HOMEWORK 4 (FOR OCTOBER 11, 2021)

This week we have defined:

- the (abstract) tangent space T_pM of any (smooth) manifold M at any point $p \in M$ (Def. 3.4).
- for any (smooth) curve $\gamma \in \text{Curves}_p(M)$ (hence $\gamma(0) = p$) the induced tangent vector
$$\frac{d\gamma}{dt}(0) \in T_pM.$$
- for any (smooth) chart χ of M around p, the induced tangent vectors
$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \in T_pM \qquad (3.2.3),$$
as the unique (abstract) tangent vectors which, via the isomorphism $T_pM \to \mathbb{R}^m$ induced by χ (Lemma 3.5), correspond to the canonical vectors
$$e_1, \ldots, e_m \in \mathbb{R}^m.$$

Exercise 1. Let be given an arbitrary (smooth) manifold M of dimension m, $p \in M$ and χ a (smooth) chart of M around p. Define the following curves in M:
$$\gamma_1, \ldots, \gamma_m : (-\epsilon, \epsilon) \to M, \quad \gamma_i(t) = \chi^{-1}\left(\chi(p) + t \cdot e_i\right).$$

(a) for ϵ small enough, all these curves are well-defined and belong to $\text{Curves}_p(M)$.

(b) the induced tangent vectors
$$\frac{d\,\gamma_1}{dt}(0), \ldots \frac{d\,\gamma_m}{dt}(0) \in T_pM$$
are precisely the tangent vectors (3.2.3).

(c) exhibit a curve $\gamma \in \text{Curves}_p(M)$ with the property that
$$\frac{d\,\gamma}{dt}(0) = \left(\frac{\partial}{\partial \chi_1}\right)_p + \ldots + \left(\frac{\partial}{\partial \chi_m}\right)_p.$$

(d) if $M_0 \subset M$ is an embedded submanifold of dimension m_0 and $p \in M_0$, can one find a chart χ as above such that, furthermore, the first m_0 tangent vectors in (3.2.3) are tangent to M_0 (recall that $T_pM_0 \subset T_pM$).