HOMEWORK 6 (TO BE HANDED IN BY OCT 27, 2021)

Exercise 1. Please do the following

(1) If $F : M \to N$ is a smooth map between two manifolds M and N and $f \in \mathcal{C}^\infty(N)$ show that the pull-back of $df \in \Omega^1(N)$ by F coincides with the differential of the function $F^*(f) := f \circ F \in \mathcal{C}^\infty(M)$:

$$F^*(df) = d\left(F^*(f)\right).$$

(2) Consider the 1-form $-ydx + xdy \in \Omega^1(\mathbb{R}^2)$ and denote by $\omega \in \Omega^1(S^1)$ its restriction to S^1:

$$\omega := -ydx + xdy \quad \text{on} \quad S^1.$$

Show that ω is nowhere vanishing (i.e., for each $p \in S^1$, the element $\omega_p \in T_p^* S^1$ is non-zero).

(3) Show that the pull-back $e^*\omega \in \Omega^1(\mathbb{R})$ of ω by the map

$$e : \mathbb{R} \to S^1, \quad e(t) = (\cos t, \sin t)$$

coincides with the 1-form $dt \in \Omega^1(\mathbb{R})$ (where t is the notation for the coordinate in $\mathbb{R}$, i.e. dt is the differential of the function $t \mapsto t$).

(4) The form ω is sometimes denoted "$d\varphi$", where "φ" refers to "the angle coordinate" (so far, just notations!). To motivate/explain this, assume that we have a smooth function $\psi : U \to \mathbb{R}$, defined on some open $U \subset S^1$, such that

$$x = \cos(\psi(x, y)), \quad y = \sin(\psi(x, y)) \quad \text{forall} \quad (x, y) \in U.$$

Show that, indeed, $\omega = d\psi$ at all points in U.

(5) However, show that, despite the notation $d\varphi$ that is sometimes used, ω cannot be written as df for some smooth function $f \in \mathcal{C}^\infty(S^1)$.

(Hint: for the last point, use (1) and (3) and then inspect the outcome!)