case of matrices with real entries:

- also the collection $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries can be seen as an Euclidean space, namely $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$.
- inside it, the space of invertible matrices
$$\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathcal{M}_n(\mathbb{C}) : \det(A) \neq 0\}$$
is an open inside $\mathcal{M}_n(\mathbb{C})$.
- inside it one has the so-called unitary group
$$\mathrm{U}(n) := \{A \in \mathrm{GL}_n(\mathbb{C}) : A \cdot A^* = \mathrm{Id}\},$$
where A^* is the conjugate transpose and, inside it, the special unitary one:
$$\mathrm{SU}(n) := \{A \in \mathrm{U}(n) : \det(A) = 1\}.$$

Exercise 1. Please do the following:

(a) (50 %) Using the regular value theorem, prove that $U(n)$ is a (smooth) embedded submanifold of $\mathcal{M}_n(\mathbb{C})$. What is its dimension? What is $\mathrm{U}(1)$?

(b) (10 %) Then prove that also $SU(n)$ is an embedded submanifold of $\mathcal{M}_n(\mathbb{C})$.

(c) (40 %) Then prove that
$$F : S^3 \to \mathrm{GL}_2(\mathbb{C}), \quad F(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$
is an embedding whose image is precisely $SU(2)$, where we interpret S^3 as $\{(\alpha, \beta) \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\}$.