

(1)

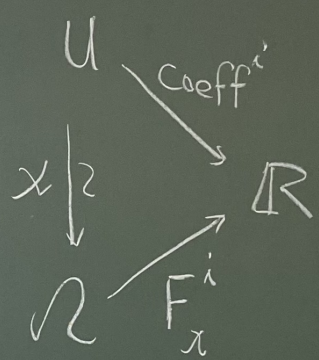
Reminder:

• vector fields on M , $X \in \mathcal{X}(M)$:

$$M \ni p \mapsto X_p \in T_p M$$

& SMOOTH: $(\forall) \chi: U \xrightarrow{\sim} \Omega$ chart,

$$X_p = \sum_i \underbrace{F^i(\chi(p))}_{\substack{\text{coeff}^i(p) \\ \uparrow \\ \text{smooth}}} \left(\frac{\partial}{\partial x_i} \right)_p$$



• for $M = \Omega \subseteq \mathbb{R}^m$ open:

$$X = \sum_i F^i \frac{\partial}{\partial x_i} \in \mathcal{X}(\Omega)$$

with $F^i \in C^\infty(\Omega)$.
Coefficients.

OR:

Reminder:

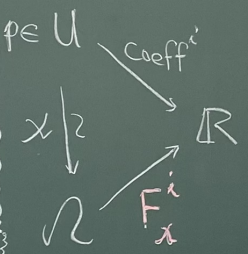
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• vector fields on M , $X \in \mathfrak{X}(M)$:

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& SMOOTH: $(\forall) \chi: U \xrightarrow{\cong} \mathbb{R}^m$ chart,

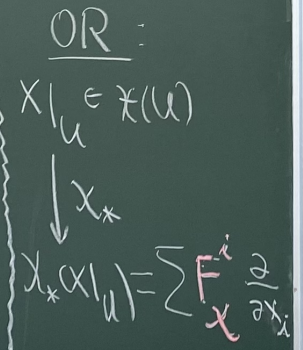
$$X_p = \sum_i \underbrace{F^i(\chi(p))}_{\substack{\text{coeff}^i(p) \\ \text{smooth}}} \left(\frac{\partial}{\partial x_i} \right)_p$$



• for $M = \mathcal{N} \subseteq \mathbb{R}^m$ open:

$$X = \sum_i F^i \left(\frac{\partial}{\partial x_i} \right) \in \mathfrak{X}(\mathcal{N})$$

with $F^i \in C^\infty(\mathcal{N})$.
coefficients.



$$\chi: U \rightarrow \mathcal{N}$$

$$\chi_*: \mathfrak{X}(U) \rightarrow \mathfrak{X}(\mathcal{N})$$

GIVEN: $F = \frac{dx}{dt}$
TO SOLVE

THM: and an coincid

s.t.

OPERATIONS INVOLVING $\mathfrak{X}(M)$: ①

a) ADDITION: $(X+Y)_p := X_p + Y_p$

b) MULTIPLICATION BY SCALARS: $(\lambda \cdot X)_p := \lambda X_p$

c) — BY SMOOTH FUNCTIONS $(f \cdot X)_p := f(p) X_p$

i.e. OPERATION $C^\infty(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$

$$(f, X) \mapsto f \cdot X$$

d) ACTIONS OF $X \in \mathfrak{X}(M)$ BY DERIVATIONS

$$\mathcal{L}_X : C^\infty(M) \rightarrow C^\infty(M)$$

e) LIE BRACKET OF VECTOR FIELDS:

$$[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

• skew symmetric: $[X, Y] = -[Y, X]$

• bilinear: $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$

• $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

JACOBI IDENTITY.

$\mathfrak{X}(M)$ is:

a vector space

a $C^\infty(M)$ -module

in 1-1 with $\text{Deriv}(C^\infty(M))$

a Lie algebra

$$X, Y \in \mathfrak{X}(M) \mapsto [X, Y] \in \mathfrak{X}(M)$$

$$\mathcal{L}_{[X, Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$$

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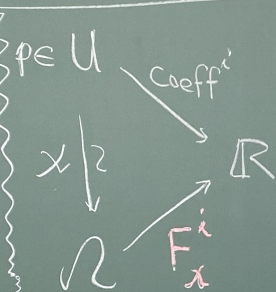
• vector fields on M , $X \in \mathfrak{X}(M)$:

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$$X_p = \sum_i \underbrace{F^i(\chi(p))}_{\text{coeff}^i(p)} \left(\frac{\partial}{\partial x_i} \right)_p$$

Smooth



OR:

$$X|_U \in \mathfrak{X}(U)$$

$\downarrow \chi_*$

$$\chi_* X|_U = \sum_i F^i \frac{\partial}{\partial x_i}$$

• for $M = \Omega \subseteq \mathbb{R}^m$ open:

$$X = \sum_i F^i \left(\frac{\partial}{\partial x_i} \right) \in \mathfrak{X}(\Omega)$$

with $F^i \in C^\infty(\Omega)$.
Coefficients.

• for $\gamma: I \rightarrow M$ curve ($I \subseteq \mathbb{R}$ open interval):

$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M$$

$$\left(\text{In } \mathbb{R}^m, \gamma = (\gamma_1, \dots, \gamma_m) \right) \left(\frac{d\gamma_i}{dt}(t) \left(\frac{\partial}{\partial x_i} \right)_{\gamma(t)} + \dots \right)$$

DEF: Given $X \in \mathfrak{X}(M)$, an INTEGRAL CURVE of X : any $\gamma: I \rightarrow M$ s.t.

s.t.:
$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall) t \in I.$$

Say: γ starts at $p \in M$ if $\gamma(0) = p$.

• for $\gamma: I \rightarrow M$ curve ($I \subseteq \mathbb{R}$ open interval)

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(In \mathbb{R}^m ,
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Ex ⁽³⁾: $M = \mathbb{R}^2$, $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$, $\lambda \in \mathbb{R}$

Int curve $\gamma(t) = (x(t), y(t))$

$$\dot{\gamma}(t) = X(\gamma(t))$$

$$\dot{x}(t) \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \dot{y}(t) \left(\frac{\partial}{\partial y} \right)_{\gamma(t)} = \left(\frac{\partial}{\partial x} \right)_{\gamma(t)} + \lambda \left(\frac{\partial}{\partial y} \right)_{\gamma(t)}$$

$$\Rightarrow \begin{cases} \dot{x}(t) = 1 \\ \dot{y}(t) = \lambda \end{cases} \Rightarrow \begin{cases} x(t) = t + a \\ y(t) = \lambda t + b \end{cases} \Rightarrow$$

\Rightarrow the i. curves are $\gamma_{a,b}$ defined by $\gamma_{a,b}(t) = (t+a, \lambda t+b)$.

Rk1: Prescribing $\gamma(0) = (a,b) \Rightarrow$ only one IC starting there

Rk2: $\gamma_{a,b}: \mathbb{R} \rightarrow M = \mathbb{R}^2$. i.e. may take $I = \mathbb{R}$

Ex: $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \mapsto$ i. curves parametrize circles.

More of

$X \in \mathfrak{X}(M)$

Curve

to solve
 $1 \leq t \leq$

Initial
condi

$$\text{Ex: } M = \mathbb{R}^2, X = -x^2 \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \quad (4)$$

Int curves $\gamma = (x, y)$ are solutions of

$$\begin{cases} \dot{x} = -x^2 \\ \dot{y} = -y \end{cases} \Rightarrow \gamma_{a,b}(t) = \left(\frac{a}{at+1}, b e^{-t} \right)$$

Rk 1: ✓

Rk 2: X

e.g., for $a > 0$:

$$\gamma_{a,b} : \underbrace{\left(-\frac{1}{a}, \infty\right)} \rightarrow M (= \mathbb{R}^2)$$

largest one (containing 0)

More generally, in any $\Omega \subseteq \mathbb{R}^m$ open:

$$\underline{X \in \mathcal{X}(\Omega)}: X = \sum_{i=1}^m F^i \frac{\partial}{\partial x_i}$$

Curve γ : $\gamma = (\gamma^1, \dots, \gamma^m)$

to solve:

$$\frac{d\gamma^i}{dt}(t) = F^i(\gamma^1(t), \dots, \gamma^m(t))$$

Initial conditions: $\gamma^1(0) = a, \gamma^2(0) = b, \dots$

GIVEN: F

TO SOLVE

THM:

and an

coincide

s.t.,

More generally, in any $\Omega \subseteq \mathbb{R}^m$ open:

$$X \in \mathcal{X}(\Omega): X = \sum_{i=1}^m F^i \frac{\partial}{\partial x_i}$$

curve $\gamma: \gamma = (\gamma^1, \dots, \gamma^m)$

to solve: $\frac{d\gamma^i}{dt}(t) = F^i(\gamma^1(t), \dots, \gamma^m(t))$

Initial conditions: $\gamma^1(0) = a, \gamma^2(0) = b, \dots$

GIVEN: $F = (F_1, \dots, F_m): \Omega \rightarrow \mathbb{R}^m$ smooth (6)

TO SOLVE: $\frac{d\gamma}{dt}(t) = F(\gamma(t)), \gamma(0) = \gamma_0$ (*) given

ODEs. Cauchy local $\exists!$:

THM: $(\forall) \gamma_0 \in \Omega, (*)$ has a solution $\gamma: I \rightarrow \Omega$ and any two solutions $\gamma_1: I_1 \rightarrow \Omega, \gamma_2: I_2 \rightarrow \Omega$ coincide in an open $I \subseteq I_1 \cap I_2$, with $0 \in I$.

Moreover, $(\forall) \gamma_0 \in \Omega$

$(\exists) \Omega_{\gamma_0} \subseteq \Omega$ open $(\exists) \varepsilon > 0$

$(\exists) \phi: \Omega_{\gamma_0} \times (-\varepsilon, \varepsilon) \rightarrow \Omega$ smooth

s.t. $(\forall) \gamma_0 \in \Omega_{\gamma_0}$,

$\phi(\gamma_0, \cdot): (-\varepsilon, \varepsilon) \rightarrow \Omega$ IS SOLUTION OF (*)

open:

GIVEN: $F = (F_1, \dots, F_m): \Omega \rightarrow \mathbb{R}^m$ smooth (6)

TO SOLVE

$\frac{dx}{dt}(t) = F(x(t)), F(0) = x_0$ (*)
given

ODEs. Cauchy local E!

THM: (forall) $x_0 \in \Omega, (*)$ has a solution $\gamma: I \rightarrow \Omega$
and any two solutions $\gamma_1: I_1 \rightarrow \Omega, \gamma_2: I_2 \rightarrow \Omega$
coincide in an open $I \subseteq I_1 \cap I_2$, with $x_0 \in I$.

Moreover, (forall) $x_0 \in \Omega$

(exists) $\Omega_{x_0} \subseteq \Omega$ open (exists) $\varepsilon > 0$
 \downarrow
 x_0

(exists) $\phi: \Omega_{x_0} \times (-\varepsilon, \varepsilon) \rightarrow \Omega$
smooth

s.t., (forall) $x \in \Omega_{x_0}$,
 $\phi(x, \cdot): (-\varepsilon, \varepsilon) \rightarrow \Omega$ IS SOLUTION OF (*)

(t)
etc.



Prop 3.70: Given $X \in \mathfrak{X}(M)$: (7)


1. $(\forall) p \in M$, \exists int curve $\gamma: I \rightarrow M$
of X , $\gamma(0) = p$.
2. any two integral curves γ_1, γ_2
of X with $\gamma_1(t_0) = \gamma_2(t_0)$ for
some $t_0 \Rightarrow \gamma_1 = \gamma_2$ in a $(t_0 - \varepsilon, t_0 + \varepsilon)$
range.

Corollary: Given $X \in \mathfrak{X}(M)$, $(\forall) p \in M$
 $\exists!$ $\gamma_p: I_p \rightarrow M$
maximal integral curve of X starting at p .

pf: Look at:
$$\begin{cases} \dot{\gamma}(t) = X_{\gamma(t)} \\ \gamma(0) = p \end{cases}$$

in a chart \mathcal{X} around p

\Downarrow
problem in $\mathcal{R} \subseteq \mathbb{R}^m$
where $F^i = F^i_X$

\Downarrow Cauchy


More general (5)

$X \in \mathfrak{X}(\mathcal{R})$:

Curve γ :

to solve:
 $1 \leq i \leq m$

Initial conditions

• for $\gamma: I \rightarrow M$ curve ($I \subseteq \mathbb{R}$ open interval):

$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M$$

(In \mathbb{R}^m , $\gamma = (\gamma_1, \dots, \gamma_m)$ $\left(\frac{d\gamma_1}{dt}(t) \frac{\partial}{\partial x_1} \Big|_{\gamma(t)} + \dots \right)$)

DEF: Given $X \in \mathfrak{X}(M)$, an INTEGRAL CURVE of X : any $\gamma: I \rightarrow M$ s.t.

s.t. $\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall t \in I)$

Say: γ starts at $p \in M$ if $\gamma(0) = p$.

Def: Maximal integral curve of X :
 $\gamma: I \rightarrow M$ which has no extension
 $\tilde{\gamma}: \tilde{I} \rightarrow M$ with $\tilde{I} \supsetneq I$ (still int. curve)

PREVIOUS EXAMPLES
 1st: $I = \mathbb{R}, \gamma: \mathbb{R} \rightarrow M$
 2nd: $I = (-\frac{1}{a}, \frac{1}{a}), a > 0$
 $I = \mathbb{R}, a = 0$
 $I = (-\infty, -\frac{1}{a}), a < 0$

• for $\gamma: I \rightarrow M$ curve ($I \subseteq \mathbb{R}$ open interval):

$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M$$

(In \mathbb{R}^m ,
 $\gamma = (\gamma_1, \dots, \gamma_m)$) $\frac{d\gamma_i}{dt}(t) \frac{\partial}{\partial x_i} \Big|_{\gamma(t)} + \dots$

DEF: Given $X \in \mathfrak{X}(M)$, an INTEGRAL CURVE of X : any $\gamma: I \rightarrow M$ s.t.

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 curve

PREVIOUS
 EXAMPLES

1st: $I = \mathbb{R}, \gamma: \mathbb{R} \rightarrow M$

2nd: $I = \mathbb{R}, \gamma: \mathbb{R} \rightarrow M$

$I = \mathbb{R}, \gamma: \mathbb{R} \rightarrow M$
 $\begin{cases} a > 0 \\ a = 0 \\ a < 0 \end{cases}$
 $\left(-\frac{1}{a}, \infty \right)$
 $\left(-\infty, -\frac{1}{a} \right)$

$$\begin{cases} \frac{d}{dt} \gamma_p(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p \end{cases}$$

Corollary: Given $X \in \mathfrak{X}(M)$, $(t) p \in M$
 $\exists!$ $\gamma_p: I_p \rightarrow M$ (notations to be used)
 maximal integral curve of X starting at p
 ((()))

~~pf~~ Look at:
 $\begin{cases} \dot{\gamma}(t) = X_{\gamma(t)} \\ \gamma(0) = p \end{cases}$
 in a chart \mathcal{X} around p
 \Downarrow
 problem in $\mathcal{R} \subseteq \mathbb{R}^m$
 where $F^i = F^i_X$
 \Downarrow Cauchy
 ☺

~~pf~~ of COROLLARY:

KEY REMARK: $\gamma_1: I_1 \rightarrow M, \gamma_2: I_2 \rightarrow M$
 integral curves, $\exists t_0 \in I_1 \cap I_2$ st. $\gamma_1(t_0) = \gamma_2(t_0)$
 $\Rightarrow \gamma_1(t) = \gamma_2(t) \forall t \in I_1 \cap I_2$

~~pf~~: $A := \{t \in I_1 \cap I_2 : \gamma_1(t) = \gamma_2(t)\}$ $\left\{ \begin{array}{l} \text{closed in } I_1 \cap I_2 \\ \text{open in } I_1 \cap I_2 \end{array} \right\}$ $A = I_1 \cap I_2$
 $\cup t_0$ $I_1 \cap I_2 = \text{connected}$

for $\gamma: I \rightarrow M$

$$\frac{d\gamma}{dt}$$

(In \mathbb{R}^m ,
 $\gamma = (\gamma_1, \dots, \gamma_m)$)

DEF: Given X
INTEGRAL CURVE

s.t. $\frac{d\gamma}{dt}$

Say: γ starts at

Def: Maximal
 $\gamma: I \rightarrow M$ wh
 $\tilde{\gamma}: \tilde{I} \rightarrow M$ wh

• for $\gamma: I \rightarrow M$ ^(γ') curve ($I \subseteq \mathbb{R}$ open interval):

$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)} M$$

(In \mathbb{R}^m ,
 $\gamma = (\gamma_1, \dots, \gamma_m)$)

$$\left(\frac{d\gamma_1}{dt}(t) \frac{\partial}{\partial x_1} \Big|_{\gamma(t)} + \dots \right)$$

DEF: Given $X \in \mathfrak{X}(M)$, an INTEGRAL CURVE of X : any $\gamma: I \rightarrow M$ s.t.

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Def: Maximal integral curve of X :

$\gamma: I \rightarrow M$ which has no extension

$\tilde{\gamma}: \tilde{I} \rightarrow M$ with $\tilde{I} \supsetneq I$ (still int.)
 curve

Def: $X \in \mathfrak{X}(M)$ called COMPLETE if $I_p = \mathbb{R} \forall p$.

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Corollary: Given $X \in \mathfrak{X}(M)$, $(\neq) p \in M$

$$\exists ! \gamma_p : I_p \rightarrow M$$

(notations to be used)

maximal integral curve of X starting at p

THEOREM: If $M = \text{compact} \Rightarrow$
 \Rightarrow all $X \in \mathfrak{X}(M)$ are complete

⑧

$$\begin{cases} \frac{d}{dt} \phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p \end{cases}$$

Given $X \in \mathfrak{X}(M)$, $(p) \in M$
 $\int_p \rightarrow M$ (notations to be used)
 integral curve of X starting at p
 "((...))"
 If $M = \text{compact} \Rightarrow$
 $\mathfrak{X}(M)$ are complete

Flows. 1st assume $X \in \mathfrak{X}(M)$ complete. ⑨

$$\gamma_p(t) =: \phi(p, t) = \phi^t(p)$$

$p \in M, t \in \mathbb{R}$

Now: $\Phi: M \times \mathbb{R} \rightarrow M$ THE FLOW OF X

$\phi^t: M \rightarrow M$ THE FLOW OF X at t .
 $(t \in \mathbb{R})$

Theorem: If $X \in \mathfrak{X}(M)$ complete \Rightarrow
 $\Rightarrow \phi^t: M \rightarrow M$ are diffeomorphisms (!!!)
 and: $\phi^t \circ \phi^s = \phi^{t+s}, \phi^0 = \text{Id.}$ (Bonus)

⚡ Bonus $\Rightarrow \phi^t$'s are diffeo: $\phi^t \circ \phi^{-t} = \text{Id} = \phi^{-t} \circ \phi^t$
 $\Rightarrow \phi^t$ is invertible with inverse $\phi^{-t} \Rightarrow \square$

Still to do: $\phi^t(\phi^s(p)) = \phi^{t+s}(p)$ $\left\{ \begin{array}{l} t \mapsto \phi^t(\phi^s(p)) \text{ starts at } \phi^s(p) \text{ i.c. of } X \\ t \mapsto \phi^{t+s}(p) \text{ starts at } p \end{array} \right.$
 Fix $s \in \mathbb{R}, p \in M$, look at $\left\{ \begin{array}{l} t \mapsto \phi^t(\phi^s(p)) \\ t \mapsto \phi^{t+s}(p) \end{array} \right.$ \Rightarrow "SAME!!"

For general X (t.b.c.):

$(p, t) \mapsto \gamma_p(t)$
 First, $\Phi: M \times \mathbb{R} \rightarrow M$
 is smooth.

a local property
 coordinates: follows from 2nd part of Cauchy

$$\begin{cases} \frac{d}{dt} \gamma_p(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p. \end{cases}$$

The defining properties
of $\gamma_p(t) = \phi(p, t) = \phi^t(p)$

$$\begin{cases} \frac{d}{dt} \phi(p, t) = X_{\phi(p, t)} \\ \phi(p, 0) = p. \end{cases}$$

$$\begin{cases} \frac{d}{dt} \phi^t(p) = X_{\phi^t(p)} \\ \phi^0(p) = p. \end{cases}$$

$(\gamma_p = \int_p \rightarrow M = \text{the (unique) maximal integral curve of } X \text{ starting at } p).$