①

Reminder:

- **vector fields** on M, $X \in \mathfrak{X}(M)$:

$$M \ni p \longmapsto X_p \in T_pM$$

& SMOOTH: $(\forall)\ x : U \xrightarrow{\simeq} \Omega$ chart,

$$X_p = \sum_i \underbrace{F^i(x(p))}_{\text{coeff}^i(p)} \left(\frac{\partial}{\partial x_i}\right)_p$$

smooth

$$\begin{tikzcd} U \arrow[dr, "\text{coeff}^i"] \arrow[d, "x"', "\wr"] & \\ \Omega \arrow[r, "F^i_x"'] & \mathbb{R} \end{tikzcd}$$

OR:

- for $M = \Omega \underset{\circ}{\subseteq} \mathbb{R}^m$ open:

$$X = \sum_i F^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(\Omega)$$

with $F^i \in C^\infty(\Omega)$.

coefficients.

①

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OR:

$$X|_U \in \mathfrak{X}(U)$$
$$\big\downarrow x_*$$
$$x_*(X|_U) = \sum F^i_x \frac{\partial}{\partial x_i}$$

- for $M = \Omega \subseteq \mathbb{R}^m$ open:

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coefficients.

in a

$$x : U \longrightarrow \Omega$$

$$x_* : \mathfrak{X}(U) \longrightarrow \mathfrak{X}(\Omega)$$

GIVEN: $F =$

To SOLVE

THM: $(\forall$

and an

coincid

s.t.

OPERATIONS INVOLVING $\mathfrak{X}(M)$: 1

a) **ADDITION**: $(X+Y)_p := X_p + Y_p$

b) MULTIPLICATION BY SCALARS $(\lambda \cdot X)_p := \lambda \cdot X_p$

c) — " — BY SMOOTH FUNCTIONS $(f \cdot X)_p = f(p) X_p$

i.e. OPERATION $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$(f, X) \longmapsto f \cdot X$$

d) ACTIONS OF $X \in \mathfrak{X}(M)$ by DERIVATIONS

$$\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$$

e) LIE BRACKET OF VECTOR FIELDS:

$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

- skew symmetric: $[Y, X] = -[X, Y]$
- bilinear: $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$
- $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$
  JACOBI IDENTITY.

$\mathfrak{X}(M)$ is:

- a vector space
- a $C^\infty(M)$-module
- in 1–1 with $\mathrm{Deriv}(C^\infty(M))$
- a Lie algebra

$X, Y \in \mathfrak{X}(M) \rightsquigarrow [X, Y] \in \mathfrak{X}(M)$

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X.$$

Reminder: (1)

- **vector fields** on M, $X \in \mathfrak{X}(M)$:

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& SMOOTH: $(\forall)\ x : U \xrightarrow{\simeq} \Omega \subseteq \mathbb{R}^m$ chart,

$$\left[\ X_p = \sum_i \underbrace{F^i_x(x(p))}_{\substack{\mathrm{coeff}^i(p) \\ \text{smooth}}} \left(\frac{\partial}{\partial x_i}\right)_p \right.$$

$$\begin{tikzcd} p \in U \arrow[dr, "\mathrm{coeff}^i"] \arrow[d, "x"', "\wr"] & \\ \Omega \arrow[r, "F^i_x"'] & \mathbb{R} \end{tikzcd}$$

OR:

$$\begin{tikzcd} X|_U \in \mathfrak{X}(U) \arrow[d, "x_*"] \\ x_*(X|_U) = \sum F^i_x \frac{\partial}{\partial x_i} \end{tikzcd}$$

- for $M = \Omega \subseteq \mathbb{R}^m$ open:

$$X = \sum_i F^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(\Omega)$$

with $F^i \in C^\infty(\Omega)$.
coefficients.

---

(2)

- for $\gamma : I \to M$ curve $(I \subseteq \mathbb{R}$ open interval$)$:

$$\boxed{\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M}$$

$$\left(\text{In } \mathbb{R}^m,\ \gamma = (\gamma_1, \ldots, \gamma_m):\quad \frac{d\gamma}{dt}(t) = \frac{d\gamma_1}{dt}(t) \left(\frac{\partial}{\partial x_1}\right)_{\gamma(t)} + \ldots \right)$$

DEF: Given $X \in \mathfrak{X}(M)$, an **INTEGRAL CURVE** of X: any $\gamma : I \to M$ s.t.

s.t.: $$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall)\ t \in I.$$

Say: γ **starts at p** $\in M$ if $\gamma(0) = p$.

- for $\gamma: I \to M$ C^∞ curve ($I \subseteq \mathbb{R}$ open interval

$$\boxed{\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M}$$

$\left(\begin{array}{l} \text{In } \mathbb{R}^m, \\ \gamma = (\gamma_1, \dots, \gamma_m) \end{array} \quad \frac{d\gamma_1}{dt}(t) \left(\frac{\partial}{\partial x_1} \right)_{\gamma(t)} + \dots \right)$

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Ex 3: $M=\mathbb{R}^2$, $X=\frac{\partial}{\partial x}+\lambda\frac{\partial}{\partial y}\in\mathfrak{X}(\mathbb{R}^2)$, $\lambda\in\mathbb{R}$

Int curve $\gamma(t)=(x(t),y(t))$

$$\dot\gamma(t)=X(\gamma(t))$$

$$\dot x(t)\left(\frac{\partial}{\partial x}\right)_{\gamma(t)}+\dot y(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)}=\left(\frac{\partial}{\partial x}\right)_{\gamma(t)}+\lambda\left(\frac{\partial}{\partial y}\right)_{\gamma(t)}$$

$$\Rightarrow\begin{cases}\dot x(t)=1\\ \dot y(t)=\lambda\end{cases}\Rightarrow\begin{cases}x(t)=t+a\\ y(t)=\lambda t+b.\end{cases}\Rightarrow$$

$\Rightarrow$ the i. curves are $\gamma_{a,b}$ defined by $\gamma_{a,b}(t)=(t+a,\lambda t+b)$.

Rk 1: Prescribing $\gamma(0)=(a,b)\Rightarrow$ only one IC starting there

Rk 2: $\gamma_{a,b}:\mathbb{R}\to M(=\mathbb{R}^2)$. I.e. may take $I=\mathbb{R}$

Ex: $y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\rightsquigarrow$ i. curves parametrize circles.

More g...

$X\in\mathfrak{X}($...

curve...

to sol...

$1\le i\le$...

Initial condi...

<u>Ex</u>: $M = \mathbb{R}^2$, $X = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ 4

Int curves $\gamma = (x, y)$ are solutions of

$$\begin{cases} \dot{x} = -x^2 \\ \dot{y} = -y \end{cases} \quad \dots \Rightarrow \gamma_{a,b}(t) = \left(\frac{a}{at+1}, b e^{-t} \right)$$

<u>Rk 1</u>: ✓

<u>Rk 2</u>: ✗ e.g., for $a > 0$:

$$\gamma_{a,b} : \underbrace{\left(-\frac{1}{a}, \infty \right)}_{\text{largest one (containing } 0)} \longrightarrow M \; (= \mathbb{R}^2)$$

5

More generally, in any $\Omega \subseteq \mathbb{R}^m$ open:

$X \in \mathfrak{X}(\Omega)$: $X = \sum_{i=1}^{m} F^i \frac{\partial}{\partial x_i}$

curve γ: $\gamma = (\gamma^1, \ldots, \gamma^m)$

to solve: $1 \le i \le m$

$$\frac{d\gamma^i}{dt}(t) = F^i(\gamma^1(t), \ldots, \gamma^m(t))$$

Initial conditions: $\gamma^1(0) = a$, $\gamma^2(0) = b$, etc.

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$1 \leq i \leq m$

Initial conditions: $\gamma^1(0) = a, \gamma^2(0) = b$, etc

6

GIVEN: $F = (F_1, \ldots, F_m) : \Omega \to \mathbb{R}^m$ smooth

To SOLVE $\boxed{\frac{d\gamma}{dt}(t) = F(\gamma(t)), \; F(0) = x}$ (*)
given

ODE's. Cauchy local $\exists!$:

THM: $(\forall)\, x \in \Omega$, (*) has a solution $\gamma : I \to \Omega$ and any two solutions $\gamma_1 : I_1 \to \Omega$, $\gamma_2 : I_2 \to \Omega$ coincide in an open $I \subseteq I_1 \cap I_2$, with $\boxed{0 \in I}$.

Moreover, $(\forall)\, x_0 \in \Omega$

$(\exists)\, \underset{\substack{\Psi \\ x_0}}{\Omega_{x_0}} \subseteq \Omega$ open, $(\exists)\, \varepsilon > 0$

$(\exists)\, \Phi : \Omega_{x_0} \times (-\varepsilon, \varepsilon) \to \Omega$ smooth

s.t., $(\forall)\, x \in \Omega_{x_0}$,

$\Phi(x, \cdot) : (-\varepsilon, \varepsilon) \to \Omega$ IS SOLUTION OF (*)

open :

$(t))$

etc.

GIVEN: $F = (F_1, \dots, F_m) : \Omega \to \mathbb{R}^m$ smooth 6

TO SOLVE $\boxed{\dfrac{d\gamma}{dt}(t) = F(\gamma(t)),\ F(0) = x}$ $(*)$

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$\Phi(x, \cdot) : (-\varepsilon, \varepsilon) \to \Omega$ IS SOLUTION OF $(*)$

7

Prop 3.70: Given $X \in \mathfrak{X}(M)$:

1. $(\forall)\, p \in M$, $\exists$ int. curve $\gamma : I \longrightarrow M$ of X, $\gamma(0) = p$.
2. any two integral curves γ_1, γ_2 of X with $\gamma_1(t_0) = \gamma_2(t_0)$ for some $t_0 \Rightarrow \gamma_1 = \gamma_2$ in a $(t_0 - \varepsilon, t_0 + \varepsilon)$ range.

Corollary: Given $X \in \mathfrak{X}(M)$, $(\forall)\, p \in M$

$\exists!\ \gamma_p : I_p \longrightarrow M$

maximal integral curve of X starting at p.

Pf: Look at:

$$\begin{cases} \dot{\gamma}(t) = X_{\gamma(t)} \\ \gamma(0) = p \end{cases}$$

in a chart χ around p

$\Downarrow$

problem in $\Omega \subseteq \mathbb{R}^m$

where $F^i = F_X^i$

$\Downarrow$ Cauchy

☺

5

More gene

$X \in \mathfrak{X}(\Omega)$:

curve γ:

to solve:

$1 \leq i \leq m$

Initial conditions

• for $\gamma : I \longrightarrow M$ curve ($I \subseteq \mathbb{R}$ open interval):

$$\boxed{\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M}$$

(In $\mathbb{R}^n$, $\gamma = (\gamma_1, \dots, \gamma_n)$ $\qquad \frac{d\gamma_1}{dt}(t)\left(\frac{\partial}{\partial x_1}\right)_{\gamma(t)} + \dots$)

Def: Given $X \in \mathfrak{X}(M)$, an INTEGRAL CURVE of X: any $\gamma : I \longrightarrow M$ s.t.

s.t.:
$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall)\, t \in I.$$

Say: γ starts at $p \in M$ if $\gamma(0) = p$.

Def: Maximal integral curve of X: $\gamma : I \longrightarrow M$ which has no extension $\tilde{\gamma} : \tilde{I} \longrightarrow M$ with $\tilde{I} \supsetneq I$ (still int. curve).

PREVIOUS EXAMPLES

1st: $I_{a,b} = \mathbb{R}$, $\gamma_{a,b} : \mathbb{R} \to M$

3rd: $I_{a,b} = \begin{cases} (-\frac{1}{a}, \infty) & a > 0 \\ \mathbb{R} & a = 0 \\ (-\infty, -\frac{1}{a}) & a < 0 \end{cases}$

roundp

$\in \mathbb{R}^m$

PREVIOUS EXAMPLES

1st: $I_{a,b} = \mathbb{R}$, $\gamma_{a,b}: \mathbb{R} \to M$

3rd: $I_{a,b} = \begin{cases} (-\frac{1}{a}, \infty) & a>0 \\ \mathbb{R} & a=0 \\ (-\infty, -\frac{1}{a}) & a<0 \end{cases}$

② • for $\gamma: I \longrightarrow M$ curve ($I \subseteq \mathbb{R}$ open interval):

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<u>DEF</u>: Given $X \in \mathfrak{X}(M)$, an <u>INTEGRAL CURVE</u> of <u>X</u>: any $\gamma: I \to M$ s.t.

s.t.:

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall) t \in I.$$

<u>Say</u>: γ starts at $p \in M$ if $\gamma(0) = p$.

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$$\begin{cases} \frac{d}{dt}\gamma_p(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p. \end{cases}$$

Corollary: Given $X \in \mathfrak{X}(M)$, $(\forall)\, p \in M$

$\exists!$ $\boxed{\gamma_p : I_p} \to M$ ← notations to be used

maximal integral curve of X starting at p

pf: Look at:

$$\begin{cases} \dot\gamma(t) = X_{\gamma(t)} \\ \gamma(0) = p \end{cases}$$

in a chart X around p

$\Downarrow$

problem in $\Omega \subseteq \mathbb{R}^n$

where $F^i = F^i_X$

$\Downarrow$ Cauchy

☺

pf of COROLLARY:

KEY REMARK: $\gamma_1 : I_1 \to M$, $\gamma_2 : I_2 \to M$

integral curves, $\exists\, t_0 \in I_1 \cap I_2$ s.t. $\gamma_1(t_0) = \gamma_2(t_0)$

$\Longrightarrow \gamma_1(t) = \gamma_2(t)\ (\forall)\, t \in I_1 \cap I_2$.

pf: $A := \{t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t)\} \ni t_0$ — closed in $I_1 \cap I_2$, open $I_1 \cap I_2$, $I_1 \cap I_2$ = connected $\Rightarrow A = I_1 \cap I_2$

• for $\gamma : I \to M$

$\boxed{\frac{d\gamma}{dt}}$

(In $\mathbb{R}^m$, $\gamma = (\gamma_1, \dots, \gamma_m)$)

DEF: Given X

INTEGRAL CURV

s.t.: $\frac{d\gamma}{dt}$

Say: γ starts at

Def: Maximal

$\gamma : I \to M$ wh

$\tilde\gamma : \tilde I \to M$ w

(8")

- for $\gamma: I \to M$ curve ($I \subseteq \mathbb{R}$ open interval):

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Def: Maximal integral curve of X: $\gamma: I \to M$ which has no extension $\tilde{\gamma}: \tilde{I} \to M$ with $\tilde{I} \supsetneq I$ (still int. curve).

Def: $X \in \mathfrak{X}(M)$ called COMPLETE if $I_p = \mathbb{R}$ $(\forall)\, p$.

8''

Corollary: Given $X \in \mathfrak{X}(M)$, $(\forall)\, p \in M$

$\exists!\ \boxed{\gamma_p : I_p} \to M$ notations to be used.

maximal integral curve of X starting at p.

"((.))"

THEOREM: If M = compact $\Rightarrow$

$\Rightarrow$ all $X \in \mathfrak{X}(M)$ are complete

8

$\phi^s(p)$

$\begin{cases} \frac{d}{dt}\phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p \end{cases}$

$X_{\phi(p,t)}$

Given $X \in \mathfrak{X}(M)$, $(\forall)\, p \in M$

$\gamma_p \to M$ — notations to be used

integral curve of X starting at p

If M = compact $\Rightarrow$ $X \in \mathfrak{X}(M)$ are complete

9

Flows

1st assume $X \in \mathfrak{X}(M)$ complete.

$$\gamma_p(t) =: \phi(p,t) = \phi^t(p)$$

$p \in M,\ t \in \mathbb{R}$

Now:

$\phi : M \times \mathbb{R} \longrightarrow M$ THE FLOW OF X

$\phi^t : M \longrightarrow M$ THE FLOW OF X at t.

$(t \in \mathbb{R})$

Theorem: If $X \in \mathfrak{X}(M)$ complete $\Rightarrow$

$\Rightarrow \phi^t : M \to M$ are diffeomorphisms (!!!!)

and: $\boxed{\phi^t \circ \phi^s = \phi^{t+s},\ \phi^0 = Id.\ (\text{Bonus})}$

Pf: Bonus $\Rightarrow$ ϕ^t's are diffeo: $\phi^t \circ \phi^{-t} = Id = \phi^{-t} \circ \phi^t$

$\Rightarrow \phi^t$ is invertible with inverse ϕ^{-t}. $\Rightarrow$ ☐

Still to do: $\phi^t(\phi^s(p)) = \phi^{t+s}(p)$

Fix $s \in \mathbb{R}, p \in M$, look at $\begin{cases} t \mapsto \phi^t(\phi^s(p)) & \text{starts at } \phi^s(p) \text{ i.c. of } X \\ t \mapsto \phi^{t+s}(p) & \text{" — "} \end{cases}$

$\Rightarrow$ SAME !!

For general X (t.b.c.):

$(p,t) \longmapsto \gamma_p(t)$

First, $\phi : M \times \mathbb{R} \longrightarrow M$ is smooth.

a local property

coordinates: follows from 2nd part of Cauchy

$$\begin{cases} \dfrac{d}{dt}\gamma_p(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p. \end{cases}$$

The defining properties of $\gamma_p(t) = \phi(p,t) = \phi^t(p)$

$$\begin{cases} \dfrac{d}{dt}\phi(p,t) = X_{\phi(p,t)} \\ \phi(p,0) = p. \end{cases}$$

$$\begin{cases} \dfrac{d}{dt}\phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p. \end{cases}$$

($\gamma_p : I_p \to M$ = the (unique) maximal integral curve of X starting at p).