(INTRO TO) Manifolds ① (WISB 342)

Keyword: smoothness

$\boxed{\text{Smoothness in } \mathbb{R}^n}$:

On $\mathbb{R}^n$: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^k$

smooth if { all partial derivatives are defined
all orders
at all points $x \in \mathbb{R}^n$ }

On opens $\Omega \subseteq \mathbb{R}^n$: $f: \Omega \longrightarrow \mathbb{R}^k$

$(\forall)\, x_0 \in \Omega$,
$\exists\, B(x_0, \varepsilon) \subseteq \Omega$ for some $\varepsilon > 0$.

On subsets $M \subseteq \mathbb{R}^n$: $f: M \longrightarrow \mathbb{R}^k$ is smooth if $(\forall)\, x \in M$

$\exists$ { $\Omega_x \subseteq \mathbb{R}^n$ open containing x
$\tilde{f}_x: \Omega_x \to \mathbb{R}^k$ smooth }

s.t. $f = \tilde{f}$ on $M \cap \Omega_x$.

$f|_{M \cap \Omega_x} = \tilde{f}|_{M \cap \Omega_x}$

• in g...

when...

Def...

D...

The...

Inverse Fct Thm: f: ...

RAINIC

Ω for some $\varepsilon > 0$

efined.

$\in M$

(2)

Rks:

- when $k=1$: $\mathbb{R}$-valued smooth fcts; $C^\infty(\mathbb{R}^n), C^\infty(\Omega), C^\infty(M)$
- in general: $f = (f_1, \ldots, f_k)$, f_i = smooth $\mathbb{R}$-valued fcts
- when $n=1$: $\gamma : \Omega \longrightarrow \mathbb{R}^k$: curves in $\mathbb{R}^k$
  $\Omega \subset \mathbb{R}$ an interval

Def: $f : N \longrightarrow M$, $N \subset \mathbb{R}^n$, $M \subset \mathbb{R}^m$, called a diffeomorphism if f = bijective, f, f^{-1} = smooth

Differentials: $f : \Omega \longrightarrow \mathbb{R}^k$ smooth, $\Omega \subset \mathbb{R}^n$ open, $x \in \Omega$ point

The differential of f at the point x,

$$(df)_x : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

- is the LINEAR ...
- given by the dir...
  $(df)_x($...
- represented by ...

Prop:

- Chain rule:
- If $f : \Omega \to$ ... ($\Omega \subset \mathbb{R}^n$)
  $\Rightarrow (df)_x =$ ...

$C^\infty(\Omega), C^\infty(M)$

$\mathbb{R}$-valued

cts

sm if

= smooth

smooth

$\in \Omega$

point

- is the LINEAR map that (3) approximates f around x.
- given by the directional derivatives at x:

$$(df)_x(\underset{\substack{\cap\\ \mathbb{R}^n \\ \text{vector}}}{v}) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} = \frac{d}{dt}\Big|_{t=0} f(x+tv)$$

- represented by:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x), & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ - & - & - \\ \frac{\partial f_k}{\partial x_1}(x) & \cdots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix}$$

$f \circ f^{-1} = \mathrm{Id}$

$f^{-1} \circ f = \mathrm{Id}.$

Prop:

- Chain rule: $\underset{\mathbb{R}^n}{\underset{\cup}{\Omega}} \xrightarrow{f} \underset{\mathbb{R}^k}{\underset{\cup}{\Omega'}} \xrightarrow{g} \underset{\mathbb{R}^l}{\underset{\cup}{\Omega''}}$ smooth, $x \in \Omega$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(df)_x"] \arrow[rr, bend right, "(d(g\circ f))_x"'] & \mathbb{R}^k \arrow[r, "(dg)_{f(x)}"] & \mathbb{R}^l \end{tikzcd}$$

- If $f: \Omega \to \Omega'$ is a diffeomorphism $\Longrightarrow$ ($\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^k$)

$\Rightarrow (df)_x$ = linear isomorphism $(\forall)\, x \in \Omega$.

Inverse Fct Thm : $f : \Omega \to \Omega'$ (4) smooth, $x \in \Omega$, $(df)_x$ = isomorphism

$\Longrightarrow$ f = a local diffeom around x, i.e.:

$$\exists \begin{cases} \Omega_x \subseteq \Omega \text{ open}, x \in \Omega_x \\ \Omega'_{f(x)} \subseteq \Omega' \text{ open}, f(x) \in \Omega'_{f(x)} \end{cases} \text{ s.t. } f|_{\Omega_x} : \Omega_x \longrightarrow \Omega'_{f(x)}$$

is a diffeomorphism.

Def: $f : \underset{\mathbb{R}^n}{\underset{\cap|\text{ open}}{\Omega}} \longrightarrow \underset{\mathbb{R}^k}{\underset{\cap|\text{ open}}{\Omega'}}$ smooth, $x \in \Omega$. Call f:

- an <u>IMMERSION</u> at x: if $(df)_x$ = injective
- a <u>SUBMERSION</u> at x: if $(df)_x$ = surjective

$=$ isomorphism
invertible
bijection

$f(x)$
morphism.

On $M \subseteq \mathbb{R}^n$: $f : M \longrightarrow \mathbb{R}^k$ smooth, $x \in M$

$$(df)_x : ? \longrightarrow \mathbb{R}^k$$

<u>Def</u>: The tangent space of M at $x \in M$ is

$$T_x^{geom} M = \left\{ \frac{d\gamma}{dt}(0) \;\Big/\; \gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^k \text{ smooth, taking values in } M \text{, s.t. } \gamma(0) = x \right\} \subseteq \mathbb{R}^n$$

<u>Exercise</u>: well defined $(df)_x : T_x^{geom} M \longrightarrow \mathbb{R}^k$

with

$\longrightarrow$ description 1: $v = \frac{d\gamma}{dt}(0) \longmapsto \frac{d\, f\circ\gamma}{dt}(0)$

$\longrightarrow$ description 2: $v \longmapsto (d\tilde{f})_x(v)$

where $\tilde{f}_x =$ smooth ext

Sm

Ex 1

represented by:

Smooth, $x \in M$

$x \in M$ is

$\mathbb{R}^k$ smooth, $\subseteq \mathbb{R}^n$

in M, s.t. $\gamma(0) = x$

$M \longrightarrow \mathbb{R}^k$

$\frac{d\gamma}{dt}(0) \longmapsto \frac{d f\circ\gamma}{dt}(0)$

$\longrightarrow (d\tilde{f}_x)_x(v)$

where $\tilde{f}_x$ = a smooth ext of f around x.

Smooth charts in $\mathbb{R}^n$ / ⑥ changing coordinates in $\mathbb{R}^n$

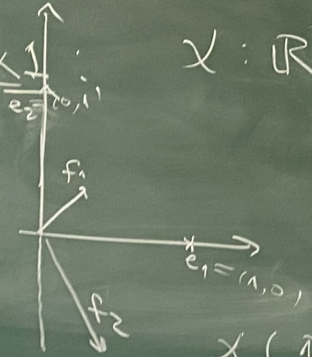
Smooth chart in $\mathbb{R}^n$: (U, x) with $U \subseteq \mathbb{R}^n$ open

and $x : U \longrightarrow \Omega$ diffeomorphism

$x = (x_1, \dots, x_n)$, $p \in U$ (domain of the chart), Ω open $\subseteq \mathbb{R}^n$

$(x_1(p), \dots, x_n(p))$ the coordinates of p w.r.t. (U, x).

Ex 1: $x : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



$$f_1 = \frac{e_1 + e_2}{3}$$

$$f_2 = \frac{e_1 - 2e_2}{3}$$

$x(p)$ = the coord (u, v) of p w.r.t. $\{f_1, f_2\}$, $p = (x, y)$

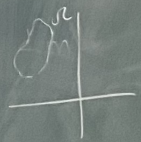
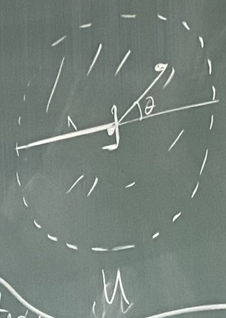
$$x(x, y) = (2x + y,\ x - y)$$

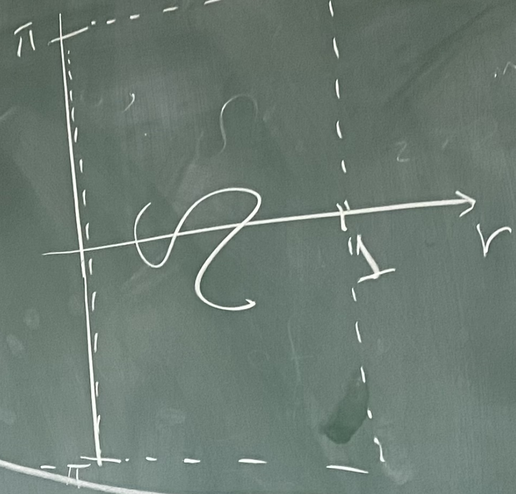
$$\begin{cases} x = \frac{u+v}{3} \\ y = \frac{u-2v}{3} \end{cases} \qquad \begin{cases} u = 2x + y \\ v = x - y \end{cases}$$

⑦

Ex 2: Polar coordinates

$x(x,y) = (r, \theta)$ defined by $\begin{cases} x = r \cdot \cos(\theta) \\ y = r \cdot \sin(\theta) \end{cases}$

Domain:  or: 



Represent fcts in "new coordinates"

$$f \; \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

chart (U, x) for $\mathbb{R}^n$ chart (U', x') for $\mathbb{R}^k$

$\Rightarrow$ f in new coord if $f^{x'}_{x} := x' \circ f \circ x^{-1}$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 5x^2 + 2xy + 2y^2 = (\underbrace{2x+y}_{u})^2 + (\underbrace{x-y}_{v})^2$

$\Rightarrow$ for x = chart from ex 1, $x' = \mathrm{Id}$, $f^{x'}_{x}(u,v) = u^2 + v^2$.

• when n...

• in general

• when $n = 1$

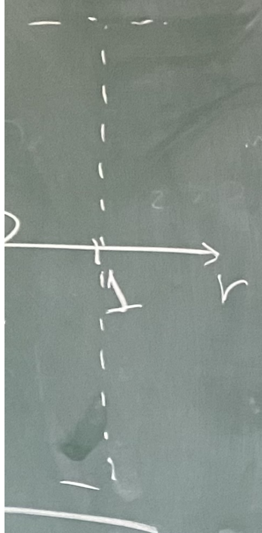
Def: f...

Differ...

The dif...

Inverse Fct Th...

7



Thm: $f : \underset{\substack{\cap \text{open} \\ \mathbb{R}^n}}{\tilde{U}} \longrightarrow \mathbb{R}^k$ smooth, $p \in \tilde{U}$ 8

Then:

(1) IMMERSION THM $(n \geq k)$ if f = submersion at p $\Longrightarrow$

$\Longrightarrow \exists\, (U, x)$ chart of $\mathbb{R}^n$ with $p \in U$

s.t. $f_x = f \circ x^{-1}$ is, near $x(p)$, given by

$$f_x(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k)$$

(2) SUBMERSION THM $(n \leq k)$ if f = immersion at p $\Rightarrow$

$\Rightarrow \exists\, (U', x')$ chart of $\mathbb{R}^k$ with $f(p) \in U'$

s.t. $f^{x'} = x' \circ f$ is, near p, given by

$$f^{x'}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{k-n})$$

- is the Li…
- given by t…
- represent…

Prop:

- Chain rul…
- If f : …

$\Rightarrow (df$…