HOMEWORK 5 (TO BE HANDED IN BY OCT 20, 2021)

First some comments- hopefully of some help. But they are not really necessary, hence you could also skip them and, if you have difficulties, come back read the relevant comment for some help.

Comment 1: We "agreed" that, for $\mathbb{R}^n$, we identify the tangent spaces

$$T_p\mathbb{R}^n \cong \mathbb{R}^n$$

using the standard isomorphism $standard_p$. But, whenever we look at $\mathbb{R}^n$ as being the tangent space at p, we still use the notation

$$\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial x_2}\right)_p, \ldots$$

for the canonical basis of $\mathbb{R}^n$ (more commonly denoted $e_1, e_2, \ldots$). In particular, a vector $v = (v_1, v_2, \ldots) \in \mathbb{R}^n$ will be written

$$v_1 \left(\frac{\partial}{\partial x_1}\right)_p + v_2 \left(\frac{\partial}{\partial x_2}\right)_p + \ldots .$$

Comment 2: Next, for $M \subset N$ embedded submanifold (usually $N = \mathbb{R}^n$), we have similar inclusions

$$T_pM \subset T_pN \quad \text{(for all } p \in M).$$

Strictly speaking, this is the differential (at p) of the inclusion map $i : M \to N$, but we do not write the inclusion all the time (well, $i(p) = p$ after all). But how do you recognize when a vector tangent to N at p, is actually tangent to M? (this is particularly interesting when $N = \mathbb{R}^n$, because then one can use explicit formulas to write down tangent vectors). Using the description of tangent vectors via speeds, one way to show that a vector $v_p \in T_pN$ is actually tangent to M would be to write v as the speed of the curve that sits inside M (and not only inside N). Another way would be to use the regular value theorem (the last version, which also gives a description of the tangent spaces).

Comment 3: Next, if you are interested in the differential of a smooth map

$$F = (F_1, \ldots, F_v) : \mathbb{R}^u \to \mathbb{R}^v$$

(or between opens in those Euclidean spaces),

$$(dF)_p : T_p\mathbb{R}^u \to T_{h(p)}\mathbb{R}^v,$$

then with the previous identifications in mind, it is a linear map $\mathbb{R}^u \to \mathbb{R}^v$ and many of you would like to think of it as a matrix- and, indeed, there is the matrix made of the partial derivatives of the components F_i of F at p that can be used to write down the explicit formula for $(dF)_p$: first

$$(dF)_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \frac{\partial F}{\partial x_i}(p) = \left(\frac{\partial F}{\partial x_1}(p), \ldots, \frac{\partial F}{\partial x_v}(p)\right) =$$

$$= \frac{\partial F_1}{\partial x_i}(p) \cdot \left(\frac{\partial}{\partial x_1}\right)_{F(p)} + \frac{\partial F_2}{\partial x_i}(p) \cdot \left(\frac{\partial}{\partial x_2}\right)_{F(p)} + \ldots$$

and then

$$(dF)_p\left(v_1\cdot\left(\frac{\partial}{\partial x_1}\right)_p+v_2\cdot\left(\frac{\partial}{\partial x_2}\right)_p+\dots\right)=v_1\cdot(dF)_p\left(\left(\frac{\partial}{\partial x_1}\right)_p\right)+v_2\cdot(dF)_p\left(\left(\frac{\partial}{\partial x_2}\right)_p\right)+\dots$$
$$=\dots$$

Moving on, if you look at the differential of a smooth map $h : M \subset N$,

$$(dh)_p : T_pM \to T_{h(p)}N,$$

where all the manifolds and formulas make sense in some Euclidean spaces, i.e. $M \subset \mathbb{R}^u$, $N \subset \mathbb{R}^v$ are embedded submanifolds and h is given by explicit formulas, then you can still make use of the ambient Euclidean spaces, but you do have to be careful. First of all: you cannot think of $(dh)_p$ as being a matrix!!! If your formulas make sense on the entire $\mathbb{R}^u$ [1] , i.e. there is a function

$$\widetilde{h} : \mathbb{R}^u \to \mathbb{R}^v$$

which extends h (i.e. $\widetilde{h}|_M = h$), then you can/should make use of it! More precisely, there should be a pretty clear guess of how to compute $(dh)_p$ using $(d\widetilde{h})_p$:

$$\begin{array}{ccc} T_pM & \xrightarrow{(dh)_p} & T_{h(p)}N \\ \downarrow & & \downarrow \\ (T_p)\mathbb{R}^u & \xrightarrow{(d\widetilde{h})_p} & (T_{h(p)})\mathbb{R}^v \end{array}$$

i.e. $(dh)_p$ is the restriction of $(d\widetilde{h})_p$ to T_pM. And, in turn, $(d\widetilde{h})_p$ can be computed as discussed above (in particular, using matrices). (of course, the guess above should be backed by a proof, but that should not be very difficult if you represent tangent vectors as speeds of curves- do it in the werkcollege!).

For instance, for $h : S^1 \to \mathbb{R}$, $h(x, y) = x^2 + xy^3$, if I have a point $p = (a, b) \in S^1$ and I am interested in computing $(dh)_p$ applied to the tangent vector

$$v_p = b\cdot\left(\frac{\partial}{\partial x}\right)_p - a\cdot\left(\frac{\partial}{\partial y}\right)_p \in T_pS^1,$$

in principle you are not even allowed to write

$$(dh)\left(\left(\frac{\partial}{\partial x}\right)_p\right)$$

because $\left(\frac{\partial}{\partial x}\right)_p$ is not even tangent to S^1. However, h is obviously the restriction of a map $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}$- you just keep the same formulas: [2]

$$\widetilde{h}(x, y) = x^2 + xy^3$$

[1] well, just an open containing M would be enough, but let's keep the story simpler

[2] of course, if you insist, you can also complicate your life and choose something like $\widetilde{h}(x, y) = x^2 + xy^3 + (x^2 + y^2 - 1)(e^{2\cos(y)} + \sin(xe^{y^5}))$, but the final outcome should really be the same!

And now you can write

$$
\begin{aligned}
(dh)_p(v_p) = (d\widetilde{h})(v_p) = (d\widetilde{h})\left(b \cdot \left(\frac{\partial}{\partial x}\right)_p - a \cdot \left(\frac{\partial}{\partial y}\right)_p\right) = \\
= b \cdot (d\widetilde{h})\left(\left(\frac{\partial}{\partial x}\right)_p\right) - a \cdot (d\widetilde{h})\left(\left(\frac{\partial}{\partial y}\right)_p\right) = \\
= b \cdot (2a + b^3)\left(\frac{\partial}{\partial t}\right)_{h(p)} - a \cdot (3ab^2)\left(\frac{\partial}{\partial t}\right)_{h(p)} \\
= (b^4 + 2ab - 3a^2b^2)\left(\frac{\partial}{\partial t}\right)_{a^2+ab^3}.
\end{aligned}
$$

Comment 4: Similar discussions (on how to use the ambient Euclidean spaces) apply also when interested in integral curves of a vector field $X \in \mathfrak{X}(M)$, where $M \subset \mathbb{R}^n$ is an embedded submanifold. As it often happens, X is given by formulas that actually make sense on the entire $\mathbb{R}^u$ (or on an open containing M). In other words, you do have a vector field

$$\widetilde{X} = F_1 \cdot \frac{\partial}{\partial x_1} + F_2 \cdot \frac{\partial}{\partial x_2} + \ldots \mathfrak{X}(\mathbb{R}^n)$$

which extends X. The point is that, if you are interested in computing an integral curve of X starting at some $p \in M$, you can just look for (small) integral curves of $\widetilde{X}$ (why? this is where local uniqueness of integral curves is very useful ... think about it in the werkcollege. Also, do you see why I was careful enough to add the word "small"?), Anyway, for $\widetilde{X}$, the integral curves $\gamma(t) = (x_1(t), x_2(t), \ldots)$ are just the solutions of the ODE

$$\dot{x}_i(t) = F_i(x_1(t), x_2(t), \ldots).$$

For instance, for $M = S^1$ and the vector field showing up above,

$$V = y \cdot \frac{\partial}{\partial x} - x \cdot \frac{\partial}{\partial y} \mathfrak{X}(S^1),$$

precisely the same formulas define a vector field

$$\widetilde{V} \in \mathfrak{X}(\mathbb{R}^2).$$

Hence, for $p = (a, b) \in S^1$, to find an integral curve for V (a curve in S^1!) you just search for an integral curve $(x(t), y(t))$ of $\widetilde{V}$ (a curve in $\mathbb{R}^2$!). You end up with solving

$$\dot{x} = y,\ \dot{y} = -x, \quad x(0) = a,\ y(0) = b$$

etc, finding eventually

$$x(t) = a\cos t + b\sin t, y(t) = -a\sin t + b\cos t.$$

You should not be surprised now that the curve you found takes values in S^1! (why?).

Comment 5: And, finally, for computing the Lie bracket $[X, Y]$ of two vector fields $X, Y \in \mathfrak{X}(M)$, when $M \subset \mathbb{R}^n$ and X and Y are given by explicit formulas. The useful remark here is that if those formulas make sense on the entire $\mathbb{R}^n$ (or an open inside it containing M), means we have two vector fields

$$\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(\mathbb{R}^n)$$

extending X and Y. As you may guess, the Lie bracket $[\widetilde{X}, \widetilde{Y}]$ (computed on $\mathbb{R}^n$!) will be an extension of $[X, Y]$ (again, this requires a proof, and it is something you can think about at home or at the werkcollege. You may want to assume first that M is closed and then you could for instance use (take for granted) that smooth functions on M can be extended to smooth functions on $\mathbb{R}^n$. Or think about different arguments).

Anyway, this says that, for $p \in M$, $[X, Y]_p = [\widetilde{X}, \widetilde{Y}]_p$ and all you have to do is to do the computation of $[\widetilde{X}, \widetilde{Y}]$ in $\mathbb{R}^n$. And there you could either use the definition or, more elegantly, use the main properties of the Lie bracket to brake it into simpler bits:

- the expressions $[X, Y]$ are linear in X, as well as in Y:
$$[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y], \quad \text{etc}$$
- skew-symmetry $[Y, X] = -[X, Y]$
- the interaction with multiplication by functions f:
$$[X, f \cdot Y] = f \cdot [X, Y] + L_X(f) \cdot Y.$$

(the proof of one of these is something else you can think about/discuss during the werkcollege).

Exercise 1. Please do the following

(1) For the sphere $S^2 \subset \mathbb{R}^3$, show that for any $q = (x, y, z) \in S^2$,
$$X_q^1 := z\left(\frac{\partial}{\partial y}\right)_q - y\left(\frac{\partial}{\partial z}\right)_q,$$
$$X_q^2 := x\left(\frac{\partial}{\partial z}\right)_q - z\left(\frac{\partial}{\partial x}\right)_q,$$
$$X_q^3 := y\left(\frac{\partial}{\partial x}\right)_q - x\left(\frac{\partial}{\partial y}\right)_q$$
belong to T_qS^2. Find two different arguments.

(2) For the sphere $S^3 \subset \mathbb{R}^4$, at an arbitrary $p = (x, y, z, t) \in S^3$, show that
$$V_p^1 := \frac{1}{2}\left(-y\left(\frac{\partial}{\partial x}\right)_p + x\left(\frac{\partial}{\partial y}\right)_p + t\left(\frac{\partial}{\partial z}\right)_p - z\left(\frac{\partial}{\partial t}\right)_p\right),$$
$$V_p^2 := \frac{1}{2}\left(-z\left(\frac{\partial}{\partial x}\right)_p - t\left(\frac{\partial}{\partial y}\right)_p + x\left(\frac{\partial}{\partial z}\right)_p + y\left(\frac{\partial}{\partial t}\right)_p\right),$$
$$V_p^3 := \frac{1}{2}\left(-t\left(\frac{\partial}{\partial x}\right)_p + z\left(\frac{\partial}{\partial y}\right)_p - y\left(\frac{\partial}{\partial z}\right)_p + x\left(\frac{\partial}{\partial t}\right)_p\right).$$
form a basis of T_pS^3.

(3) Consider now the so called Hopf map $h : S^3 \to S^2$,
$$h(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt))$$
and show that, for $p \in S^3$ and $q = h(p) \in S^2$, $(dh)_p : T_pS^3 \to T_qS^2$ sends V_p^1 to X_q^1, V_p^2 to X_q^2 and V_p^3 to X_q^3.

(4) Deduce that h is a submersion.

Then, interpreting V^1, V^2 and V^3 as vector fields on S^3,

(5) Find the maximal integral curve of V^1 starting at $p = (a, b, c, d)$ arbitrary.

(6) show that $[V^1, V^2] = V^3, \quad [V^3, V^2] = V^1, \quad [V^2, V^1] = V^3$.